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Magnus force acting upon a rotating sphere passing in an incompressible viscous flow

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I would like to express sincere appreciation and gratitude to my supervisor Professor Zdeněk Martinec who was taking care of me in Dublin where main ideas of this thesis were mainly written. I am also thankful to my family and close friends supporting me on my personal endless journey toward knowledge...

Title: Magnus force acting upon a rotating sphere passing in an incompressible viscous flow

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Abstract: Classical results of hydrodynamics such as Stokes' force law and Kirchhoff's moment (torque) law are re-derived for laminar viscous flow in the framework of modern compact simplified vector calculus notation. First perturbations of these laws are found and compared visually with experiments. The Magnus drag force on a rotating and moving sphere surrounded by an incompressible viscous Newtonian fluid is derived from the perturbation series of the Navier-Stokes equations in low speed regimes with a small Reynolds number.

Abstrakt (v češtině): Jsou znovuodvozeny klasické výsledky hydromechaniky jako např. Stokesův a Kirchhofův zákon ve zjednodušené kompaktní vektorové symbolice. Tyto zákony jsou opraveny o první perturbaci malého Reynoldsova čísla a poté vizuálně porovnány s experimentem. Pro dokonalou rotující a pohybující se kouli obklopenou Newtonovskou vizkózní kapalinou je pro malé podélné rychlosti a malé rotace (malá Reynoldsova čísla) odvozena Magnusova síla z perturbační řady Navierových-Stokesových rovnic.

Keywords: Magnus Force, Navier-Stokes Equations, Hydrodynamics

Contents

List of symbols

$\mathcal{O}$	Big O notation
$a, b, c \ldots$	scalars/scalar fields
Δ	Laplacian operator
ϱ	density (of a fluid)
v_∞	far-distance speed
ω	angular speed
Γ	ratio of angular versus far-distance speed
t	time
η	dynamical viscosity
p	pressure
Re	Reynolds number
x, y, z	Cartesian coordinates
ρ, φ, z	cylindrical coordinates
r, θ, φ	spherical coordinates
w, y, s	rotated Cartesian coordinates
σ, ϕ, s	rotated cylindrical coordinates
r, ϑ, ϕ	rotated spherical coordinates
$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \ldots$	vectors/vector fields
$\mathbf{0}$	null vector
$\boldsymbol{r}$	position vector
$\boldsymbol{v}$	velocity vector field
$\boldsymbol{\omega}$	angular velocity vector
$\boldsymbol{\nabla}$	nabla vector operator
$\boldsymbol{F}$	force vector
$\boldsymbol{f}$	force vector field
$\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}} \ldots$	unit vectors/ unit vector fields
$\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$	Cartesian coordinate unit vectors
$\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{z}}$	cylindrical coordinate unit vectors
$\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$	spherical coordinate unit vectors
$\hat{\boldsymbol{w}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{s}}$	rotated Cartesian coordinate unit vectors
$\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{s}}$	rotated cylindrical coordinate unit vectors
$\hat{\boldsymbol{r}}, \hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\phi}}$	rotated spherical coordinate unit vectors
$\mathbb{A}, \mathbb{B}, \mathbb{C} \ldots$	tensors/tensor fields
$\mathbb{I}$	identity tensor
$\mathbb{T}$	Cauchy stress tensor

Introduction

The field of theoretical hydrodynamics is still very active today. The importance of semi-exact solutions is clear when considering that simulations of many hydrodynamical problems are often numerically unstable and require a lot of computational power. Semi-exact solutions form an essential tool for understanding the global evolution of the motion of a fluid as a whole. Moreover, they are often used to test whether a numerical model or simulation gives the correct result for certain flow regimes.

The drag force acting on a moving sphere in a Newtonian fluid with a constant viscosity was first obtained in an analytical form for zero Reynolds number (the so-called Stokes flow) by Stokes [1851] using the so-called Stokes equations, which now bare the name after him. The Stokes equations are a special case of the so-called Navier-Stokes equations describing the motion of various fluids for a non-zero Reynolds number. The solution of the Navier-Stokes equations does not, in general, exist in a closed form. However, a semi-exact solution given as a perturbation series in terms of Reynolds number was proposed by Lamb [1916]. The solution is found separately in the region near the surface of a sphere (the so-called Stokes expansion) and in the far-field (the so-called Oseen expansion), which are then assymptotically matched via the matching principle given by Proudman and Pearson [1957] and Van Dyke [1975].

Several modifications of the problem of finding the drag and moment due to Stokes flow have been proposed and solved in literature. The solutions for specific axially symmetric bodies were found by Happel et al. [1983] and Datta and Srivastava [1999]. The slip flow solution was obtained by Deo and Datta [2002] for a spheroid. The drag force due to a slip flow of a micropolar fluid around a sphere was obtained by Ramkissoon and Majumdar [1976]. The flow around a spherical particle coated with a porous layer was solved by Cichocki and Felderhof [2009]. The Stokes flow around a torus has been also studied, the drag force on a torus can be found in a semi-exact form as an infinite series involving toroidal harmonic functions, see Majumdar and O'Neill [1977] for analytical derivation and Amarakoon et al. [1982] for experimental results.

In this thesis, we will present the derivation of the first two terms of a Stokes expansion of a low Reynolds number flow around a rotating sphere in transverse motion (the so-called Magnus flow). These terms were suggested by Rubinov and Keller [1961]. To accomplish this, we will first treat the cases of translation and rotation separately and solve them up to the second order in accordance with the matching principle. From these solutions, we will then derive the formulae for the Magnus flow, from which the formula for the drag force follows subsequently (the Magnus force). Throughout the thesis, we will assume the flow is due to the motion of an incompressible homogeneous viscous Newtonian fluid with density and viscosity being constant. On the surface of a sphere we assume the no-slip condition.

The thesis is divided into **Chapters 1, 2, 3** and **Appendices A, B, C, D, E**.

The capital letters F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V are mostly reserved for functions dependent on r. Therefore, in the cases it does not cause ambiguity, we will, for brevity, leave out the r-dependence notation. For example, we write R instead of $R(r)$.

1. World as a continuum

1.1 Stress tensor

Suppose that a force $\boldsymbol{F}$ acts on a cross-section of a continuum body. In the limit, when the area S of the cross-section with a normal vector $\hat{\boldsymbol{n}}$ approaches $\mathrm{d}S$, we assume there exists $\boldsymbol{t}^{(\hat{\boldsymbol{n}})}$ such that, for any infinitesimal force $\mathrm{d}\boldsymbol{F}$ acting on the infinitesimal cross-section, it holds that $\mathrm{d}\boldsymbol{F} = \boldsymbol{t}^{(\hat{\boldsymbol{n}})}\,\mathrm{d}S$. The force and moment balance can be then used to define the **stress tensor** $\mathbb{T}$ by a relation known as the **Cauchy force formula**

$$\boldsymbol{t}^{(\hat{\boldsymbol{n}})} = \hat{\boldsymbol{n}} \cdot \mathbb{T}, \tag{1.1}$$

or, written as a sum of three terms

$$\mathbb{T} = \hat{\boldsymbol{x}}\boldsymbol{t}^{(\hat{\boldsymbol{x}})} + \hat{\boldsymbol{y}}\boldsymbol{t}^{(\hat{\boldsymbol{y}})} + \hat{\boldsymbol{z}}\boldsymbol{t}^{(\hat{\boldsymbol{z}})}, \tag{1.2}$$

where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ are the unit vectors in the direction of Cartesian coordinates x, y and z, respectively. Its components are often graphically depicted on the surface of an elementary cube.

1.2 Cauchy momentum balance equations

The conservation of momentum applied to an infinitesimal volume of the body gives the **Cauchy momentum balance equations**

$$\varrho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \boldsymbol{f} + \boldsymbol{\nabla} \cdot \mathbb{T}, \tag{1.3}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative with respect to time, ϱ is the **density** of the continuum at a point in space and time, $\boldsymbol{v}$ is the **velocity vector field** describing the motion of material particles of the body and $\boldsymbol{f}$ is the external forcing per unit volume due to the volume forces. All these variables are generally dependent on time, temperature et cetera. In this thesis, we will assume there is no dependence on temperature nor any other measurable quantity except the position and time[1].

Similarly, the conservation of mass applied to an infinitesimal volume gives a relation known as the **continuity equation**

$$\varrho_{,t} + \boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) = 0, \tag{1.4}$$

where $_{,t}$ stands for the partial derivative with respect to time. For an incompressible fluid ϱ is a constant, so the relation will simplify to

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{1.5}$$

[1] dependence on time will be later also omitted

1.3 Material relations

To complete the description of the motion of a continuous medium, its material relation is necessary to specify. In this thesis, we will deal with an **incompressible isotropic homogeneous Newtonian fluid**, for which the relation between the stress and strain tensor has the form

$$\mathbb{T} = -p\mathbb{I} + \eta \left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}} \right), \tag{1.6}$$

where p is called the **pressure**, η is the so-called **dynamical viscosity** and $\mathbb{I}$ is the identity tensor. In this thesis, we will assume that η is a constant *independent* of position and time.

1.4 Navier-Stokes equations

To derive the Navier-Stokes equations from the Cauchy momentum equation, we first apply divergence to the material relation of a Newtonian fluid (1.3), to obtain

$$\nabla\cdot\mathbb{T} = \nabla \cdot \left(-p\mathbb{I}+\eta \left(\nabla \boldsymbol{v}+\nabla \boldsymbol{v}^{\mathrm{T}}\right)\right) = -\nabla p+\eta\Delta \boldsymbol{v}+\eta\nabla\nabla \cdot \boldsymbol{v} = -\nabla p+\eta\Delta \boldsymbol{v}. \tag{1.7}$$

By the continuity equation for an incompressible fluid (1.5), this simplifies to the **Navier-Stokes equations** for an incompressible Newtonian fluid

$$\varrho \boldsymbol{v}_{,t} + \varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{f} - \nabla p + \eta \Delta \boldsymbol{v}, \tag{1.8}$$

these equations describe the behaviour of the fluid in time and space under external forcing $\boldsymbol{f}$. Note that if the flow is **steady** (i.e., neither p nor $\boldsymbol{v}$ depend on time), we have $\boldsymbol{v}_{,t} = \boldsymbol{0}$, and the corresponding equations are called the **steady-state Navier-Stokes equations**

$$\varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{f} - \nabla p + \eta \Delta \boldsymbol{v}. \tag{1.9}$$

2. Stokes flow

Definition 1 (Stokes Flow)**.** *A Stokes flow (or creeping flow) is the flow described by the steady-state Navier-Stokes equations in which the inertial term* $\varrho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}$ *is omitted.*

Stokes flow is described by the **Stokes flow equations**. In this thesis, we will assume there is no external forcing $\boldsymbol{f}$. The Stokes flow equations then have the following form

$$\boldsymbol{\nabla} p = \eta \Delta \boldsymbol{v}, \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{2.2}$$

2.1 Transverse flow

Let us consider the case of transverse flow. The solution presented here is a modification of the solution by Landau and Lifshitz [1984]. The sphere of a radius $r = a$ surrounded by a Newtonian fluid (with viscosity η) moves with a constant speed $\boldsymbol{v}_\infty$ in the direction of $\hat{\boldsymbol{z}}$. Assuming the flow is Stokesian, we will calculate the drag force $\boldsymbol{F}$ exerted on the sphere (see Figure 2.1).

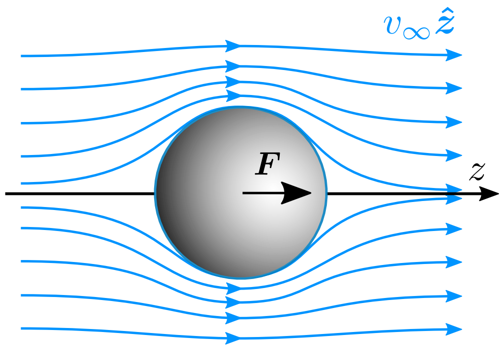

Figure 2.1: Creeping flow around a sphere

Boundary conditions

We place the center of the reference coordinate system to the center of the moving sphere. Since the velocity of the sphere is constant, the reference frame is inertial. Hence, the flow surrounding the sphere is described by the equations introduced above. We additionally assume the flow is stationary far from the sphere. However, from the point of view of our reference frame, the flow appears to be moving at the speed v_∞ in the direction of the axis z far from the sphere (see Figure 2.1). We also assume the fluid does not move on the surface of the sphere (the so-called **no-slip condition**). In summary, the boundary and far-field conditions have the form

$$\boldsymbol{v}|_{r\to\infty} = v_\infty \hat{\boldsymbol{z}}, \tag{2.3}$$

$$\boldsymbol{v}|_{r=a} = \boldsymbol{0}, \tag{2.4}$$

$$p|_{r\to\infty} = p_\infty = 0, \tag{2.5}$$

where we have defined, without a loss of generality, $p_\infty = 0$.

Velocity

Since the velocity is divergence free, the ansatz $\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$ holds, where $\boldsymbol{\psi}$ is the so-called **stream vector field**. This solves (2.2) immediately. Inserting this relation into (2.1) and taking the curl of the result, we get by the commutativity of partial derivatives (see (C.14)):

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Delta\boldsymbol{\psi}=\mathbf{0}. \tag{2.6}$$

The rotational symmetry of the problem implies that $\boldsymbol{v}=v_1\left(r,\rho\right)\hat{\boldsymbol{\theta}}+v_2\left(r,\theta\right)\hat{\boldsymbol{z}}$ (note we have used mixed coordinates – spherical r,θ and cylindrical z,ρ, see Appendix D on coordinates). Therefore, $\boldsymbol{\psi}=\psi\left(r,\rho\right)\hat{\boldsymbol{\varphi}}$; otherwise the rotational symmetry of the velocity will be broken. Moreover, $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\psi}=\boldsymbol{\nabla}\boldsymbol{\cdot}(\psi\hat{\boldsymbol{\varphi}})=\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\hat{\boldsymbol{\varphi}}=0$ since $\boldsymbol{\nabla}\psi$ is perpendicular to $\hat{\boldsymbol{\varphi}}$ and $\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{\varphi}}=0$. Therefore, (2.6) becomes (using (C.15))

$$\Delta^2\left(\psi\hat{\boldsymbol{\varphi}}\right)=\mathbf{0}. \tag{2.7}$$

In view of (E.3), we may use the ansatz $\psi(r,\rho)=\rho R$, where $R=R(r)$ is an unknown function of r. By (E.6), the velocity is expressed as

$$\boldsymbol{v}=\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}})=2R\hat{\boldsymbol{z}}-\rho R'\hat{\boldsymbol{\theta}}. \tag{2.8}$$

The boundary and far-field conditions (2.3) – (2.5) are satisfied when $R(a)=R'(a)=0$ (the no-slip condition) and $R(\infty)=\frac{1}{2}v_\infty$. In view of (E.3), the solution of (2.7) is

$$\mathrm{E}_4^2\left[R\right]=0, \tag{2.9}$$

where we define $\mathrm{E}_\alpha\left[f(r)\right]\overset{\text{def}}{=}f''(r)+\frac{\alpha}{r}f'(r)$. This is an Euler differential equation (see (B.6) for its general solution). Instead of solving a system of equations for the constants of integration associated with a solution of the Euler differential equation, we will proceed differently and integrate (2.9) directly (an approach discussed in Appendix B, Section B.1). This is possible to apply due to the arrangement $\mathrm{E}_\alpha\left[f\right]=r^{-\alpha}\left(r^\alpha f'\right)'$, integrating the previous equation twice with respect to r, we obtain (see (B.3))

$$Q\overset{\text{def}}{=}\mathrm{E}_4\left[R\right]=\frac{A}{r^3}+B, \tag{2.10}$$

where Q has been introduced for the sake of simplicity of further calculations. The constant B must vanish, otherwise an additional integration of Q with respect to r will produce a term r^2 which diverges for $r\to\infty$. In view of the arrangement $\mathrm{E}_4\left[R\right]=\frac{1}{r^4}\left(r^4R'\right)'$, integrating Q with respect to r gives

$$r^4R'=a^4R'(a)+\int_a^r r^4Q\ \mathrm{d}r=\int_a^r r^4Q\ \mathrm{d}r, \tag{2.11}$$

since $R(a)=0$ and $R'(a)=0$. Furthermore, in view of (2.10), we have

$$R=\int_a^r\frac{1}{r^4}\int_a^r r^4Q\ \mathrm{d}r\mathrm{d}r=\frac{A}{2}\int_a^r r^{-4}\left(r^2-a^2\right)\mathrm{d}r=\frac{A}{2}\left(\frac{2}{3a}-\frac{1}{r}+\frac{a^2}{3r^3}\right). \tag{2.12}$$

When r approaches to infinity, R takes the value

$$R(\infty)=\frac{A}{3a}. \tag{2.13}$$

Therefore, since $R(\infty)=\frac{1}{2}v_\infty$, $A=\frac{3}{2}av_\infty$. For Q then

$$Q=\frac{A}{r^3}=\frac{3av_\infty}{2r^3}. \tag{2.14}$$

Hence,

$$R = \frac{v_\infty}{2}\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) = \frac{v_\infty}{2}\left(1 + \frac{a}{2r}\right)\left(1 - \frac{a}{r}\right)^2. \tag{2.15}$$

Substituting (2.15) into (2.8), we get for the velocity

$$\boldsymbol{v} = v_\infty\left(1 - \frac{a}{r}\right)\left[\left(1 + \frac{a}{2r}\right)\left(1 - \frac{a}{r}\right)\hat{\boldsymbol{z}} - \frac{3a\rho}{4r^2}\left(1 + \frac{a}{r}\right)\hat{\boldsymbol{\theta}}\right]. \tag{2.16}$$

Pressure

By (2.1) and in view of (E.6), the pressure gradient becomes

$$\frac{1}{\eta}\boldsymbol{\nabla} p = \Delta\boldsymbol{v} = \boldsymbol{\nabla}\times\Delta\boldsymbol{\psi} = \boldsymbol{\nabla}\times(\rho Q\hat{\boldsymbol{\varphi}}) = 2Q\hat{\boldsymbol{z}} - \rho Q'\hat{\boldsymbol{\theta}}. \tag{2.17}$$

Since $\boldsymbol{\nabla} p$ is *conservative*, we choose an oriented straight line γ from r ($r > a$) to infinity in accordance with the boundary conditions (2.5), and obtain

$$p = \int_\gamma \boldsymbol{\nabla} p \boldsymbol{\cdot} \mathrm{d}\boldsymbol{r} = -\eta\int_r^\infty \left(2Q\hat{\boldsymbol{z}} - \rho Q'\hat{\boldsymbol{\theta}}\right)\boldsymbol{\cdot}\hat{\boldsymbol{r}}\ \mathrm{d}r = -2\eta\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{r}}\int_r^\infty Q\ \mathrm{d}r, \tag{2.18}$$

where $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{r}} = \cos\theta$ (see Table D.3) has been taken out of the integral since it does not depend on r. The integration in (2.18) yields

$$p = -\frac{3\eta a z v_\infty}{2r^2}\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{r}} = -\frac{3\eta a z v_\infty}{2r^3}. \tag{2.19}$$

Force

Let us calculate the drag force exerted on the surface of the sphere. A force $\boldsymbol{t}^{(\hat{\boldsymbol{n}})}\ \mathrm{d}S$ acts on a very small element of the surface with normal $\hat{\boldsymbol{n}}$ (for the sphere $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$). By Cauchy's formula $\boldsymbol{t}^{(\hat{\boldsymbol{n}})} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\mathbb{T}$, the net force acting on the sphere is given by a sum of all these infinitesimal forces:

$$\boldsymbol{F} = \iint_{r=a} \boldsymbol{t}^{(\hat{\boldsymbol{n}})}\mathrm{d}S = \iint_{r=a} \hat{\boldsymbol{r}}\boldsymbol{\cdot}\mathbb{T}\ \mathrm{d}S = \iint_{r=a} -p\hat{\boldsymbol{r}} + \eta\hat{\boldsymbol{r}}\boldsymbol{\cdot}\left(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\right)\ \mathrm{d}S. \tag{2.20}$$

Let us simplify the last expression using the fact that $R = R' = 0$ at $r = a$. It means that only the second- and higher-order derivatives of R contribute to the force (2.20). Therefore,

$$\boldsymbol{\nabla}\boldsymbol{v}|_{r=a} = \boldsymbol{\nabla}(2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}})|_{r=a} = -\rho\left(\boldsymbol{\nabla}R'\right)\hat{\boldsymbol{\theta}}|_{r=a} = -\rho R''(a)\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} = -\frac{3\rho v_\infty}{2a^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}, \tag{2.21}$$

where $R''(a) = R''(a) + \frac{4}{r}R'(a) = \mathrm{E}_4\left[R\right]|_{r=a} = Q(a) = \frac{3v_\infty}{2a^2}$ since $R'(a) = 0$. Substituting this together with the pressure into (2.20), we obtain the **Stokes' law**

$$\boldsymbol{F} = \frac{3\eta v_\infty}{2a^2}\iint_{r=a} z\hat{\boldsymbol{r}} - \rho\hat{\boldsymbol{\theta}}\ \mathrm{d}S = 6\pi a\eta v_\infty\hat{\boldsymbol{z}}, \tag{2.22}$$

where $z\hat{\boldsymbol{r}} - \rho\hat{\boldsymbol{\theta}} = r\hat{\boldsymbol{z}}$, i.e. (D.5), has been used. Hence, the force points in the direction of the unit vector $\hat{\boldsymbol{z}}$, which is in agreement with the rotational symmetry of the setting. Figure 2.2 shows the flow pattern (velocity field $\boldsymbol{v}$ given by (2.16)) plotted by Mathematica.

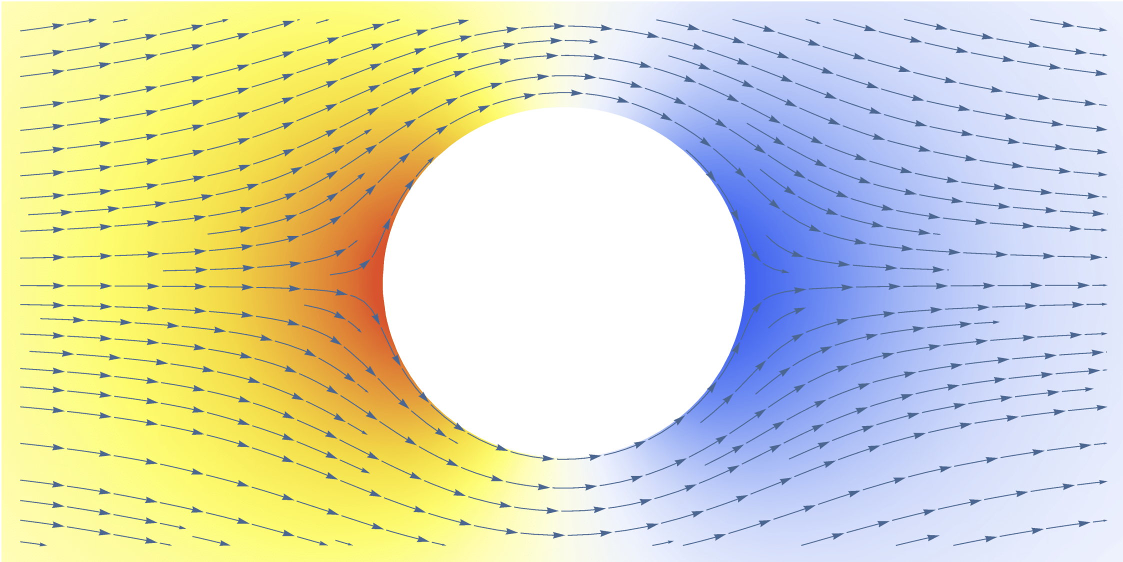

Figure 2.2: Cross-section of creeping flow around a sphere

2.2 Rotating flow

We now consider the case where the sphere rotates with **angular velocity** $\boldsymbol{\omega}$ pointing in the direction of $\hat{\boldsymbol{z}}$, i.e. $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$, where ω is the **angular speed**. Assuming the flow is Stokesian, we will calculate the moment $\boldsymbol{M}$ exerted on the sphere (see Figure 2.3).

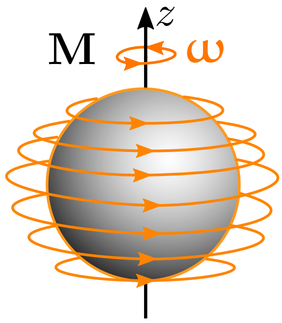

Figure 2.3: Creeping rotational flow around a sphere

Boundary conditions

We use the same inertial frame as in the previous case, although we now assume no transverse motion. The boundary condition and the asymptotic conditions at infinity are

$$\boldsymbol{v}|_{r\to\infty} = \boldsymbol{0}, \tag{2.23}$$

$$\boldsymbol{v}|_{r\to a} = \rho\omega\hat{\boldsymbol{\varphi}}, \tag{2.24}$$

$$p|_{r\to\infty} = 0. \tag{2.25}$$

Velocity and pressure

Stokes flow equations (2.1) and (2.2) still hold in this reference frame. Due to rotational symmetry, a solution has a form $\boldsymbol{v} = v\,(r,\rho)\,\hat{\boldsymbol{\varphi}}$. In view of (E.3), we suggest the ansatz $\boldsymbol{v} = \rho P\hat{\boldsymbol{\varphi}}$, where $P = P(r)$ is an unknown function of r. Therefore, (2.1) becomes

$$\frac{1}{\eta}\boldsymbol{\nabla}p = \Delta\boldsymbol{v} = \Delta\,(\rho P\hat{\boldsymbol{\varphi}}) = \rho \mathrm{E}_4\,[P]\,\hat{\boldsymbol{\varphi}}. \tag{2.26}$$

Since $p = p\,(r,\rho)$ due to rotational symmetry, $\boldsymbol{\nabla}p = p_{,r}\hat{\boldsymbol{r}} + p_{,\rho}\hat{\boldsymbol{\rho}}$. Hence, p is constant, since the right-hand side of (2.26) cannot be expressed as a linear combination of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\rho}}$. The assumption $p = 0$ at infinity then gives $p = 0$ everywhere. Therefore, we get from (2.26) that

$$\mathrm{E}_4\,[P] = 0. \tag{2.27}$$

This is an Euler differential equation with a solution (B.3), i.e.

$$P = \frac{A}{r^3} + B. \tag{2.28}$$

The constants A and B are determined from the boundary conditions (2.23) and (2.24). Since $\boldsymbol{v}$ vanishes for $r \to \infty$, $B = 0$. Assuming the no-slip condition on the surface, we obtain $A = \omega a^3$, and thus

$$P = \omega \frac{a^3}{r^3}. \tag{2.29}$$

In summary,

$$\boldsymbol{v} = \rho\omega \frac{a^3}{r^3} \hat{\boldsymbol{\varphi}}. \tag{2.30}$$

Note that, in case the sphere rotates around a general $\hat{\boldsymbol{s}}$-axis, the solution of the velocity is given, using the rotated coordinates (see Appendix D), by

$$\boldsymbol{v} = \sigma\omega \frac{a^3}{r^3} \hat{\boldsymbol{\phi}}. \tag{2.31}$$

Force and moment

The drag force exerted on the sphere is zero by the rotational symmetry. However, the drag force has a moment that slows down the rotation of the sphere. Again, the total moment is a sum of the infinitesimal moments $\boldsymbol{r} \times \boldsymbol{t}^{(\hat{\boldsymbol{n}})} \mathrm{d}S = r\hat{\boldsymbol{r}} \times \mathbb{T} \cdot \hat{\boldsymbol{r}} \mathrm{d}S$. Therefore,

$$\boldsymbol{M} = \iint_{r=a} \boldsymbol{r} \times \boldsymbol{t}^{(\hat{\boldsymbol{n}})} \mathrm{d}S = a\eta \iint_{r=a} \hat{\boldsymbol{r}} \times \left(\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^{\mathrm{T}} \right) \cdot \hat{\boldsymbol{r}}\ \mathrm{d}S. \tag{2.32}$$

Let us simplify the argument of the integral by using (E.4):

$$\hat{\boldsymbol{r}} \times (\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^{\mathrm{T}}) \cdot \hat{\boldsymbol{r}} \Big|_{r=a} = \rho P'(a)\, \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{r}} = \rho P'(a)\, \hat{\boldsymbol{r}} \times \hat{\boldsymbol{\varphi}} = -\rho P'(a) \hat{\boldsymbol{\theta}}. \tag{2.33}$$

In view of (2.29), we have $P'(a) = -\frac{3\omega}{a}$. Finally, we get the **Kirchhoff's moment law**

$$\boldsymbol{M} = a\eta \iint_{r=a} \rho P'(a) \hat{\boldsymbol{\theta}}\ \mathrm{d}S = -3\omega\eta \iint_{r=a} \rho \hat{\boldsymbol{\theta}}\ \mathrm{d}S = -8\pi\eta\omega a^3 \hat{\boldsymbol{z}}, \tag{2.34}$$

where we have used (E.12) for the evaluating the integral. This formula was originally presented in Kirchhoff [1876]. In a similar way, it is possible to derive the result given in Lamb [1916], Art. 334., for concentric spheres of radii $a < b$, for which a coupled moment, assuming the inner sphere is rotating with angular speed ω and the outer sphere is fixed, is given by

$$\boldsymbol{M} = 8\pi\eta\omega \frac{a^3 b^3}{a^3 - b^3} \hat{\boldsymbol{z}}. \tag{2.35}$$

In particular, when $b \to \infty$, we get Kirchhoff's result.

2.3 Combined flow

We briefly discuss the case of combined flow, i.e., Stokes flow around a sphere that travels through a fluid with a constant speed v_∞ along the z-axis and rotates with a constant angular speed ω around the s-axis, $\boldsymbol{\omega} = \omega \hat{\boldsymbol{s}}$ (see Figure 3.5, Section 3.6 on Magnus flow). This flow solves the Stokes flow equations (2.1) and (2.2) with boundary conditions (2.3), (2.24)[1]. Due to linearity in $\boldsymbol{v}$ and p, the combined flow is constructed by superposing transverse and rotating flow (see Sections 2.1 and 2.2). Since there is no contribution of pressure from the rotating flow, the total pressure is given by (2.19). Summing (2.16) with (2.31), we get for the velocity:

$$\boldsymbol{v} = v_\infty \left[\left(1 - \frac{a}{r}\right) \left[\left(1 + \frac{a}{2r}\right) \left(1 - \frac{a}{r}\right) \hat{\boldsymbol{z}} - \frac{3a\rho}{4r^2} \left(1 + \frac{a}{r}\right) \hat{\boldsymbol{\theta}} \right] + \sigma \Gamma \frac{a^2}{r^3} \hat{\boldsymbol{\phi}} \right], \qquad (2.36)$$

where $\Gamma \stackrel{\text{def}}{=} \frac{a\omega}{v_\infty}$. Moreover, since the net drag force given by (2.20) is linear in $\boldsymbol{v}$ and p, only the drag parallel to the motion of the sphere due to transverse flow exists. Similarly, the net moment is due to rotating flow only and thus given by (2.34) (replacing $\hat{\boldsymbol{z}}$ by $\hat{\boldsymbol{s}}$). The flow given by (2.36) is no longer rotationally symmetrical, Figure 2.4 shows only the yz-plane cross-section of the flow in which we have chosen $\hat{\boldsymbol{s}} = \hat{\boldsymbol{x}}$ and $\Gamma = 3$. In this set up, the sphere rotates counter-clockwise around the x-axis pointing perpendicularly out of the plane of the page.

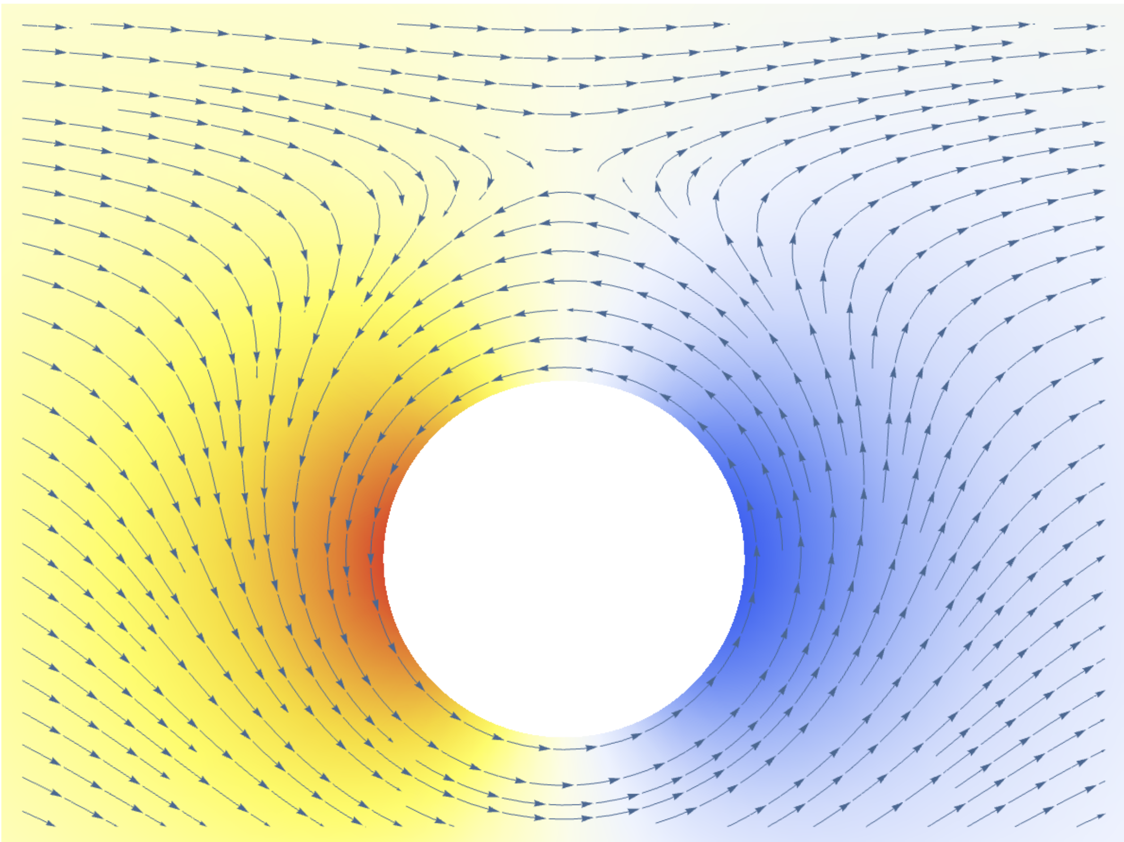

Figure 2.4: yz-plane cross-section of combined flow for $\Gamma = 3$

[1]replacing ρ by σ and $\hat{\boldsymbol{\varphi}}$ by $\hat{\boldsymbol{\phi}}$ in (2.24)

3. First perturbation in Reynolds number

3.1 Lamb flow

Definition 2 (Lamb Flow)**.** *In terms of a perturbation series in* $\alpha \stackrel{\text{def}}{=} \varrho/\eta$ *small, a Lamb flow is a flow which solves the steady-state Navier-Stokes equations up to the first order.*

The steady-state Navier-Stokes equations (1.9) rewritten in terms of α have the form:

$$\alpha \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{1}{\eta} \boldsymbol{\nabla} p + \Delta \boldsymbol{v}, \tag{3.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{3.2}$$

Let us assume that α is a small number. The solution of (3.1) and (3.2) will be searched in the form of a so-called **perturbations series**

$$\boldsymbol{v} = \boldsymbol{v}_0 + \alpha \boldsymbol{v}_1 + \alpha^2 \boldsymbol{v}_2 + \cdots, \tag{3.3}$$

$$p = p_0 + \alpha p_1 + \alpha^2 p_2 + \cdots. \tag{3.4}$$

This perturbation series is called the **Stokes expansion** (see Gavnholt et al. [2004]). In this thesis, we will find only the first terms of the perturbation series (3.3) and (3.4), i.e. $\boldsymbol{v} = \boldsymbol{v}_0 + \alpha \boldsymbol{v}_1$ and $p = p_0 + \alpha p_1$. Substituting these series into (3.1), we obtain, up to the first power of α, the following equations

$$\alpha \boldsymbol{v}_0 \cdot \boldsymbol{\nabla} \boldsymbol{v}_0 + \mathcal{O}\left(\alpha^2\right) = -\frac{1}{\eta} \boldsymbol{\nabla} p_0 - \frac{\alpha}{\eta} \boldsymbol{\nabla} p_1 + \Delta \boldsymbol{v}_0 + \alpha \Delta \boldsymbol{v}_1, \tag{3.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_0 + \alpha \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0, \tag{3.6}$$

where $\mathcal{O}\left(\alpha^2\right)$ contains quadratic and higher-order terms of α (Big O notation). Neglecting these terms, we get a set of four partial differential equations. The first two are the Stokes flow equations for $\boldsymbol{v}_0$ and p_0, i.e.

$$\boldsymbol{\nabla} p_0 = \eta \Delta \boldsymbol{v}_0, \tag{3.7}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_0 = 0. \tag{3.8}$$

The remaining two equations for the functions $\boldsymbol{v}_1$ and p_1 are called the **Lamb equations**,

$$\Delta \boldsymbol{v}_1 - \frac{1}{\eta} \boldsymbol{\nabla} p_1 = \boldsymbol{v}_0 \cdot \boldsymbol{\nabla} \boldsymbol{v}_0, \tag{3.9}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0. \tag{3.10}$$

3.2 Oseen flow

We should note that any real k-multiple of functions $\tilde{\boldsymbol{v}}_0$ and $\tilde{p}_0$ that satisfy the Stokes flow equations can be added to $\boldsymbol{v}_1$ and p_1 so the functions

$$\tilde{\boldsymbol{v}}_1 = \boldsymbol{v}_1 + k\tilde{\boldsymbol{v}}_0, \tag{3.11}$$

$$\tilde{p}_1 = p_1 + k\tilde{p}_0 \tag{3.12}$$

also satisfy the Lamb equations (3.9) and (3.10). Due to failing of the perturbation series expansion in α at far-distance, it is not guaranteed that there is a solution $\boldsymbol{v}_1$ and p_1 of the Lamb equations that satisfies simultaneously both the boundary conditions (2.3) and the asymptotic condition (2.4) (or both (2.23) and (2.24), respectively). The Stokes expansion is a good approximation of the actual flow only for $r \sim a$. However, for $r \gg 1/\alpha$, the inertial term $\alpha\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v}$ in (3.1) is no longer small compared with $\Delta\boldsymbol{v}$ and $\boldsymbol{\nabla}p$ and the Stokes expansion is no longer valid (the argument known as the **Oseen criticism**, see Lamb [1916]).

Therefore, besides the Stokes expansion, another expansion valid at far-distance has to be constructed. These expansions are then joined by a matching condition. We will proceed to find the matching condition according to Proudman and Pearson [1957] via the so-called **method of asymptotic expansion** (the method extensively described for example in Zhu [2009], Holmes [2012] or Lagerstrom [2013], for within fluid mechanics context, see Van Dyke [1975], Happel et al. [1983] or Steinrück [2012]). In order to match our solution, we first find a governing equation for far-distance flows. We begin with defining the so-called **Oseen variable** r^*:

$$r^* \overset{\text{def}}{=} \alpha r, \tag{3.13}$$

where α now plays a role of a *scale factor*. Similarly, other quantities scaled by α are $\rho^* = \alpha\rho$, $z^* = \alpha z$, $x^* = \alpha x$, $y^* = \alpha y$. The position vector $\boldsymbol{r}$ scales as $\boldsymbol{r}^* = \alpha\boldsymbol{r}$. However, angles do not scale, i.e. $\theta^* = \theta$, $\varphi^* = \varphi$. Also, the unit vectors do not scale, i.e. $\hat{\boldsymbol{r}}^* = \hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\rho}}^* = \hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{z}}^* = \hat{\boldsymbol{z}}$ et cetera. Clearly, in view of (3.13), when $\alpha \to 0$, we have for fixed r^* that $r \to \infty$. That means an expansion in α assuming r^* fixed might be indeed valid at far-distance. Note that care should be taken with boundary condition (2.4) on the surface, since for every finite r we have $r^* \to 0$ when $\alpha \to 0$. Only the boundary conditions for $r \to \infty$ are applied. Moreover,

$$\boldsymbol{\nabla}\left(\frac{1}{r^*}\right) = \boldsymbol{\nabla}\left(\frac{1}{r\alpha}\right) = -\frac{\alpha}{\alpha^2 r^2}\hat{\boldsymbol{r}} = -\frac{\alpha}{r^{*2}} = \alpha\boldsymbol{\nabla}^*\left(\frac{1}{r^*}\right), \tag{3.14}$$

where we have introduced $\boldsymbol{\nabla}^*$ such that it acts on rescaled coordinates the same way as $\boldsymbol{\nabla}$ on unscaled coordinates. This a useful notation since the formulae derived for the unscaled quantities can be used in the same form for the scaled quantities. Clearly, in view of the chain-rule of differentiation, we obtain for any function

$$\boldsymbol{\nabla}f(r^*) = \alpha\boldsymbol{\nabla}^* f(r^*). \tag{3.15}$$

Generally,

$$\boldsymbol{\nabla} = \alpha\boldsymbol{\nabla}^*. \tag{3.16}$$

The solution of (3.1) valid at far-distance is then found by the perturbation series in α, similarly as in case of the Lamb equations. Assume that $\boldsymbol{v}_0$ solves the unperturbed problem (3.11) and (3.12). However, a perturbation series for $\boldsymbol{v}$ satisfying the Navier-Stokes equations in far-distance will not simply be $\boldsymbol{v} = \boldsymbol{v}_0 + \alpha \boldsymbol{v}_1$ since $\boldsymbol{v}_0$ is a function of $\boldsymbol{r}$ rather than $\boldsymbol{r}^*$. First, we will write $\boldsymbol{v}_0$ in terms of $\boldsymbol{r}^*$ as a series in α

$$\boldsymbol{v}_0 = \alpha^h \boldsymbol{v}_{00}^*(\boldsymbol{r}^*) + \alpha^{h+1} \boldsymbol{v}_{01}^*(\boldsymbol{r}^*) + \cdots , \tag{3.17}$$

where h is the lowest power of α and $\boldsymbol{v}_{00}^*(\boldsymbol{r}^*)$ is its corresponding coefficient depending on $\boldsymbol{r}^*$ (equivalently, for $\boldsymbol{v}$ given by $\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$, we expand $\boldsymbol{\psi}$ in terms of series in α). Then, according to the formula (3.17), we seek a perturbation series solution in the form

$$\boldsymbol{v} = \alpha^h \boldsymbol{v}_{00}^*(\boldsymbol{r}^*) + \alpha^{h+1} \boldsymbol{v}_1^*(\boldsymbol{r}^*), \tag{3.18}$$

where $\boldsymbol{v}_1^*(\boldsymbol{r}^*)$ is an unknown function of $\boldsymbol{r}^*$. This perturbation series is called the **Oseen expansion** (see Gavnholt et al. [2004]). Equivalently, for $\boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$, we have $\boldsymbol{v} = \alpha \boldsymbol{\nabla}^* \times \boldsymbol{\psi}$ and $\boldsymbol{\psi}$ is sought in the form

$$\boldsymbol{\psi} = \alpha^{h-1} \boldsymbol{\psi}_{00}^*(\boldsymbol{r}^*) + \alpha^h \boldsymbol{\psi}_1^*(\boldsymbol{r}^*) \tag{3.19}$$

to match the relation for $\boldsymbol{v}$, where $\boldsymbol{\psi}_1^*(\boldsymbol{r}^*)$ is an unknown function of $\boldsymbol{r}^*$.

For the pressure, the Stokes equation $\boldsymbol{\nabla} p_0 = \eta \Delta \boldsymbol{v}_0$ gives $\alpha \boldsymbol{\nabla}^* p_0 = \eta \alpha^2 \Delta^* \boldsymbol{v}_0$, so p_0 is expanded in the series of α with the lowest power of $h+1$; otherwise the equation cannot be matched. Hence,

$$p_0 = \alpha^{h+1} p_{00}^*(\boldsymbol{r}^*) + \alpha^{h+2} p_{01}^*(\boldsymbol{r}^*) + \cdots . \tag{3.20}$$

Moreover, since the first Stokes flow equation (2.1) is linear with respect to superposition in velocity, every pair $\alpha^{h+k} \boldsymbol{v}_{0k}^*, \alpha^{h+1+k} p_{0k}^*$ satisfies the Stokes equation (3.11). Equivalently,

$$\boldsymbol{\nabla}^* p_{0k}^* = \eta \Delta^* \boldsymbol{v}_{0k}^* \tag{3.21}$$

for any $k = 0, 1, 2, \ldots$ Similarly as in (3.20), the perturbed pressure is searched in the form

$$p = \alpha^{h+1} p_{00}^*(\boldsymbol{r}^*) + \alpha^{h+2} p_1^*(\boldsymbol{r}^*). \tag{3.22}$$

where $p_1^*(\boldsymbol{r}^*)$ is an unknown function of $\boldsymbol{r}^*$. Substituting these ansatzes for $\boldsymbol{v}$ and p into the steady-state Navier-Stokes equations (3.1), comparing the terms at the same power of h, we obtain the so-called **Oseen equations** for $\boldsymbol{v}_1^*$. In particular, for $h = 0$ and $\boldsymbol{v}_{00}^* = \mathbf{const}.$, for which $\boldsymbol{v}_{00}^* \cdot \boldsymbol{\nabla}^* \boldsymbol{v}_{00}^* = 0$, the Oseen equations (see Oseen [1910]) take the form

$$\Delta^* \boldsymbol{v}_1^* - \frac{1}{\eta} \boldsymbol{\nabla}^* p_1^* = \boldsymbol{v}_{00}^* \cdot \boldsymbol{\nabla}^* \boldsymbol{v}_1^*, \tag{3.23}$$

$$\boldsymbol{\nabla}^* \cdot \boldsymbol{v}_1^* = 0. \tag{3.24}$$

The case of transverse flow, in which $\boldsymbol{v}_{00}^* = v_\infty \hat{\boldsymbol{z}}$, has been solved by Oseen [1910] up to the first order of α. The solution for $\boldsymbol{\psi}_1^*$ given by Lamb [1916], Art. 340, is

$$\boldsymbol{\psi}_1^* = -3 a v_\infty \frac{1 + \cos\theta}{\sin\theta} \left(1 - e^{-\frac{\rho^* v_\infty}{2}(1 - \cos\theta)} \right) \hat{\boldsymbol{\varphi}}, \tag{3.25}$$

higher-order terms are found in Proudman and Pearson [1957]. This solution enables us to perform appropriate matching with the first order Stokes expansion (3.3).

3.3 Matching condition

We are not able to determine the most appropriate form of the approximate solution of the steady-state Navier-Stokes equations (1.9) from the Stokes expansion only. For example, a solution with any k given by (3.9), (3.10) will solve the Lambs equations as well. To determine unknown constats, the matching of Stokes and Oseen expansions is needed. Due to Proudman and Pearson [1957], this matching condition is

$$\lim_{r\to\infty} \boldsymbol{v}(\boldsymbol{r}) = \lim_{r^*\to 0} \boldsymbol{v}^*(\boldsymbol{r}^*). \tag{3.26}$$

According to Van Dyke [1975], we may additionally require matching of coefficients of

$$\boldsymbol{v}\left(\frac{\boldsymbol{r}}{\alpha}\right) \sim \boldsymbol{v}^*(\alpha\boldsymbol{r}). \tag{3.27}$$

That means, the coefficients of Taylor expansion of both left- and right-hand side of (3.27) in powers of α equal. Under certain circumstances (see Proudman and Pearson [1957] or Datta and Singhal [2011]), it is possible to write $\tilde{\boldsymbol{v}}_0 = \boldsymbol{v}_0$ and $\tilde{p}_0 = p_0$ and choose the constant k such that the matching (3.26) is satisfied. Since (3.25) is the exact solution of Oseen equation, the condition for the first-order expansion of the transverse flow reduces to a simple condition $\boldsymbol{\psi}_1|_{r\to\infty} = 0$ at $\theta = 0$. The flow vanishes in the direction of the z^+ halfaxis. In case of the Magnus flow (see Section 3.6) around a rotating sphere with the angular speed ω and moving with the velocity $v_\infty\hat{\boldsymbol{z}}$, we will also assume that the flow vanishes in a particular direction, which is justified by employing the exact solution of the Oseen equations for the Magnus flow found in Rubinov and Keller [1961].

3.4 Transverse flow

In this section, we will find the next term in the Stokes expansion corresponding to the transverse flow discussed in Section 2.1.

Boundary conditions

The boundary conditions for transverse flow are given by (2.3) – (2.5). Since $\boldsymbol{v} = \boldsymbol{v}_0 + \alpha\boldsymbol{v}_1$ and $p = p_0 + \alpha p_1$, the perturbation functions $\boldsymbol{v}_1$ and p_1 satisfy

$$\boldsymbol{v}_1|_{r\to\infty} = \boldsymbol{0}, \tag{3.28}$$

$$\boldsymbol{v}_1|_{r=a} = \boldsymbol{0}, \tag{3.29}$$

$$p_1|_{r\to\infty} = 0. \tag{3.30}$$

Lamb equations

We will now solve the transverse flow around a sphere. As in case of the Stokes equation (2.1), let us take the curl of the first Lamb equation (3.9):

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(\boldsymbol{v}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}_0)\,. \tag{3.31}$$

In view of $\boldsymbol{v}_0 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_0 = 0$, the right-hand side of (3.31) can be simplified by (C.16) and (C.18) as

$$\boldsymbol{\nabla}\times(\boldsymbol{v}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}_0) = (\Delta\boldsymbol{\psi}_0)\cdot\boldsymbol{\nabla}\boldsymbol{v}_0 - \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\Delta\boldsymbol{\psi}_0, \tag{3.32}$$

where $\boldsymbol{v}_0$ is given by the transverse Stokes flow solution (2.8) (more specifically, by (2.16)), i.e.

$$\boldsymbol{v}_0 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_0 = \boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}}) = 2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}} \tag{3.33}$$

and R is given by (2.15). To get $\boldsymbol{\nabla}\boldsymbol{v}_0$, we use (E.7):

$$\boldsymbol{\nabla}\boldsymbol{v}_0 = \frac{R'}{r}\left(3r\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} + \rho\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}} - z\mathbb{I}\right) - \rho R''\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}}. \tag{3.34}$$

Similarly, by (E.3), we have

$$\Delta\boldsymbol{\psi}_0 = \Delta\left(\rho R\hat{\boldsymbol{\varphi}}\right) = \rho Q\hat{\boldsymbol{\varphi}}, \tag{3.35}$$

where $Q = \mathrm{E}_4\left[R\right]$. By the identity (E.4), we have then

$$\boldsymbol{\nabla}\Delta\boldsymbol{\psi}_0 = \rho Q'\hat{\boldsymbol{r}}\hat{\boldsymbol{\varphi}} + Q\left(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}\right). \tag{3.36}$$

Substituting (3.33) – (3.36) into the identity (3.32), we get

$$\boldsymbol{\nabla}\times(\boldsymbol{v}_0\cdot\boldsymbol{\nabla}\boldsymbol{v}_0) = -\frac{2z\rho}{r}RQ'\hat{\boldsymbol{\varphi}} = \frac{9az\rho v_\infty^2}{2r^5}\left(1+\frac{a}{2r}\right)\left(1-\frac{a}{r}\right)^2\hat{\boldsymbol{\varphi}}. \tag{3.37}$$

Velocity

In order to solve (3.31), let us write a particular solution $\boldsymbol{v}_1$ in the form $\boldsymbol{v}_1 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_1$ which satisfies the equation (3.10). Due to the form of the right-hand side of (3.31), we will search for $\boldsymbol{\psi}_1$ in the form $\boldsymbol{\psi}_1 = z\rho S\hat{\boldsymbol{\varphi}}$, where $S = S(r)$ is an unknown function of r. Using the identity (E.9), we obtain

$$\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(z\rho S\hat{\boldsymbol{\varphi}}) = 2Sz\hat{\boldsymbol{z}} - S\rho\hat{\boldsymbol{\rho}} - z\rho S'\hat{\boldsymbol{\theta}}. \tag{3.38}$$

The boundary condition (3.29) gives $S = S' = 0$ at $r = a$. Similarly, (3.28) gives $S(\infty) = 0$. Function S is now determined in terms of $\boldsymbol{v}_0$ by (3.31). Using (E.8), we have

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_1 = \Delta\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(z\rho S\hat{\boldsymbol{\varphi}}) = -\Delta^2\left(z\rho S\hat{\boldsymbol{\varphi}}\right) = -z\rho\mathrm{E}_6^2\left[S\right]\hat{\boldsymbol{\varphi}}. \tag{3.39}$$

Therefore, we have

$$\mathrm{E}_6^2\left[S\right] = -\frac{9av_\infty^2}{2r^5}\left(1+\frac{a}{2r}\right)\left(1-\frac{a}{r}\right)^2 = -\frac{9av_\infty^2}{2}\left(\frac{1}{r^5} - \frac{3a}{2r^6} + \frac{a^3}{2r^8}\right) \tag{3.40}$$

with boundary conditions $S(a) = S'(a) = 0$ and $S(\infty) = 0$. This is an Euler differential equation with a non-zero right-hand side. Let us denote $T = \mathrm{E}_6\left[S\right]$, i.e. $\Delta\boldsymbol{\psi}_1 = \Delta\left(z\rho S\hat{\boldsymbol{\varphi}}\right) = z\rho T\hat{\boldsymbol{\varphi}}$. The solution for T is given by (B.8):

$$\begin{aligned} T &= \int r^{-6}\int r^6\, \mathrm{E}_6\left[T\right] \mathrm{d}r\mathrm{d}r \\ &= -\frac{9av_\infty^2}{2}\int r^{-6}\int\left(r - \frac{3a}{2} + \frac{a^3}{2r^2}\right)\mathrm{d}r\mathrm{d}r \\ &= -\frac{9av_\infty^2}{2}\int\left(\frac{1}{2r^4} - \frac{3a}{2r^5} - \frac{a^3}{2r^7} + \frac{\tilde{C}}{r^6}\right)\mathrm{d}r \\ &= \frac{9av_\infty^2}{4}\left(\frac{1}{3r^3} - \frac{3a}{4r^4} + \frac{C}{r^5} - \frac{a^3}{6r^6}\right). \end{aligned} \tag{3.41}$$

Note that T does not contain a constant of integration since T approaches zero for $r \to \infty$; otherwise S will contain r^2 term and thus diverging for $r \to \infty$. We now use the condition $S(a) = S'(a) = 0$. Therefore, S might be found using (B.12) (or (B.19)). We first find the constant C by applying (B.20) and the condition $S(\infty) = 0$:

$$0 = 5S(\infty) = \int_a^\infty r\,T\ \mathrm{d}r = \frac{9av_\infty^2}{4}\left(\frac{1}{3a} - \frac{3a}{4\cdot 2a^2} + \frac{C}{3a^3} - \frac{a^3}{6\cdot 4a^6}\right). \tag{3.42}$$

Hence,

$$C = \frac{a^2}{4}, \tag{3.43}$$

which gives T in the form

$$T = \frac{9av_\infty^2}{4}\left(\frac{1}{3r^3} - \frac{3a}{4r^4} + \frac{a^2}{4r^5} - \frac{a^3}{6r^6}\right) = \frac{3av_\infty^2}{4r^3}\left(1 - \frac{2a}{r}\right)\left(1 - \frac{a}{4r} + \frac{a^2}{4r^2}\right). \tag{3.44}$$

Note that

$$T(a) = -\frac{3v_\infty^2}{4a^2}. \tag{3.45}$$

Function S is derived by using (B.12):

$$\begin{aligned} S &= \int_a^r r^{-6}\int_a^r r^6\, T\ \mathrm{d}r\ \mathrm{d}r \\ &= \frac{9av_\infty^2}{4}\int_a^r r^{-6}\int_a^r r^6\left(\frac{1}{3r^3} - \frac{3a}{4r^4} + \frac{a^2}{4r^5} - \frac{a^3}{6r^6}\right)\ \mathrm{d}r\ \mathrm{d}r \\ &= \frac{9av_\infty^2}{4}\int_a^r r^{-6}\left(\frac{r^4}{12} - \frac{ar^3}{4} + \frac{a^2r^2}{8} - \frac{a^3r}{6} + \frac{5a^4}{24}\right)\ \mathrm{d}r \\ &= -\frac{3av_\infty^2}{16}\left(\frac{1}{r} - \frac{3a}{2r^2} + \frac{a^2}{2r^3} - \frac{a^3}{2r^4} + \frac{a^4}{2r^5}\right) \\ &= -\frac{3av_\infty^2}{16r}\left(1 - \frac{a}{r}\right)^2\left(1 + \frac{a}{2r} + \frac{a^2}{2r^2}\right). \end{aligned} \tag{3.46}$$

Pressure

The pressure term p_1 is derived in a similar way as for transverse Stokes flow, see Section 2.1. In view of (3.9), we have

$$p_1 = \int_\gamma \boldsymbol{\nabla} p_1 \boldsymbol{\cdot} \mathrm{d}\boldsymbol{r} = -\eta\int_r^\infty\left(\Delta\boldsymbol{v}_1 - \boldsymbol{v}_0 \boldsymbol{\cdot} \boldsymbol{\nabla}\boldsymbol{v}_0\right)\boldsymbol{\cdot}\hat{\boldsymbol{r}}\ \mathrm{d}r. \tag{3.47}$$

The integrand can be arranged by (E.9):

$$\Delta \boldsymbol{v}_1 = \boldsymbol{\nabla} \times \Delta \boldsymbol{\psi}_1 = 2Tz\hat{\boldsymbol{z}} - T\rho\hat{\boldsymbol{\rho}} - z\rho T'\hat{\boldsymbol{\theta}}. \tag{3.48}$$

Therefore,

$$\hat{\boldsymbol{r}} \cdot \Delta \boldsymbol{v}_1 = \frac{T}{r}\left(2z^2 - \rho^2\right). \tag{3.49}$$

Similarly, to arrange $(\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\boldsymbol{v}_0) \cdot \hat{\boldsymbol{r}}$, we start with $(\boldsymbol{\nabla}\boldsymbol{v}_0) \cdot \hat{\boldsymbol{r}}$, where $\boldsymbol{\nabla}\boldsymbol{v}_0$ is given by (3.34):

$$(\boldsymbol{\nabla}\boldsymbol{v}_0) \cdot \hat{\boldsymbol{r}} = \frac{R'}{r}\left(2z\hat{\boldsymbol{r}} + \rho\hat{\boldsymbol{\theta}}\right). \tag{3.50}$$

Taking the scalar product with $\boldsymbol{v}_0$ given by (3.33), we get

$$(\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\boldsymbol{v}_0)\cdot\hat{\boldsymbol{r}} = \left(2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}}\right)\cdot\frac{R'}{r}\left(2z\hat{\boldsymbol{r}} + \rho\hat{\boldsymbol{\theta}}\right) = \frac{2RR'}{r^2}\left(2z^2-\rho^2\right)-\frac{\rho^2}{r}R'^2. \tag{3.51}$$

Now, p_1 takes the form

$$p_1 = -\eta\int_r^\infty \frac{T}{r}\left(2z^2 - \rho^2\right) - \frac{2RR'}{r^2}\left(2z^2 - \rho^2\right) + \frac{\rho^2}{r}\left(R'\right)^2 \,\mathrm{d}r. \tag{3.52}$$

Substituting $\rho = r\sin\theta$ and $z = r\cos\theta$ and keeping in mind that θ does not change along the path of integration, we obtain, after cumbersome algebra, the final expression for p_1:

$$p_1 = -\frac{\eta q^2 v_\infty^2}{64}\left(q\left(14-24q+5q^3\right)-3\left(12-14q+12q^2-q^4\right)\cos\left(2\theta\right)\right), \tag{3.53}$$

where $q \stackrel{\text{def}}{=} a/r$. At $r = a$, $q = 1$, so

$$p_1|_{r=a} = \frac{\eta v_\infty^2}{64}\left(5 + 27\cos\left(2\theta\right)\right) = \frac{\eta v_\infty^2}{64a^2}\left(5a^2 + 27\left(z^2 - \rho^2\right)\right). \tag{3.54}$$

Force

Now, we compute the drag force. Similarly, as in case of the Stokes transverse Newtonian fluid flow (equation (2.20)), the drag force is

$$\boldsymbol{F} = \iint_{r=a} -p\hat{\boldsymbol{r}} + \eta\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\right)\mathrm{d}S = \boldsymbol{F}_0 + \alpha\iint_{r=a} -p_1\hat{\boldsymbol{r}} + \eta\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}_1 + \boldsymbol{\nabla}\boldsymbol{v}_1^{\mathrm{T}}\right)\mathrm{d}S. \tag{3.55}$$

The Stokes' drag force $F_0 = 6\pi a\eta v_\infty\hat{\boldsymbol{z}}$ is present due to the expansion $\boldsymbol{v} = \boldsymbol{v}_0 + \alpha\boldsymbol{v}_1$ and $p = p_0 + \alpha p_1$. Let us simplify the integrand using the fact that $S = S' = 0$ at $r = a$. It means that only the second and higher-order derivatives of S at $r = a$ are contained in the result. Hence,

$$\boldsymbol{\nabla}\boldsymbol{v}_1|_{r=a} = \boldsymbol{\nabla}\left(2Sz\hat{\boldsymbol{z}} - S\rho\hat{\boldsymbol{\rho}} - z\rho S'\hat{\boldsymbol{\theta}}\right)|_{r=a} = -z\rho S''(a)\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} = z\rho\frac{3v_\infty^2}{4a^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} \tag{3.56}$$

since $S''(a) = \mathrm{E}_6\left[T\right]|_{r=a} = -\frac{3v_\infty^2}{4a^2}$. Substituting this together with the pressure term p_1 at $r = a$ given by (3.54) into (3.55), we get, in view of integral formulae (E.10), (E.13), (E.14) and (E.15), that

$$\boldsymbol{F} = \boldsymbol{F}_0 + \frac{\alpha\eta v_\infty^2}{64a^2}\iint_{r=a}\left(-5a^2 - 27z^2 + 27\rho^2\right)\hat{\boldsymbol{r}} + 48z\rho\hat{\boldsymbol{\theta}}\,\mathrm{d}S = \boldsymbol{F}_0. \tag{3.57}$$

Surprisingly, the additional drag force exerted on the surface of the sphere created by perturbation terms is zero. We would expect an increase in drag, but (3.57) does not show it. However, from physical point of view, there must be a non-zero correction drag force to $\boldsymbol{F}_0$ such that $\boldsymbol{F}$ and $\boldsymbol{F}_0$ differ. The reason of missing additional drag force in the first-order perturbation theory of the Stokes expansion is its failure in far-field, which was demonstrated by Oseen [1910] (Note that $\boldsymbol{v}_1|_{r\to\infty} \neq \boldsymbol{0}$). For $r \sim a$, $\boldsymbol{v}_1$ is only a particular solution, the general solution, which vanishes at the surface, is

$$\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(z\rho S\hat{\boldsymbol{\varphi}} + k\rho R\hat{\boldsymbol{\varphi}}) = 2Sz\hat{\boldsymbol{z}} - S\rho\hat{\boldsymbol{\rho}} - z\rho S'\hat{\boldsymbol{\theta}} + k\left(2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}}\right) \tag{3.58}$$

or, equivalently,

$$\boldsymbol{\psi}_1 = z\rho S\hat{\boldsymbol{\varphi}} + k\rho R\hat{\boldsymbol{\varphi}} = z\rho S\hat{\boldsymbol{\varphi}} + k\boldsymbol{\psi}_0. \tag{3.59}$$

This form of the solution of the Stokes expansion is then asymptotically matched according to the matching principle of Van Dyke [1975]. Employing the exact solution of the Oseen equations, the matching principle reduces to the condition $\boldsymbol{\psi}_1|_{r\to\infty} = \boldsymbol{0}$ in the direction of the z^+ halfaxis ($\theta = 0$). Since $\hat{\boldsymbol{r}}|_{\rho=0} = \hat{\boldsymbol{z}}$, $r = z$ and $\hat{\boldsymbol{\theta}}|_{\rho=0} = \hat{\boldsymbol{\rho}}$ in $z > 0$, it suffices that we ensure $\boldsymbol{\psi}_1$ to vanish for $r \to \infty$. Since

$$\boldsymbol{\psi}_1 = \rho zS\hat{\boldsymbol{\varphi}} + k\rho R\hat{\boldsymbol{\varphi}} = \rho\,(rS + kR)\,\hat{\boldsymbol{\varphi}}|_{z=r}, \tag{3.60}$$

we have

$$k = \lim_{r\to\infty} -\frac{rS(r)}{R(r)} = \frac{3}{8}av_\infty \tag{3.61}$$

by simple calculation. Therefore,

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \alpha\boldsymbol{\psi}_1 = \boldsymbol{\psi}_0\left(1 + \frac{3}{8}av_\infty\alpha\right) + \alpha z\rho S\hat{\boldsymbol{\varphi}}. \tag{3.62}$$

The velocity field $\boldsymbol{v}$ is then given by $\boldsymbol{v} = \boldsymbol{\nabla}\times\boldsymbol{\psi}$:

$$\boldsymbol{v} = \left(1 + \frac{3\mathrm{Re}}{8}\right)\left(2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}}\right) + \frac{\mathrm{Re}}{av_\infty}\left(2Sz\hat{\boldsymbol{z}} - S\rho\hat{\boldsymbol{\rho}} - z\rho S'\hat{\boldsymbol{\theta}}\right), \tag{3.63}$$

where $\mathrm{Re} \stackrel{\text{def}}{=} a\varrho v_\infty/\eta = av_\infty\alpha$ is the **Reynolds number**. By linearity of (3.55) with respect to $\boldsymbol{v}$ and p, the **Oseen correction** has the form

$$\boldsymbol{F} = \boldsymbol{F}_0(1 + k\alpha) = 6\pi a\eta v_\infty\left(1 + \frac{3}{8}\mathrm{Re}\right)\hat{\boldsymbol{z}}. \tag{3.64}$$

The higher-order corrections of the force have been found using the method of matched asymptotic expansion (see Proudman and Pearson [1957]). The solution (3.62) is also derived in Van Dyke [1975] and in Proudman and Pearson [1957]. Higher-order perturbations terms $\boldsymbol{v}_2, \boldsymbol{v}_3, \ldots$ are derived in Datta and Singhal [2011]. The flow pattern for $\mathrm{Re} = 100$ is shown in Figure 3.2. Figure 3.1 compares visually the cross-sections of flow patterns for various values of the Reynolds number (right column) with the photographs of experimental results performed by Taneda [1956] (left column). Radius of the sphere was 19.8 cm (note that Taneda's definition of the Reynolds number uses the diameter of the sphere, while, in this thesis, the Reynolds number is defined as $\mathrm{Re} = \frac{a\rho v_\infty}{\eta}$, hence, its value is a half of Taneda's definition). Despite of similarities, Taneda's photographs show the real flow appears to be slightly wider than predicted by (3.63).

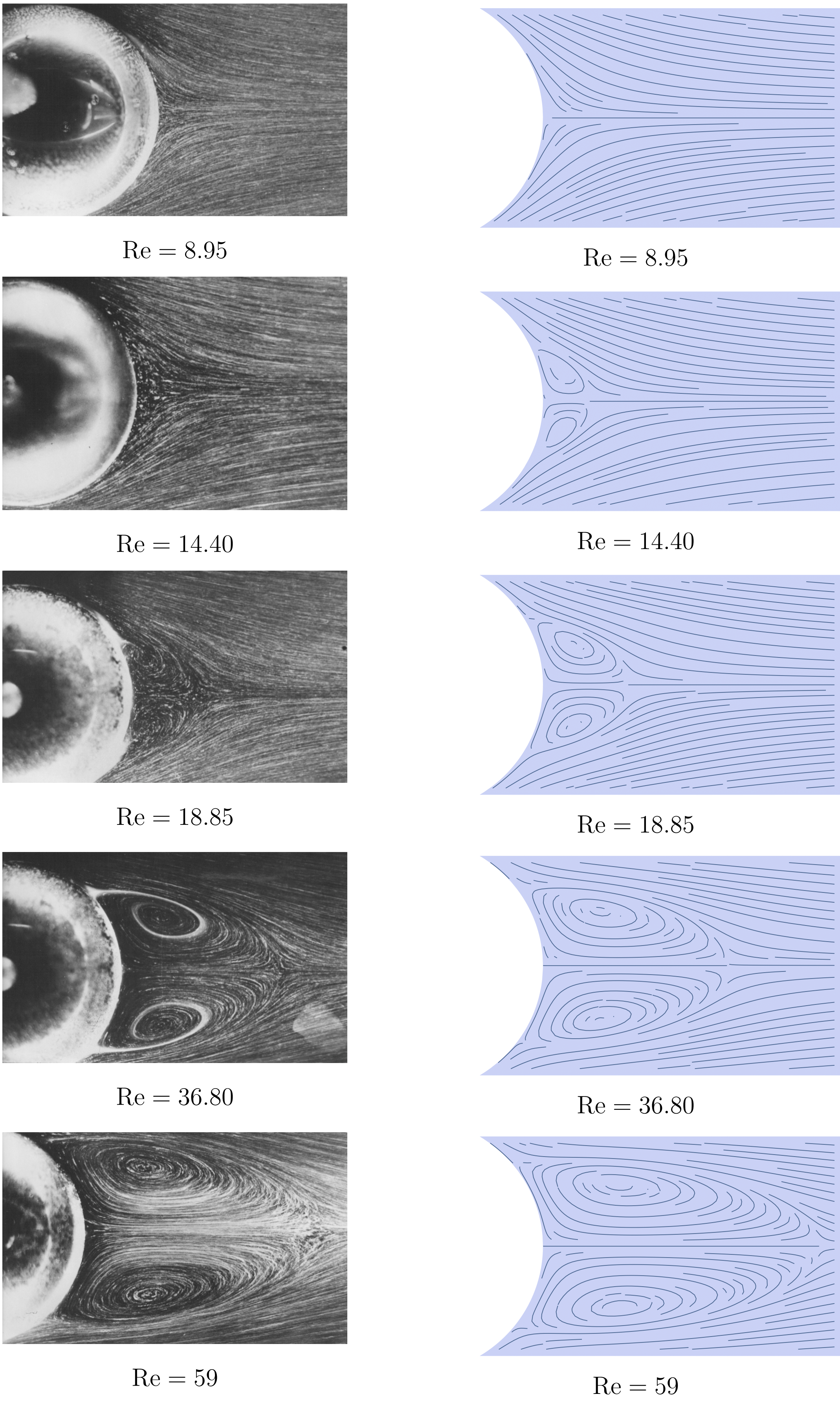

Figure 3.1: Visual comparison of real and model transverse flow for different Reynolds numbers

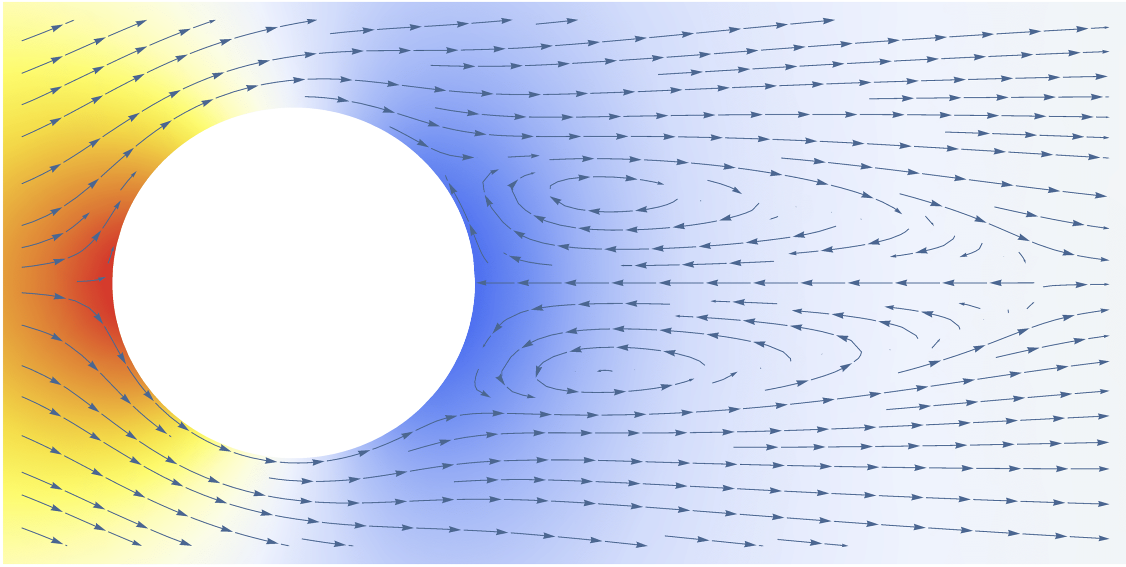

Figure 3.2: Cross-section of perturbated transverse flow for Re = 100

3.4.1 Geometrical properties

Let us find mathematical formulae describing certain geometric properties of the transverse flow. Figure 3.3 shows the visualisation of the set up. Let us denote θ_{sep} the angle between the z axis and the region on the surface of the sphere, where the flow has zero tangential component at $r = a$. We will refer to θ_{sep} as the **angle of separation**. The streamlines which start at the separation (in red) enclose the so-called **vortex region**. At the cross-section (see 3.3) the centre of the vortex corresponds to two nodes with cylindrical coordinates $[z_{ver}, \rho_{ver}]$ and $[z_{ver}, -\rho_{ver}]$, respectively. The outermost point of the vortex region will be called an **antipode point** with coordinates $[z_{ant}, 0]$. The antipode point is a stagnation point.

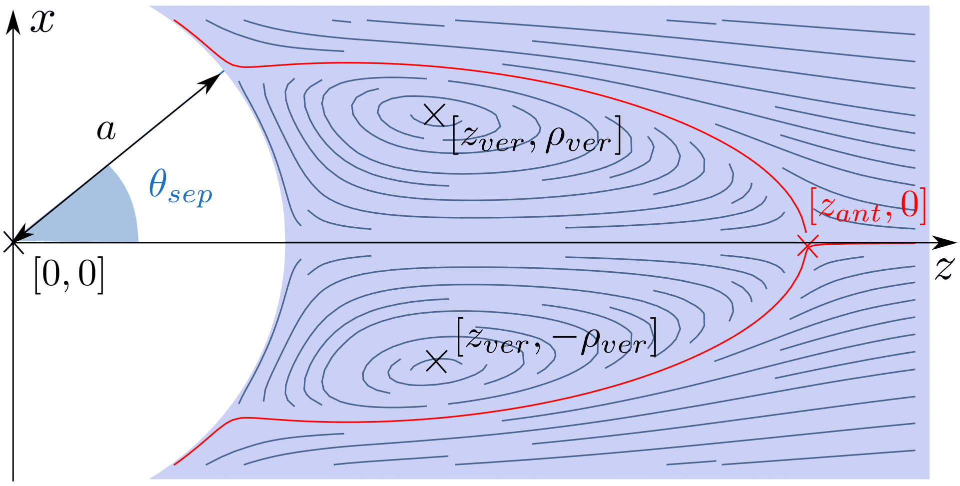

Figure 3.3: Geometrical characteristics of perturbated transverse flow

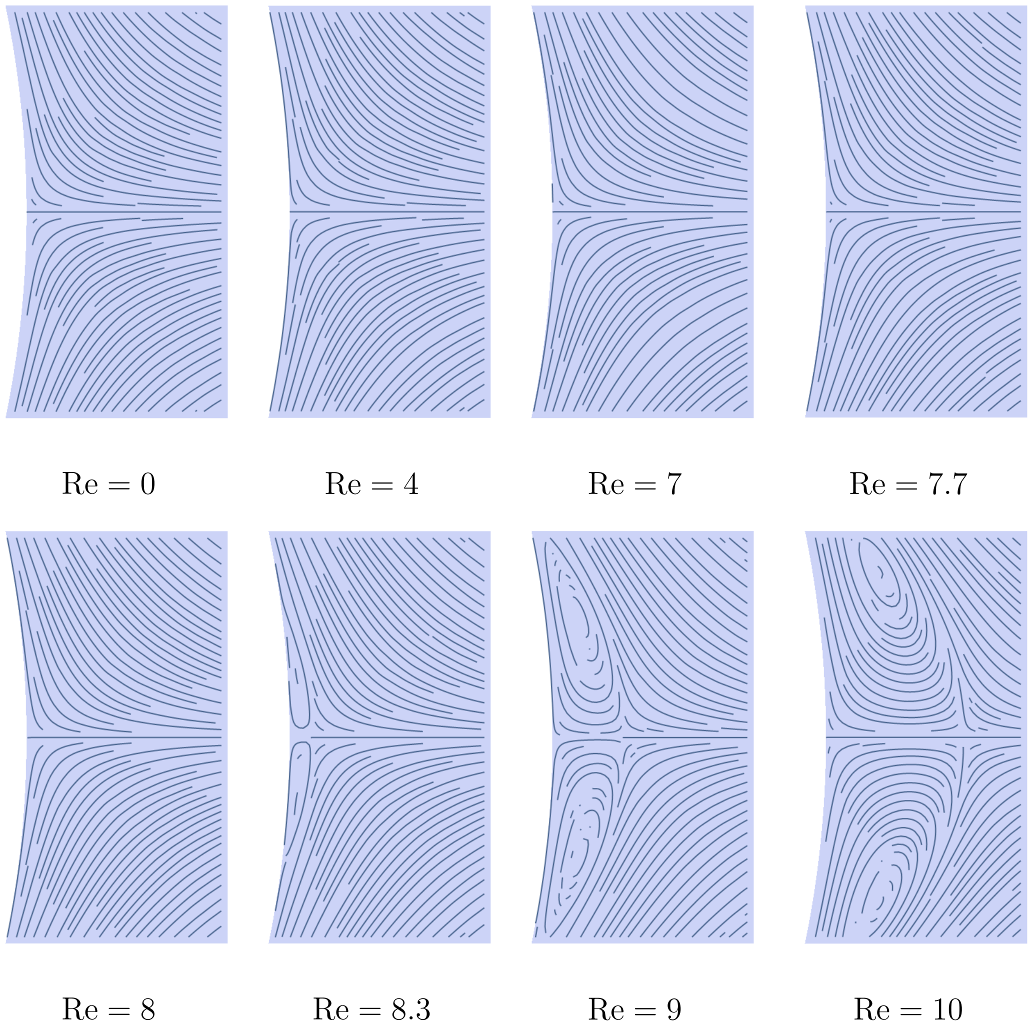

Figure 3.4: Flow patterns near the surface of the sphere for specific Reynolds numbers

Vortex formation

Let us first find the coordinates of the antipode point. The antipode point lies on the axis of symmetry z, that is, when $z = r, \rho = 0, \theta = 0, \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\rho}}$. With this setting, we get using (3.63):

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{v}|_{\rho=0} = v_\infty \left(1-\frac{a}{r}\right)^2 \left(\left(1+\frac{3\mathrm{Re}}{8}\right)\left(1+\frac{a}{2r}\right)-\frac{3\mathrm{Re}}{8}\left(1+\frac{a}{2r}+\frac{a^2}{2r^2}\right)\right). \quad (3.65)$$

Since the fluid is not moving at the antipode point, its z coordinate, z_{ant}, is determined by (3.65) for $\boldsymbol{v} = \boldsymbol{0}$, which results in a quadratic equation for z_{ant}. Its solution is

$$z_{ant} = \frac{a}{4}\left(-1+\sqrt{1+3\mathrm{Re}}\right). \quad (3.66)$$

So, $z_{ant} > a$ only for $\mathrm{Re} > 8$, while for $\mathrm{Re} \leq 8$ there are no vortices present, so $\mathrm{Re} = 8$ has a meaning of a *bifurcation point* which separates two geometrically different flows. Figures 3.4 illustrate this transition. At first, when $\mathrm{Re} < 8$, there is no vortex region. Thus, there are no stagnation points in the flow except the surface of the sphere itself. The transition happens when $\mathrm{Re} = 8$. The flow patterns for Re > 8 contain a vortex as seen in Figures 3.4. Moreover, the only stagnation point outside the sphere except the centres of vortices is the antipode point $[z_{ant}, 0]$ seen on the right of Figures 3.4 for which Re > 8.

Angle of separation

A similar condition holds at the angle of separation, where the velocity vanishes. By Taylor expansion of (3.63) at $r = a$ in terms of $1 - a/r$, we obtain

$$\boldsymbol{v} = -\rho\hat{\boldsymbol{\theta}}\left(1 - \frac{a}{r}\right)\left(aR''(a)\left(1 + \frac{3\mathrm{Re}}{8}\right) + zS''(a)\frac{\mathrm{Re}}{v_\infty}\right) + \mathcal{O}\left(\left(1 - \frac{a}{r}\right)^2\right). \quad (3.67)$$

The term $aR''(a)\left(1 + \frac{3\mathrm{Re}}{8}\right) + zS''(a)\frac{\mathrm{Re}}{v_\infty}$ vanishes at the flow separation. Since $z = a\cos\theta$, we have

$$\theta_{sep} = \arccos\left(-\frac{(8 + 3\mathrm{Re})R''(a)v_\infty}{8S''(a)\mathrm{Re}}\right) = \arccos\left(\frac{8 + 3\mathrm{Re}}{4\mathrm{Re}}\right) \quad (3.68)$$

since $R''(a) = Q(a) = \frac{3v_\infty}{2a^2}$ and $S''(a) = T(a) = -\frac{3v_\infty^2}{4a^2}$. For Re $= 8$, we have

$$\theta_{sep}|_{\mathrm{Re}=8} = \arccos 1 = 0 \quad (3.69)$$

as expected (since a vortex forms for Re > 8, see Figures 3.4). For Re < 8, the formula (3.68) does not make sense mathematically (there is no vortex present). An interesting limit happens for Re $\to \infty$, we get

$$\theta_{sep}|_{\mathrm{Re}\to\infty} = \arccos\left(\frac{3}{4}\right) \approx 41^\circ\, 25'. \quad (3.70)$$

However, this is not a valid limit of the separation angle since the assumption that Re is small no longer applies. Although, numerical results have been found for the sphere in the regime of large Reynolds numbers. From the numerical solution of the Navier-Stokes equations (1.8) for transverse flow Achenbach [1972] estimates θ_{sep} in the limit as

$$\theta_{sep}|_{\mathrm{Re}\to\infty} \approx 59^\circ. \quad (3.71)$$

This result is actually not so surprising since from Figures 3.1 we see that the real flow has indeed a wider vortex zone.

3.5 Rotating flow

In this section, we find the next term in the Stokes expansion corresponding to the rotating flow discussed in Section 2.2.

Boundary conditions

The boundary conditions for the flow around a spinning sphere are given by (2.23) – (2.25). For the perturbation functions $\boldsymbol{v}_1$ and p_1 they have the following form

$$\boldsymbol{v}_1|_{r\to\infty} = \mathbf{0}, \tag{3.72}$$

$$\boldsymbol{v}_1|_{r=a} = \mathbf{0}, \tag{3.73}$$

$$p_1|_{r\to\infty} = 0. \tag{3.74}$$

Lamb equations

Taking the curl of the first of Lamb equation (3.9), we have

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}_0)\,, \tag{3.75}$$

where $\boldsymbol{v}_0 = \rho P\hat{\boldsymbol{\varphi}} = \rho\omega\frac{a^3}{r^3}\hat{\boldsymbol{\varphi}}$ from (2.30). First, for $\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}_0$ standing on the right-hand side of (3.75):

$$\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}_0 = \rho P\hat{\boldsymbol{\varphi}}\boldsymbol{\cdot}\left(\rho P'\hat{\boldsymbol{r}}\hat{\boldsymbol{\varphi}} + P\left(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}\right)\right) = -\rho P^2\hat{\boldsymbol{\rho}} = -\rho\omega^2\frac{a^6}{r^6}\hat{\boldsymbol{\rho}}. \tag{3.76}$$

Taking the curl of the last expression and using the identity (E.5), we get

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v}_0) = z\rho\frac{6a^6\omega^2}{r^8}\hat{\boldsymbol{\varphi}}. \tag{3.77}$$

Velocity

Equation (3.77) is a partial differential equation for $\boldsymbol{v}_1$. The form of the equation suggests to search for the solution in the form $\boldsymbol{v}_1 = \boldsymbol{\nabla}\times\boldsymbol{\psi}_1$. We chose

$$\boldsymbol{\psi}_1 = z\rho O\hat{\boldsymbol{\varphi}}, \tag{3.78}$$

where $O = O(r)$ is an unknown function of r. Note that this particular form satisfies $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\psi}_0 = 0$. Then,

$$\boldsymbol{v}_1 = \boldsymbol{\nabla}\times(z\rho O\hat{\boldsymbol{\varphi}}) = 2Oz\hat{\boldsymbol{z}} - O\rho\hat{\boldsymbol{\rho}} - z\rho O'\hat{\boldsymbol{\theta}}. \tag{3.79}$$

The boundary condition (3.73) gives $O(a) = O'(a) = 0$. Similarly, the boundary condition (3.72) gives $O(\infty) = 0$. Since

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_1 = -\Delta^2\boldsymbol{\psi}_1 = -\Delta^2\left(z\rho O\hat{\boldsymbol{\varphi}}\right) = -z\rho\mathrm{E}_6^2\left[O\right]\hat{\boldsymbol{\varphi}}, \tag{3.80}$$

we get

$$\mathrm{E}_6^2\left[O\right] = -\frac{6a^6\omega^2}{r^8}, \tag{3.81}$$

which is an Euler differential equation with a non-zero right-hand side. Let us denote $I \stackrel{\text{def}}{=} \mathrm{E}_6\,[O]$. By (B.8), we get for I:

$$I = \int r^{-6} \int r^6\, \mathrm{E}_6^2\,[O]\, \mathrm{d}r\mathrm{d}r = 6a^6\omega^2 \int r^{-6} \left(\frac{1}{r} - \tilde{C}\right) \mathrm{d}r = a^6\omega^2 \left(\frac{C}{r^5} - \frac{1}{r^6}\right), \quad (3.82)$$

where the integration constant for the second integral is zero, otherwise O diverges for $r \to \infty$. From (B.20) we find the constant C:

$$0 = 5O(\infty) = \int_a^\infty rI\mathrm{d}r = a^6\omega^2 \left(\frac{C}{3a^3} - \frac{1}{4a^4}\right). \quad (3.83)$$

Hence,

$$C = \frac{3}{4a}. \quad (3.84)$$

Therefore,

$$I = a^5\omega^2 \left(\frac{3a}{4r^5} - \frac{a}{r^6}\right). \quad (3.85)$$

The function O is given by (B.12):

$$\begin{aligned} O &= \int_a^r r^{-6} \int_a^r r^6\, I\mathrm{d}r\mathrm{d}r \\ &= a^5\omega^5 \int_a^r r^{-6} \int_a^r r^6 \left(\frac{3a}{4r^5} - \frac{a}{r^6}\right) \mathrm{d}r\ \mathrm{d}r \\ &= a^5\omega^2 \int_a^r r^{-6} \left(\frac{3r^2}{8} - ar + \frac{5a^2}{8}\right) \mathrm{d}r \\ &= -\frac{a^5\omega^2}{8} \left(\frac{1}{r^3} - \frac{2a}{r^4} + \frac{a^2}{r^5}\right) \\ &= -\frac{a^5\omega^2}{8r^3} \left(1 - \frac{a}{r}\right)^2. \end{aligned} \quad (3.86)$$

Pressure

The pressure term p_1 is given by (3.47), we get

$$p_1 = \int_\gamma \boldsymbol{\nabla} p_1 \cdot \mathrm{d}\boldsymbol{r} = -\eta \int_r^\infty (\Delta \boldsymbol{v}_1 - \boldsymbol{v}_0 \cdot \boldsymbol{\nabla} \boldsymbol{v}_0) \cdot \hat{\boldsymbol{r}}\ \mathrm{d}r. \quad (3.87)$$

Simplifying the integrand by using (3.49), i.e.

$$\hat{\boldsymbol{r}} \cdot \Delta \boldsymbol{v}_1 = \frac{I}{r}\left(2z^2 - \rho^2\right), \quad (3.88)$$

we get

$$p_1 = -\eta \int_r^\infty \frac{I}{r}\left(2z^2 - \rho^2\right) + \rho^2\omega^2\frac{a^6}{r^7}\ \mathrm{d}r = -\frac{1}{8}\eta a^2\omega^2 q^3\left(1 + (3 - 4q)\cos(2\theta)\right), \quad (3.89)$$

where $q \stackrel{\text{def}}{=} a/r$. At $r = a$, we have $q = 1$, so

$$p_1|_{r=a} = -\frac{1}{8}\eta a^2\omega^2\left(1 - \cos(2\theta)\right). \quad (3.90)$$

Force and moment

Since the form of the solution $\boldsymbol{\psi}_1$ (see (3.78)) and p_1 is the same as in Section 3.4, there is no additional force nor moment due to the perturbation terms $\boldsymbol{v}_1$ and p_1.

3.6 Magnus flow

We now consider the sphere that travels through a fluid with a constant speed v_∞ along the z-axis and rotates with a constant angular speed ω around the s-axis, $\boldsymbol{\omega} = \omega\hat{\boldsymbol{s}}$ (see Figure 3.5). The angle between the s- and z-axes is β (see Appendix D on coordinates).

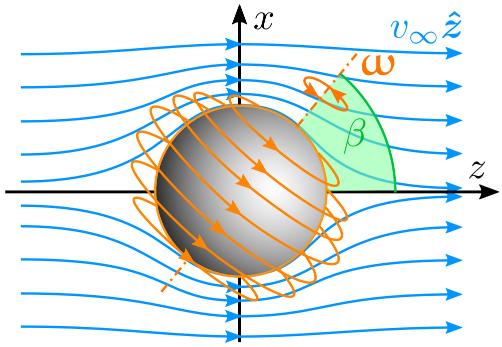

Figure 3.5: Magnus flow as the first combined flow perturbation

Boundary conditions

We place the center of the reference frame in the center of the moving sphere. Since the transverse velocity of the moving sphere is constant, this reference frame is inertial. Let us denote $\boldsymbol{v}$ the velocity of the fluid and and p the pressure. Since $\boldsymbol{v} = \boldsymbol{v}_0 + \alpha\boldsymbol{v}_1$ and $p = p_0 + \alpha p_1$, the boundary conditions take the form:

$$\boldsymbol{v}_0|_{r\to\infty} = v_\infty\hat{\boldsymbol{z}} \qquad (3.91) \qquad\qquad \boldsymbol{v}_1|_{r\to\infty} = \boldsymbol{0} \qquad (3.94)$$

$$\boldsymbol{v}_0|_{r=a} = \omega\sigma\hat{\boldsymbol{\phi}} \qquad (3.92) \qquad\qquad \boldsymbol{v}_1|_{r=a} = \boldsymbol{0} \qquad (3.95)$$

$$p_0|_{r\to\infty} = 0 \qquad (3.93) \qquad\qquad p_1|_{r\to\infty} = 0 \qquad (3.96)$$

Lamb equations

The fields $\boldsymbol{v}_0$ and p_0 satisfy the Stokes flow equations (2.1) and (2.2). Since these equations are linear with respect to linear combination in velocity, $\boldsymbol{v}_0$ can be expressed as the superposition of the transverse and rotating flow, i.e. $\boldsymbol{v}_0 = \boldsymbol{v}_{0t} + \boldsymbol{v}_{0r}$. Similarly, p_0 can be expressed as $p_0 = p_{0t} + p_{0r}$ (see Sections 2.1 and 2.2 for the derivation of $\boldsymbol{v}_{0t}$, $\boldsymbol{v}_{0r}$, p_{0t} and p_{0r}). Substituting this form into the Lamb equation (3.9), we get

$$\Delta\boldsymbol{v}_1 - \frac{1}{\eta}\boldsymbol{\nabla}p_1 = \boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t} + \boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} + \boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} + \boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t}. \qquad (3.97)$$

Let us write $\boldsymbol{v}_1$ and p_1 as $\boldsymbol{v}_1 = \boldsymbol{v}_{1t} + \boldsymbol{v}_{1r} + \boldsymbol{v}_{1m}$ and $p_1 = p_{1t} + p_{1r} + p_{1m}$, where $\boldsymbol{v}_{1t}, \boldsymbol{v}_{1r}, p_{1t}, p_{1r}$ are the perturbation terms derived in previous sections for transverse and rotating flow, respectively (Sections 3.4 and 3.5). Substituting this ansatz for $\boldsymbol{v}_1$ and p_1 into the equation (3.97), and since each of the terms $\boldsymbol{v}_{1t}, \boldsymbol{v}_{1r}, p_{1t}, p_{1r}$ satisfies its corresponding Lamb equation separately, we get for unknown functions $\boldsymbol{v}_{1m}$ and p_{1m}:

$$\Delta\boldsymbol{v}_{1m} - \frac{1}{\eta}\boldsymbol{\nabla}p_{1m} = \boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} + \boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t}. \qquad (3.98)$$

Taking the curl of (3.98), we get

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m} = \boldsymbol{\nabla}\times(\boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} + \boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t}). \qquad (3.99)$$

This is simplified by using (C.17) and (C.18):

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m} = (\Delta\boldsymbol{\psi}_{0t})\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} - \boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\Delta\boldsymbol{\psi}_{0t} + \boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}_{0r} - (\boldsymbol{\nabla}\times\boldsymbol{v}_{0r})\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t}, \quad (3.100)$$

where $\boldsymbol{v}_{0t} = \boldsymbol{\nabla}\times\boldsymbol{\psi}_{0t}$ with $\boldsymbol{\psi}_{0t} = \rho R\hat{\boldsymbol{\varphi}}$. The function R is given by (2.15). Due to the boundary condition (3.92), the solution $\boldsymbol{v}_{0r}$ of the Stokes equations (2.1) and (2.2) is given in the rotated spherical system by (2.30), where ρ is replaced by σ and $\hat{\boldsymbol{\varphi}}$ by $\hat{\boldsymbol{\phi}}$ (see (2.31)). Hence, $\boldsymbol{v}_{0r} = \sigma P\hat{\boldsymbol{\phi}}$, where P is given by (2.29). In view of (E.6), we get

$$\boldsymbol{\nabla}\boldsymbol{v}_{0r} = \boldsymbol{\nabla}\left(\sigma P\hat{\boldsymbol{\phi}}\right) = \sigma P'\hat{\boldsymbol{r}}\hat{\boldsymbol{\phi}} + P\left(\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\sigma}}\right). \quad (3.101)$$

Substituting this expression into the first term of the right-hand side of (3.100) and using $\Delta\left(\rho R\hat{\boldsymbol{\varphi}}\right) = \rho Q\hat{\boldsymbol{\varphi}}$ with $Q = \mathrm{E}_4\left[R\right]$, we get

$$(\Delta\boldsymbol{\psi}_{0t})\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0r} = \rho PQ\left(\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\sigma}}\right)\hat{\boldsymbol{\phi}} - \rho PQ\left(\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\phi}}\right)\hat{\boldsymbol{\sigma}}. \quad (3.102)$$

Substituting (3.34) into the second term of the right-hand side of (3.100), we get

$$\boldsymbol{v}_{0r}\cdot\boldsymbol{\nabla}\Delta\boldsymbol{\psi}_{0t} = \sigma PQ\left(\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\rho}}\right)\hat{\boldsymbol{\varphi}} - \sigma PQ\left(\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\varphi}}\right)\hat{\boldsymbol{\rho}}. \quad (3.103)$$

For $\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}_{0r}$ term, we use (E.7) to get

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}_{0r} = \frac{P'}{r}\left(3r\hat{\boldsymbol{r}}\hat{\boldsymbol{s}} + \sigma\hat{\boldsymbol{\vartheta}}\hat{\boldsymbol{r}} - s\mathbb{I}\right) - \sigma P''\hat{\boldsymbol{r}}\hat{\boldsymbol{\vartheta}}. \quad (3.104)$$

Similarly, the third term on the right-hand side of (3.100) simplifies by using (3.33):

$$\begin{aligned}\boldsymbol{v}_{0t}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\times\boldsymbol{v}_{0r} =& \frac{6zP'R}{r}\hat{\boldsymbol{s}} + \frac{2\sigma P'R}{r}\left(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\vartheta}}\right)\hat{\boldsymbol{r}} - \frac{2sP'R}{r}\hat{\boldsymbol{z}}\\ &- \frac{2z\sigma P''R}{r}\hat{\boldsymbol{\vartheta}} - \frac{\rho\sigma R'P'}{r}\left(\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\vartheta}}\right)\hat{\boldsymbol{r}} + \frac{s\rho R'P'}{r}\hat{\boldsymbol{\theta}}.\end{aligned} \quad (3.105)$$

Finally, in view of (E.6), we get

$$\boldsymbol{\nabla}\times\boldsymbol{v}_{0r} = 2P\hat{\boldsymbol{s}} - \sigma P'\hat{\boldsymbol{\vartheta}}. \quad (3.106)$$

Hence, by (3.34), the fourth term on the right-hand side of (3.100) becomes

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{v}_{0r})\cdot\boldsymbol{\nabla}\boldsymbol{v}_{0t} =& \frac{6sPR'}{r}\hat{\boldsymbol{z}} + \frac{2\rho PR'}{r}\left(\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{\theta}}\right)\hat{\boldsymbol{r}} - \frac{2zPR'}{r}\hat{\boldsymbol{s}}\\ &- \frac{2s\rho PR''}{r}\hat{\boldsymbol{\theta}} - \frac{\sigma\rho P'R'}{r}\left(\hat{\boldsymbol{\vartheta}}\cdot\hat{\boldsymbol{\theta}}\right)\hat{\boldsymbol{r}} + \frac{z\sigma P'R'}{r}\hat{\boldsymbol{\vartheta}}.\end{aligned} \quad (3.107)$$

The first two terms (3.102) and (3.103) subtract to $PQ\boldsymbol{u}$, where

$$\boldsymbol{u} \overset{\text{def}}{=} \rho\hat{\boldsymbol{\phi}}\left(\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\sigma}}\right) + \sigma\hat{\boldsymbol{\rho}}\left(\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\phi}}\right) - \sigma\hat{\boldsymbol{\varphi}}\left(\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\rho}}\right) - \rho\hat{\boldsymbol{\sigma}}\left(\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\varphi}}\right). \quad (3.108)$$

Let us express $\boldsymbol{u}$ as a linear combination of three base vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\vartheta}}$ (i.e. $\boldsymbol{u} = A\hat{\boldsymbol{r}} + B\hat{\boldsymbol{\theta}} + C\hat{\boldsymbol{\vartheta}}$). Taking the scalar product of $\boldsymbol{u}$ with base $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{\phi}}$ and expressing the coefficients of the linear combination, we get

$$\boldsymbol{u} = \rho\hat{\boldsymbol{\theta}}\left(\frac{\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\sigma}}}{\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\theta}}}\right) - \sigma\hat{\boldsymbol{\vartheta}}\left(\frac{\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\rho}}}{\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\vartheta}}}\right), \quad (3.109)$$

where the coefficient A at vector $\hat{\boldsymbol{r}}$ vanishes since $\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{r}} = \frac{\rho}{r}$ and $\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{r}} = \frac{\sigma}{r}$, so $\boldsymbol{u}\cdot\hat{\boldsymbol{r}} = 0$. In addition, in view of (D.3) and (D.2), we can simplify (3.109):

$$\boldsymbol{u} = \rho\hat{\boldsymbol{\theta}}\left(\frac{\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\sigma}}}{\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\theta}}}\right) - \sigma\hat{\boldsymbol{\vartheta}}\left(\frac{\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\rho}}}{\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\vartheta}}}\right) = -\frac{\rho s}{r}\hat{\boldsymbol{\theta}} + \frac{\sigma z}{r}\hat{\boldsymbol{\vartheta}} = s\hat{\boldsymbol{z}} - z\hat{\boldsymbol{s}}. \quad (3.110)$$

Balancing relations

Therefore, in view of (3.110), the third and fourth term on the right-hand side of (3.100) are expressible as a linear combination of another three base vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{s}}$. Moreover, in view of (D.2), the first and second term on the right-hand side of (3.100) are expressible in $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{s}}$ as well. Simplifying all four terms, we finally obtain

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m} = szF\hat{\boldsymbol{r}} + zG\hat{\boldsymbol{s}} + sH\hat{\boldsymbol{z}} + J\,(\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{z}})\,\hat{\boldsymbol{r}}, \tag{3.111}$$

where we have introduced

$$F = \frac{2}{r^2}\,(P'R - PR') - \frac{2}{r}\,(P''R - PR''), \tag{3.112}$$

$$G = -PQ + \frac{6P'R}{r} + 2P''R + \frac{2PR'}{r} + P'R', \tag{3.113}$$

$$H = PQ - \frac{6PR'}{r} - 2PR'' - \frac{2P'R}{r} - P'R', \tag{3.114}$$

$$J = -2\,(P'R - PR')\,. \tag{3.115}$$

Note that they are rational functions of r. Moreover, they are not independent on each other since

$$0=\boldsymbol{\nabla}\cdot(\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m}) = sz\left(F'+\frac{4}{r}F+\frac{G'}{r}+\frac{H'}{r}\right)+\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{z}}\left(J'+\frac{2}{r}J+G+H\right), \tag{3.116}$$

where we have used $s\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = z\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{s}} = \frac{sz}{r}$ (see Table D.3). Since (3.116) must hold for all sz and $\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{z}}$, the terms standing in the front of sz and $\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{z}}$ must vanish:

$$F' + \frac{4}{r}F + \frac{1}{r}\,(G' + H') = 0, \tag{3.117}$$

$$J' + \frac{2}{r}J + G + H = 0. \tag{3.118}$$

By differentiation of (3.115), $J' = -2\,(P''R - PR'')$. Substituting this result into (3.112), we obtain

$$F = -\frac{J}{r^2} + \frac{J'}{r} = \left(\frac{J}{r}\right)', \tag{3.119}$$

which is the first so-called **balancing relation**. The second balancing relation is obtained by summing up (3.113) and (3.114):

$$G + H = \frac{4}{r}\,(P'R - PR') + 2\,(P''R - PR'') = -\frac{2J}{r} - J'. \tag{3.120}$$

Velocity

The solution of (3.100) will be searched in the form $\boldsymbol{v}_{1m} = \boldsymbol{\nabla}\times\boldsymbol{\psi}_{1m}$ with

$$\boldsymbol{\psi}_{1m} = zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}, \tag{3.121}$$

where $K = K(r)$ and $L = L(r)$ are functions of r only (note that generally $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_{1m} \neq 0$). In view of the product rule of differentiation for the curl, we get

$$\boldsymbol{v}_{1m} = \boldsymbol{\nabla}\times(zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}) = K\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}} + zK'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}} + L\hat{\boldsymbol{s}}\times\hat{\boldsymbol{z}} + sL'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}. \tag{3.122}$$

The boundary condition (3.95) gives $K(a) = K'(a) = L(a) = L'(a) = 0$. Moreover, this form of solution for velocity satisfies the second Lamb equation (3.10). To check that, taking the divergence of (3.122), we get

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}_{1m} &= \boldsymbol{\nabla}\boldsymbol{\cdot}(K\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}} + zK'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}} + L\hat{\boldsymbol{s}}\times\hat{\boldsymbol{z}} + sL'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}) \\ &= K'\left(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}\right)\boldsymbol{\cdot}\hat{\boldsymbol{r}} + K'\hat{\boldsymbol{z}}\boldsymbol{\cdot}(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}}) + L'\left(\hat{\boldsymbol{s}}\times\hat{\boldsymbol{z}}\right)\boldsymbol{\cdot}\hat{\boldsymbol{r}} + L'\hat{\boldsymbol{s}}\boldsymbol{\cdot}(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}) = 0,\end{aligned} \tag{3.123}$$

where $(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\boldsymbol{\cdot}\hat{\boldsymbol{r}} + \hat{\boldsymbol{z}}\boldsymbol{\cdot}(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}})$ and $(\hat{\boldsymbol{s}}\times\hat{\boldsymbol{z}})\boldsymbol{\cdot}\hat{\boldsymbol{r}} + \hat{\boldsymbol{s}}\boldsymbol{\cdot}(\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}})$ vanish by using (C.7). The divergence of $\boldsymbol{r}\times\hat{\boldsymbol{z}}$ and $\boldsymbol{r}\times\hat{\boldsymbol{s}}$, respectively, is zero by the formula (E.1). Now, we simplify the left-hand side of (3.111). Since partial derivatives commute, $\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m} = \Delta\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi}_{1m} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Delta\boldsymbol{\psi}_{1m}$. First, we evaluate the Laplacian:

$$\begin{aligned}\Delta\boldsymbol{\psi}_{1m} &= \Delta\left(zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}\right) = \boldsymbol{\nabla}\cdot\left(K\hat{\boldsymbol{z}}\hat{\boldsymbol{s}} + zK'\hat{\boldsymbol{r}}\hat{\boldsymbol{s}} + L\hat{\boldsymbol{s}}\hat{\boldsymbol{z}} + sL'\hat{\boldsymbol{r}}\hat{\boldsymbol{z}}\right) \\ &= z\left(K'' + \frac{4}{r}K'\right)\hat{\boldsymbol{s}} + s\left(L'' + \frac{4}{r}L'\right)\hat{\boldsymbol{z}},\end{aligned} \tag{3.124}$$

since $\Delta = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}$. Let us introduce $M \overset{\text{def}}{=} K'' + \frac{4}{r}K' = \mathrm{E}_4[K]$ and $N \overset{\text{def}}{=} L'' + \frac{4}{r}L' = \mathrm{E}_4[L]$. Hence,

$$\Delta\boldsymbol{\psi}_{1m} = zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}. \tag{3.125}$$

In view of (C.15), we get

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{v}_{1m} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Delta\boldsymbol{\psi}_{1m} = \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}(zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}) - \Delta\left(zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}\right). \tag{3.126}$$

The second term reduces to $\Delta\left(zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}\right) = z\mathrm{E}_4[M]\,\hat{\boldsymbol{s}} + s\mathrm{E}_4[N]\,\hat{\boldsymbol{z}}$. Let us simplify the first term. Since $\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}(zM\hat{\boldsymbol{s}}) =$

$$\boldsymbol{\nabla}\left(M\hat{\boldsymbol{s}}\boldsymbol{\cdot}\hat{\boldsymbol{z}} + sz\frac{M'}{r}\right) = M'\left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\right)\hat{\boldsymbol{r}} + \left(z\hat{\boldsymbol{s}} + s\hat{\boldsymbol{z}}\right)\frac{M'}{r} + sz\left(\frac{M'}{r}\right)'. \tag{3.127}$$

Replacing non-rotated coordinates with rotated coordinates and M with L, we get $\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}(zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}) =$

$$(M'+N')\left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\right)\hat{\boldsymbol{r}} + (z\hat{\boldsymbol{s}}+s\hat{\boldsymbol{z}})\left(\frac{M'}{r}+\frac{N'}{r}\right) + sz\left(\frac{M'}{r}+\frac{N'}{r}\right)'. \tag{3.128}$$

Substituting the last expressions into (3.126) and comparing the result with (3.111), we obtain a system of four ordinary differential equations:

$$F = \left(\frac{M'}{r} + \frac{N'}{r}\right)', \tag{3.129}$$

$$G = -\mathrm{E}_4[M] + \frac{M'}{r} + \frac{N'}{r}, \tag{3.130}$$

$$H = -\mathrm{E}_4[N] + \frac{M'}{r} + \frac{N'}{r}, \tag{3.131}$$

$$J = M' + N'. \tag{3.132}$$

Let us introduce $U \overset{\text{def}}{=} M + N$ and $V \overset{\text{def}}{=} M - N$. We then obtain an equivalent system

$$F = \left(\frac{U'}{r}\right)', \tag{3.133}$$

$$G + H = -\mathrm{E}_4\left[U\right] + \frac{2U'}{r}, \tag{3.134}$$

$$G - H = -\mathrm{E}_4\left[V\right], \tag{3.135}$$

$$J = U' \tag{3.136}$$

of four differential equations for two unknown functions U and V. This system seems to be overdetermined due to a larger number of equations than the number of unknown functions. However, substituting (3.136) into (3.133), we obtain

$$F = \left(\frac{J}{r}\right)', \tag{3.137}$$

which is exactly the first balancing relation (3.119). Moreover, by (3.134) and (3.136), we have

$$G + H = -\mathrm{E}_4\left[U\right] + \frac{2U'}{r} = -U'' - \frac{2}{r}U' = -J' - \frac{2}{r}J, \tag{3.138}$$

which is exactly the second balancing relation (3.120). It means that three equations for U, namely (3.133), (3.134) and (3.136), are *all dependent* dependent on each other. Form the three equations, we will employ (3.136). By (2.15) and (2.29), U can be computed by (3.132), (2.15) and (2.29) as follows:

$$U = \int J \,\mathrm{d}r = -2\int P'R - PR'\mathrm{d}r = \omega v_\infty \left(\frac{3a^4}{4r^4} - \frac{a^3}{r^3}\right). \tag{3.139}$$

Note that U does not contain the constant of integration since U approaches zero for $r \to \infty$; otherwise $K + L$ will contain r^2 term and thus diverging for $r \to \infty$. Unlike the previous flows (see Sections 3.4 and 3.5), where $S(\infty) = O(\infty) = 0$, we cannot put $(K + L)(\infty) = 0$ since by (B.20):

$$K + L|_{r\to\infty} = \frac{1}{3}\int_a^\infty rU\mathrm{d}r = \frac{1}{3}\omega v_\infty \int_a^\infty \left(\frac{3a^4}{4r^3} - \frac{a^3}{r^2}\right)\mathrm{d}r = -\frac{5}{24}\omega v_\infty a^2. \tag{3.140}$$

It is a similar argument as in Section 3.4, where $\boldsymbol{\psi}_1 \neq \mathbf{0}$ as r approaches infinity. The boundary condition (3.95) gives $K(a) + L(a) = K'(a) + L'(a) = 0$. Since $U = \mathrm{E}_4\left[K + L\right]$, in view of (B.12), we get

$$\begin{aligned}
K + L &= \int_a^r r^{-4}\int_a^r r^4\, U\,\mathrm{d}r\mathrm{d}r \\
&= \omega v_\infty \int_a^r r^{-4}\int_a^r r^4\left(\frac{3a^4}{4r^4} - \frac{a^3}{r^3}\right)\,\mathrm{d}r\,\mathrm{d}r \\
&= \omega v_\infty a^2 \int_a^r r^{-4}\left(\frac{3a^2 r}{4} - \frac{ar^2}{2} - \frac{a^3}{4}\right)\,\mathrm{d}r \\
&= \frac{\omega v_\infty a^2}{4}\left(-\frac{3a^2}{2r^2} + \frac{2a}{r} + \frac{a^3}{3r^3} - \frac{5}{6}\right) \\
&= -\omega v_\infty a^2 \frac{5}{24}\left(1 - \frac{a}{r}\right)^2\left(1 - \frac{2a}{5r}\right).
\end{aligned} \tag{3.141}$$

Now, we will find the function V. First, let us simplify the left-hand side of (3.135), taking the difference of (3.113) and (3.114) and simplifying by (2.10) and (2.27), we get:

$$\begin{aligned} G - H &= -2PQ + 2\left(PR'' - P''R\right) + \frac{8}{r}\left(P'R - PR'\right) + 2P'R' \\ &= 2P''R + \frac{8}{r}PR' + 2P'R' = 2\mathrm{E}_4\left[P\right]R + 2P'R' = 2P'R'. \end{aligned} \tag{3.142}$$

Finally, in view of (B.8), we can successively write

$$\begin{aligned} V &= -2\int r^{-4}\int r^4\, P'R'\ \mathrm{d}r\ \mathrm{d}r \\ &= \omega v_\infty \frac{3}{2}\int r^{-4}\int \frac{3a^4}{r^2} - \frac{3a^6}{r^4}\mathrm{d}r\ \mathrm{d}r \\ &= \omega v_\infty \frac{3}{2}\int r^{-4}\left(-\frac{3a^4}{r} + \frac{a^6}{r^7} + \frac{\tilde{C}}{r^4}\right)\mathrm{d}r \\ &= \omega v_\infty \frac{3}{2}\left(\frac{3a^4}{4r^4} - \frac{a^6}{6r^6} - \frac{C}{r^3}\right). \end{aligned} \tag{3.143}$$

Matching condition

The solution $zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}$ is not a general solution of (3.100) since we can add a multiple of a solution of the Stokes equation (2.1) to (3.121) and (3.100) is still satisfied. The general solution, used for matching, is

$$\boldsymbol{\psi}_{1m} = zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}} + kR\left(z\hat{\boldsymbol{s}} - s\hat{\boldsymbol{z}}\right). \tag{3.144}$$

The second term corresponds to Stokes flow in the direction $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}$ since $R(z\hat{\boldsymbol{s}} - s\hat{\boldsymbol{z}}) = R\left(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}\right)\times\boldsymbol{r}$. This is indeed a transverse flow in the direction $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}$ since a transverse flow in the direction $\hat{\boldsymbol{z}}$ is $\boldsymbol{\psi} = \rho R\hat{\boldsymbol{\varphi}} = R\hat{\boldsymbol{z}}\times\boldsymbol{r}$. Hence, for the Stokes flow in the direction of the unit vector $\hat{\boldsymbol{n}}$ is given by $\boldsymbol{\psi} = R\hat{\boldsymbol{n}}\times\boldsymbol{r}$. In accordance with the matching principle, there should be no flow in the direction $\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}$. There is no k in (3.143), the constant C in (3.143) is chosen such $K - L$ vanishes for $r \to \infty$. In view of (B.20), we get

$$0 = 3\left(K - L\right)|_{r\to\infty} = \int_a^\infty r\left(K - L\right)\mathrm{d}r = \omega v_\infty \frac{3}{2}\int_a^\infty \left(\frac{3a^4}{4r^3} - \frac{a^6}{6r^5} - \frac{C}{r^2}\right)\mathrm{d}r. \tag{3.145}$$

Hence,

$$C = \frac{a^3}{3}, \tag{3.146}$$

which gives V in the form

$$V = \omega v_\infty \frac{3}{2}\left(\frac{3a^4}{4r^4} - \frac{a^6}{6r^6} - \frac{a^3}{2r^3}\right). \tag{3.147}$$

Since $K(a) - L(a) = K'(a) - L'(a) = 0$ and $\mathrm{E}_4[K - L] = V$, in view of (B.12), we get

$$\begin{aligned} K - L &= \int_a^r r^{-4} \int_a^r r^4 V \,\mathrm{d}r\,\mathrm{d}r \\ &= \omega v_\infty \frac{3}{2} \int_a^r r^{-4} \int_a^r r^4 \left(\frac{3a^4}{4r^4} - \frac{a^6}{6r^6} - \frac{a^3}{2r^3}\right) \mathrm{d}r\,\mathrm{d}r \\ &= \omega v_\infty \frac{3}{2} \int_a^r r^{-4} \left(\frac{3a^4 r}{4} + \frac{a^6}{6r} - \frac{a^3 r^2}{6} - \frac{3a^5}{4}\right) \mathrm{d}r \\ &= \omega v_\infty \frac{1}{8} \left(-\frac{9a^4}{2r^2} - \frac{a^6}{2r^4} + \frac{2a^3}{r} + \frac{3a^5}{r^3}\right) \\ &= \frac{\omega v_\infty a^2}{4r} \left(1 - \frac{a}{r}\right)^2 \left(1 - \frac{a}{4r}\right). \end{aligned} \tag{3.148}$$

Hence, by adding and subtracting (3.141) and (3.148), we obtain separate expression for K and L:

$$K = \frac{\omega v_\infty a^2}{96r^2} \left(1 - \frac{a}{r}\right)^2 \left(-3a^2 + 16ar - 10r^2\right), \tag{3.149}$$

$$L = \frac{\omega v_\infty a^2}{96r^2} \left(1 - \frac{a}{r}\right)^2 \left(3a^2 - 8ar - 10r^2\right), \tag{3.150}$$

which, together with $\psi_{1m} = zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}$, solves the equation (3.100).

Pressure

Now, let us find the expression for the pressure. By (3.98), we get

$$p_{1m} = \int_\gamma \boldsymbol{\nabla} p_1 \boldsymbol{\cdot} \mathrm{d}\boldsymbol{r} = \eta \int_r^\infty \boldsymbol{v}_{0t} \boldsymbol{\cdot} \boldsymbol{\nabla} \boldsymbol{v}_{0r} \boldsymbol{\cdot} \hat{\boldsymbol{r}} + \boldsymbol{v}_{0r} \boldsymbol{\cdot} \boldsymbol{\nabla} \boldsymbol{v}_{0t} \boldsymbol{\cdot} \hat{\boldsymbol{r}} - \hat{\boldsymbol{r}} \boldsymbol{\cdot} \Delta \boldsymbol{v}_{1m} \,\mathrm{d}r, \tag{3.151}$$

where $\boldsymbol{v}_{0t}$ is given by (2.8), $\boldsymbol{\nabla}\boldsymbol{v}_{0t} \boldsymbol{\cdot} \hat{\boldsymbol{r}}$ by (3.50), and $\boldsymbol{\nabla}\boldsymbol{v}_{0r}$ by (3.101), respectively. For $\boldsymbol{\nabla}\boldsymbol{v}_{0r} \boldsymbol{\cdot} \hat{\boldsymbol{r}}$, we can successively write

$$\boldsymbol{\nabla}\boldsymbol{v}_{0r} \boldsymbol{\cdot} \hat{\boldsymbol{r}} = \left(\sigma P' \hat{\boldsymbol{r}}\hat{\boldsymbol{\phi}} + P\left(\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\sigma}}\right)\right) \boldsymbol{\cdot} \hat{\boldsymbol{r}} = -\sigma P \hat{\boldsymbol{\phi}}\,\hat{\boldsymbol{\sigma}} \boldsymbol{\cdot} \hat{\boldsymbol{r}} = -\frac{\sigma}{r} P \hat{\boldsymbol{\phi}}. \tag{3.152}$$

Then,

$$\boldsymbol{v}_{0t} \boldsymbol{\cdot} \boldsymbol{\nabla}\boldsymbol{v}_{0r} \boldsymbol{\cdot} \hat{\boldsymbol{r}} = -\left(2R\hat{\boldsymbol{z}} - \rho R' \hat{\boldsymbol{\theta}}\right) \boldsymbol{\cdot} \frac{\sigma}{r} P \hat{\boldsymbol{\phi}} = -\left(2PR + rPR'\right)\left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}\right) \boldsymbol{\cdot} \hat{\boldsymbol{r}}, \tag{3.153}$$

where we have used $\sigma\hat{\boldsymbol{\phi}} = r\hat{\boldsymbol{s}} \times \hat{\boldsymbol{r}}$ and identity (D.3). Similarly, for the second term in (3.151):

$$\boldsymbol{v}_{0r} \boldsymbol{\cdot} \boldsymbol{\nabla}\boldsymbol{v}_{0t} \boldsymbol{\cdot} \hat{\boldsymbol{r}} = \frac{PR'}{r} \sigma\rho \hat{\boldsymbol{\phi}} \boldsymbol{\cdot} \hat{\boldsymbol{\theta}} = -rPR' \left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}\right) \boldsymbol{\cdot} \hat{\boldsymbol{r}}. \tag{3.154}$$

Finally, in view of $\hat{\boldsymbol{r}} \boldsymbol{\cdot} \Delta \boldsymbol{v}_{1m} = \hat{\boldsymbol{r}} \boldsymbol{\cdot} \boldsymbol{\nabla} \times \Delta \boldsymbol{\psi}_{1m}$, the third term in (3.151) can be arranged as follows:

$$\hat{\boldsymbol{r}} \boldsymbol{\cdot} \Delta \boldsymbol{v}_{1m} = \hat{\boldsymbol{r}} \boldsymbol{\nabla} \times \left(zM\hat{\boldsymbol{s}} + sN\hat{\boldsymbol{z}}\right) = (M - N)\left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}\right) \boldsymbol{\cdot} \hat{\boldsymbol{r}} = V\left(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}\right) \boldsymbol{\cdot} \hat{\boldsymbol{r}}. \tag{3.155}$$

In summary,

$$p_{1m} = (\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\cdot\hat{\boldsymbol{r}} \int_r^\infty -2rPR' - 2PR - V \ \mathrm{d}r. \tag{3.156}$$

Substituting for R, P, V and from (2.15), (2.29) and (3.147), we get

$$p_{1m} = -\frac{v_\infty \omega a^3 \eta}{4r^2}\left(1 + \frac{3a}{2r} - \frac{a^3}{r^3}\right)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\cdot\hat{\boldsymbol{r}}. \tag{3.157}$$

At $r = a$, we have

$$p_{1m}(a) = -\frac{3a\eta v_\infty \omega}{8}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\cdot\hat{\boldsymbol{r}}. \tag{3.158}$$

Magnus force

Let us now calculate the force. By linearity of Lambs equations in $\boldsymbol{v}_1$ and $\boldsymbol{p}_1$, we have for the force:

$$\boldsymbol{F} = \boldsymbol{F}_0\left(1 + \frac{3\mathrm{Re}}{8}\right) + \boldsymbol{F}_M, \tag{3.159}$$

where

$$\boldsymbol{F}_M = \alpha \iint_{r=a} -p_{1m}\hat{\boldsymbol{r}} + \eta\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}_{1m} + \boldsymbol{\nabla}\boldsymbol{v}_{1m}^{\mathrm{T}}\right)\mathrm{d}S \tag{3.160}$$

we have denoted the so-called **Magnus force**. Integrating the pressure term $-p_{1m}\hat{\boldsymbol{r}}$ and using (E.16), we obtain

$$\iint_{r=a} -p_{1m}\hat{\boldsymbol{r}}\ \mathrm{d}S = \frac{3a\eta v_\infty\omega}{8}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\cdot\iint_{r=a}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\ \mathrm{d}S = \frac{\pi a^3\eta v_\infty\omega}{2}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}. \tag{3.161}$$

Now, let us simplify the second term of the equation (3.160) using the fact that $K = K' = L = L' = 0$ at $r = a$. Since

$$\boldsymbol{\nabla}\boldsymbol{v}_{1m} = \boldsymbol{\nabla}\boldsymbol{\nabla}\times(zK\hat{\boldsymbol{s}} + sL\hat{\boldsymbol{z}}), \tag{3.162}$$

therefore,

$$\boldsymbol{\nabla}\boldsymbol{v}_{1m}|_{r=a} = \boldsymbol{\nabla}\left(zK'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}} + sL'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}\right)|_{r=a} = zK''(a)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}} + sL''(a)\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}. \tag{3.163}$$

Hence,

$$\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}_{1m} + \boldsymbol{\nabla}\boldsymbol{v}_{1m}^{\mathrm{T}}\right)|_{r=a} = \hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\boldsymbol{v}_{1m} + \boldsymbol{\nabla}\boldsymbol{v}_{1m}\cdot\hat{\boldsymbol{r}}|_{r=a} = zK''(a)\hat{\boldsymbol{r}}\times\hat{\boldsymbol{s}} + sL''(a)\hat{\boldsymbol{r}}\times\hat{\boldsymbol{z}}, \tag{3.164}$$

since $\hat{\boldsymbol{r}}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{a}) = 0$ for a constant vector $\boldsymbol{a}$. Integrating $z\hat{\boldsymbol{r}}$ and $s\hat{\boldsymbol{r}}$, respectively. In view of the formula (E.11), we get:

$$\iint_{r=a}\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}_{1m} + \boldsymbol{\nabla}\boldsymbol{v}_{1m}^{\mathrm{T}}\right)\mathrm{d}S = \frac{4}{3}\pi a^3\left(K''(a) - L''(a)\right)\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}} = \frac{4}{3}\pi a^3 V(a)\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}, \tag{3.165}$$

since $V(a) = \mathrm{E}_4\left[K - L\right]|_{r=a} = K''(a) + \frac{4}{r}K'(a) - L''(a) - \frac{4}{r}L'(a) = K''(a) - L''(a)$. Substituting $r = a$ into V (see (3.147)), we get

$$V(a) = \frac{3av_\infty\omega}{8}. \tag{3.166}$$

Hence,

$$\iint_{r=a}\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}_{1m} + \boldsymbol{\nabla}\boldsymbol{v}_{1m}^{\mathrm{T}}\right)\ \mathrm{d}S = \frac{\pi a^3 v_\infty\omega}{2}\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}. \tag{3.167}$$

Finally, summing up (3.161) and (3.167), we get the for the Magnus force:

$$\boldsymbol{F}_M = \pi\varrho a^3 v_\infty \omega\, \hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}. \tag{3.168}$$

This is the formula obtained by Rubinov and Keller [1961]. Note that both the pressure and the viscous term contribute to the Magnus drag force formula in the ratio 1 : 1 (compare (3.161) with (3.167)). In contrast, in transverse Stokes flow they contribute in the ratio 1 : 2 (see (2.22) in view of (E.11) and (E.12)).

Visualisation

We define a non-dimensional parameter $\Gamma \stackrel{\text{def}}{=} a\omega/v_\infty$. Generally, Magnus flow is 3-dimensional and lacks rotational symmetry. Therefore, only cross-sections can be well-visualised. If we choose $\beta = 90°$ (see Figure 3.5), we get a flow with the flow pattern in yz-plane ($x = 0$) shown for Re = 100 and $\Gamma = 0.5$ in Figure 3.6. In this set up, the sphere rotates counter-clockwise around the x-axis pointing perpendicularly out of the plane of the page. Flow patterns with $\Gamma = 0.5$ for various values of the Reynold numbers are given in Figure 3.8 on the next page.

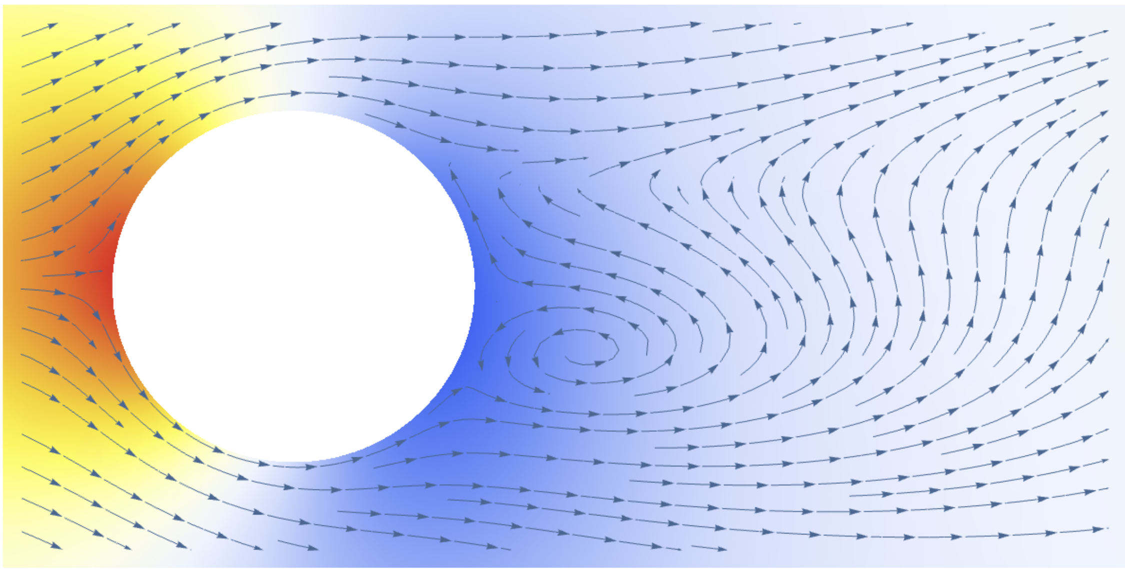

Figure 3.6: yz-plane cross-section of Magnus flow with Re = 100 and $\Gamma = 0.5$

The streamlines seem to point to the right at the top of Figure 3.6. However, since the sphere is spinning counter-clockwise, we would expect the streamlines to point to the left. Close-up inspection of Figure 3.6 reveals they indeed point to the left as the flow posses a boundary layer near the surface of the sphere (see Figure 3.7, the units are given such that $a = 1$). Qualitativelly, the presence of a vortex as well as the boundary layer coincides with the (steady) numerical simulation by Bagchi and Balachandar [2002]. However, Magnus force obtained by Bagchi and Balachandar [2002] is nearly half of Rubinov and Keller [1961] and we need to consider more perturbation terms.

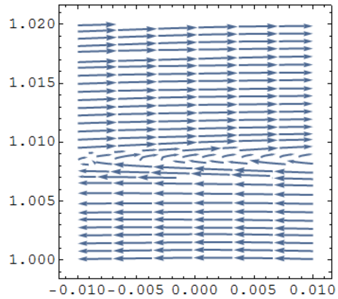

Figure 3.7: Magnus flow - Boundary layer

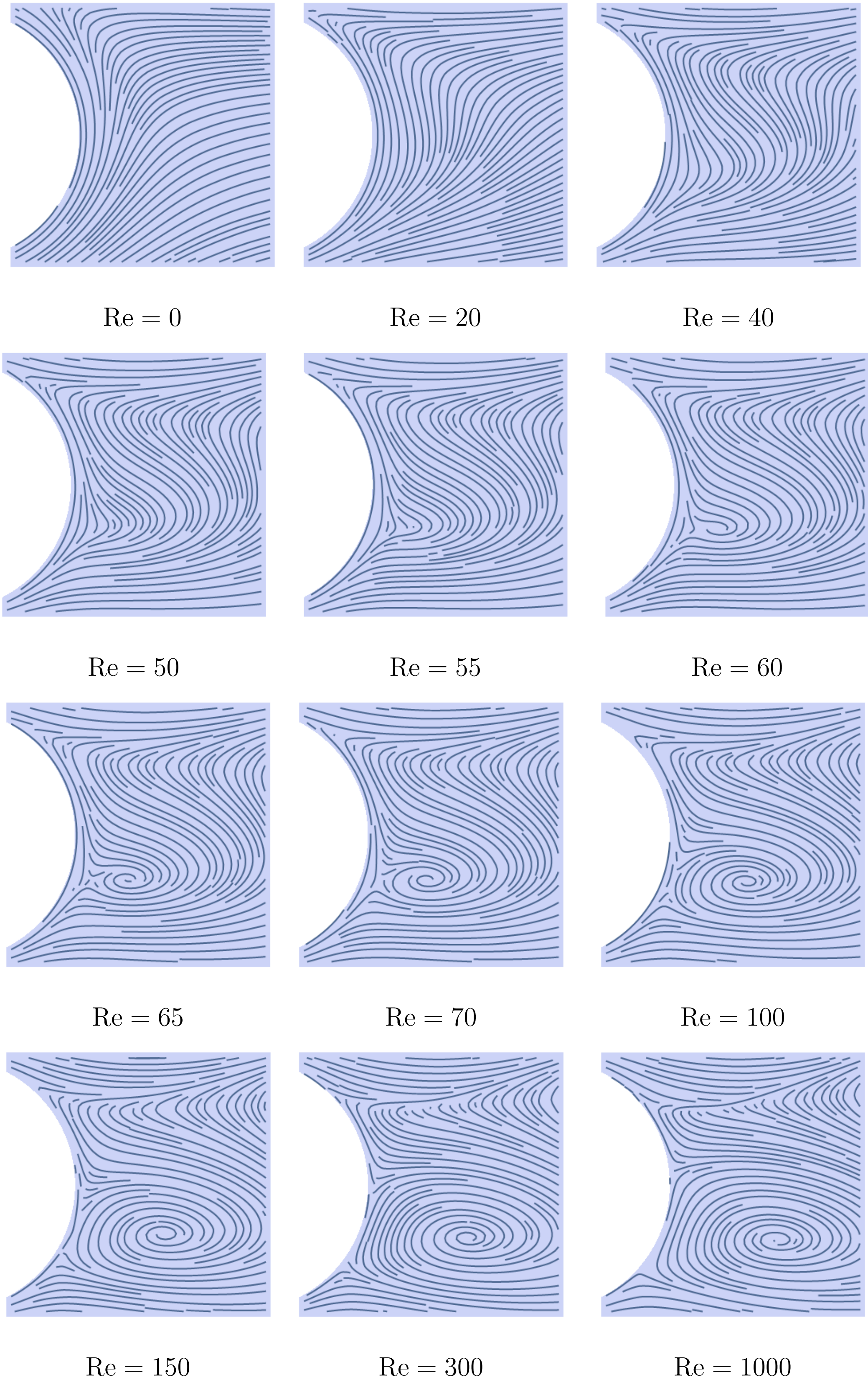

Figure 3.8: yz-plane cross-section of Magnus flow with $\Gamma = 0.5$ for various values of the Reynolds number

Summary

In the first chapter, we have outlined a framework for the thesis. General governing Navier-Stokes equations are derived for an incompressible homogeneous isotropic Newtonian fluid with density ϱ and dynamical viscosity η, assuming being constant.

In the second chapter, we have found the explicit solution of the Stokes equations for the transverse flow around a sphere with a radius a moving with a small constant speed v_∞ (formulae (2.16) and (2.19))

$$\boldsymbol{v} = v_\infty \left(1 - \frac{a}{r}\right) \left[\left(1 + \frac{a}{2r}\right)\left(1 - \frac{a}{r}\right)\hat{\boldsymbol{z}} - \frac{3a\rho}{4r^2}\left(1 + \frac{a}{r}\right)\hat{\boldsymbol{\theta}}\right],$$

$$p = -\frac{3\eta a z v_\infty}{2r^3}$$

and for the flow around a rotating sphere with a small constant angular speed ω (formula (2.30))

$$\boldsymbol{v} = \rho\omega\frac{a^3}{r^3}\hat{\boldsymbol{\varphi}},$$

$$p = 0.$$

Using these explicit formulae, we have re-derived classical results of hydrodynamics, that is, the Stokes' law (formula (2.22))

$$\boldsymbol{F} = 6\pi a\eta v_\infty \hat{\boldsymbol{z}},$$

giving us the force due to the transverse flow, and the Kirchhoff's law (formula (2.34))

$$\boldsymbol{M} = -8\pi\eta\omega a^3 \hat{\boldsymbol{z}},$$

giving us the moment due to the rotating flow.

In the third chapter, we have seen how a simple asymptotic matching condition can be used for the derivation of approximate solutions of perturbated problems. We have found the second term in the perturbation series for transverse, rotational and combined (Magnus) flow around a sphere. Section 3.4 summarises the basic characteristics of transverse flow around a sphere for low Reynolds numbers. An exact solution for the velocity and pressure field of a perturbated problem has been derived (formulae (3.62) and (3.53)):

$$\begin{aligned}\boldsymbol{v} =& \frac{v_\infty}{2}\left(1 + \frac{3}{8}a v_\infty \alpha\right)\boldsymbol{\nabla}\times\left(\rho\left(1 + \frac{a}{2r}\right)\left(1 - \frac{a}{r}\right)^2\hat{\boldsymbol{\varphi}}\right) \\ &- \frac{3a v_\infty^2}{16}\alpha\boldsymbol{\nabla}\times\left(\frac{z\rho}{r}\left(1 - \frac{a}{r}\right)^2\left(1 + \frac{a}{2r} + \frac{a^2}{2r^2}\right)\hat{\boldsymbol{\varphi}}\right),\end{aligned}$$

$$\begin{aligned}p =& -\frac{3\eta q^2 v_\infty}{2a}\left(1 + \frac{3}{8}a v_\infty \alpha\right)\cos\theta \\ &- \frac{\alpha\eta q^2 v_\infty^2}{64}\left(q\left(14 - 24q + 5q^3\right) - 3\left(12 - 14q + 12q^2 - q^4\right)\cos(2\theta)\right),\end{aligned}$$

where $q = a/r$ and $\alpha = \varrho/\eta$. Using this formula, we have derived the Oseen correction for the force (formula (3.64))

$$\boldsymbol{F} = 6\pi a\eta v_\infty \left(1 + \frac{3}{8}\mathrm{Re}\right)\hat{\boldsymbol{z}},$$

where $\mathrm{Re} = a\varrho v_\infty/\eta = a v_\infty \alpha$ is the Reynolds number. Unlike the case of the transverse Stokes flow where streamlines are nearly parallel to the motion of the sphere, the case in which a Reynolds number is larger the streamlines form a vortex behind the sphere. This transition has been obtained from the exact solution of the perturbated problem. The corresponding critical Reynolds number turned out to be

$$\mathrm{Re} = 8.$$

For $\mathrm{Re} > 8$, a vortex is formed. Similarly, we have derived the distance z_{ant} from the center of the sphere to the point behind the sphere where the fluid stands still, and the angle θ_{sep} where the fluid separates from the surface (formulae (3.66) and (3.68)):

$$z_{ant} = \frac{a}{4}\left(-1 + \sqrt{1 + 3\mathrm{Re}}\right),$$
$$\theta_{sep} = \arccos\left(\frac{8 + 3\mathrm{Re}}{4\mathrm{Re}}\right).$$

The flow pattern obtained from this solution was then compared visually for various values of the Reynolds number with the photographs of experimental results performed by Taneda [1956] (see Figure 3.1). The correction of the flow around a rotating sphere has been also derived (formulae (3.86) and (3.89))

$$\boldsymbol{v} = \rho\omega\frac{a^3}{r^3}\hat{\boldsymbol{\varphi}} - \alpha\boldsymbol{\nabla}\times\left(z\rho\frac{a^5\omega^2}{8r^3}\left(1 - \frac{a}{r}\right)^2\hat{\boldsymbol{\varphi}}\right),$$
$$p = -\frac{1}{8}\varrho a^2\omega^2 q^3\left(1 + (3 - 4q)\cos(2\theta)\right).$$

In fact, these additional terms of rotating flow do not contribute to force nor moment. In accordance with the matching principle, we have found the explicit expressions for the correction of velocity and the pressure due to the Magnus flow around a rotating sphere with a small angular speed ω and moving with a small constant velocity $v_\infty\hat{\boldsymbol{z}}$ (formulae (3.149), (3.150) and (3.157)):

$$\boldsymbol{v}_{1m} = \frac{\omega v_\infty a^2}{96}\boldsymbol{\nabla}\times\left(\left(1-\frac{a}{r}\right)^2\left(\frac{s\hat{\boldsymbol{z}}}{r^2}\left(3a^2-8ar-10r^2\right)-\frac{z\hat{\boldsymbol{s}}}{r^2}\left(3a^2-16ar+10r^2\right)\right)\right)$$
$$p_{1m} = -\frac{v_\infty\omega a^3\eta}{4r^2}\left(1 + \frac{3a}{2r} - \frac{a^3}{r^3}\right)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}})\cdot\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{s}}$ is a unit vector pointing in the direction of the axis of rotation. Then, using these results, we have re-derived the Magnus force acting on the sphere (the formula (3.168)) for low Reynolds numbers:

$$\boldsymbol{F}_M = \pi\varrho a^3 v_\infty\omega\,\hat{\boldsymbol{z}}\times\hat{\boldsymbol{s}}.$$

Conclusion

We have derived a formula for the Magnus force acting upon a rotating sphere moving in an incompressible viscous Newtonian fluid for low Reynolds numbers in a new way. We have also studied certain properties of the vortex formation of transverse flow and presented corresponding formulae.

Recent research and the discovery of new artificial materials offers a great opportunity for numerical analysts to study their properties. These materials are not always placed in a vacuum and therefore a shear flow may exist around them. The study of a behaviour of the materials has to be then extended for an interaction with liquid mediums. In case of the liquid medium itself is a new material and the properties of the medium are known, a numerical simulation can be used to study behaviour of the material in more complex and general situations. Formulae derived in the thesis can then be used, for example, for the validation of numerical simulations when simplified initial and boundary conditions, considered in this thesis, are applied.

Solving the Stokes flow around a sphere for the no-slip boundary condition is a special case of a broader class of problems, which have been studied in literature, some of which are yet unsolved analytically. It turns out the force (and moment, respectively) due to a Stokes flow with a prescribed initial velocity on the surface of a sphere can be expressed in the form of a single integral formula. These formulae offer an easy proof of the so-called Faxén's laws and can be used to approximate the force and moment due to transverse or rotational flow exerted on almost spherical particles. The proof of these statements involves a use of vector harmonic functions and goes beyond the scope of this thesis.

The third- and higher-order terms of the Stokes expansion can be found for transverse flow around a non-spinning sphere (see Proudman and Pearson [1957] and Datta and Singhal [2011]). Similarly, the third- and higher-order terms of the expansion series could be also found for the Magnus flow. Although the aim of this thesis was not to consider higher-order terms, it would be interesting to find them and compare them with experimental results by Briggs [1959] and (steady as well as unsteady) numerical simulations by Bagchi and Balachandar [2002].

Appendices

A Some formulae on distributions

This section presenting some formulae on distribution is rather informal, for more formal modern approach see Schwartz [1957]. The **Dirac delta distribution** $\delta(x)$ is defined by

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x)\ \mathrm{d}x = \varphi(0) \tag{A.1}$$

for any sufficiently smooth test function $\varphi(x)$. Moreover, we define a shifted delta distribution $\delta(x-a)$ such that

$$\int_{-\infty}^{\infty} \varphi(x)\delta(x-a)\ \mathrm{d}x = \varphi(a). \tag{A.2}$$

For any sufficiently smooth $f(x)$, it also holds

$$f(x)\delta(x-a) = f(a)\delta(x-a), \tag{A.3}$$

since

$$\int_{-\infty}^{\infty} \varphi(x)f(x)\delta(x-a)\ \mathrm{d}x = \varphi(a)f(a) = \int_{-\infty}^{\infty} f(a)\varphi(x)\delta(x-a)\ \mathrm{d}x. \tag{A.4}$$

Let us derive the action of the derivative of the delta distribution $\delta'(x)$ on a smooth function $f(x)$. Integration by parts yields

$$\int_{-\infty}^{\infty} \varphi(x)\delta'(x)\ \mathrm{d}x = -\int_{-\infty}^{\infty} \varphi'(x)\delta(x)\ \mathrm{d}x = -\varphi'(0). \tag{A.5}$$

Hence, in view of the product rule of differentiation, we get

$$f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x). \tag{A.6}$$

Therefore,

$$\delta'(x)f(x) + \delta(x)f'(0) = \delta'(x)f(x) + \delta(x)f'(x) = (\delta(x)f(x))' = \delta'(x)f(0). \tag{A.7}$$

More formally, the last formula follows from

$$\int_{-\infty}^{\infty} \varphi(x)f(x)\delta'(x)\ \mathrm{d}x = -\int_{-\infty}^{\infty} (\varphi(x)f(x))'\,\delta(x)\ \mathrm{d}x \tag{A.8}$$

$$= -\int_{-\infty}^{\infty} (\varphi'(x)f(x) + \varphi(x)f'(x))\,\delta(x)\ \mathrm{d}x = -\varphi'(0)f(0) - \varphi(0)f'(0) \tag{A.9}$$

$$= \int_{-\infty}^{\infty} f(0)\varphi(x)\delta(x)\ \mathrm{d}x - \int_{-\infty}^{\infty} f'(0)\varphi(x)\delta(x)\ \mathrm{d}x. \tag{A.10}$$

Let us define the **Heaviside step function** $\theta(x)$ such that $\frac{\mathrm{d}}{\mathrm{d}x}\theta(x) = \delta(x)$ and

$$\theta(x) = \int_{-\infty}^{x} \delta(t)\ \mathrm{d}t, \tag{A.11}$$

thus:

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{A.12}$$

Hence,

$$\int_{-\infty}^{x} \delta(x-a)\ \mathrm{d}t = \theta(x-a) + C. \tag{A.13}$$

Moreover, $F'(x) = f(x)$, so

$$\int f(x)\delta(x-a)\ \mathrm{d}x = f(a)\theta(x-a) + C = -f(a)\theta(a-x) + D, \tag{A.14}$$

where (A.3) was used. Therefore,

$$\int f(x)\delta(x-a)\ \mathrm{d}x = \int f(a)\delta(x-a) = f(a)\theta(x-a) + C. \tag{A.15}$$

The two apparently different results in the last expression are actually the same since $\theta(a-x) = 1 - \theta(x-a)$. Also,

$$\int f(x)\theta(x-a)\ \mathrm{d}x = (F(x) - F(a))\,\theta(x-a) + C \tag{A.16}$$

since, differentiating and applying the product rule of differentiation,

$$((F(x)-F(a))\theta(x-a))' = f(x)\theta(x-a) + (F(x)-F(a))\delta(x-a) = f(x)\theta(x-a), \tag{A.17}$$

where the second term $(F(x) - F(a))\,\delta(x-a) = (F(a) - F(a))\,\delta(x-a) = 0$ vanishes by (A.3). Another form includes

$$\int f(x)\theta(a-x)\ \mathrm{d}x = (F(x) - F(a))\,\theta(a-x) + C. \tag{A.18}$$

Special cases include (assuming $s > a$ and $r > a$)

$$\int_{a}^{r} f(u)\theta(u-s)\ \mathrm{d}u = (F(r) - F(s))\,\theta(r-s) \tag{A.19}$$

and

$$\int_{a}^{r} f(u)\theta(s-u)\ \mathrm{d}u = (F(r) - F(s))\,\theta(s-r) - (F(a) - F(s))\,, \tag{A.20}$$

which both follow from the previous cases.

B Euler differential equation

Definition 3. *A linear ordinary differential equation* $a_n r^n R^{(n)} + a_{n-1} r^{n-1} R^{(n-1)} + \ldots + a_0 R = f(r)$ *with constant coefficients* a_k*, where* $f(r)$ *is some known function and* $R = R(r)$ *is a function to be solved, is called an Euler differential equation.*

We will assume that $r > 0$. The Euler differential equation's characteristic solution is of the form r^λ (or $r^\lambda \ln^k(r)$ if λ is a root of the characteristic polynomial with multiplicity k). In this paper, we are mostly solving the equations of the form $\mathrm{E}_\alpha[R] \stackrel{\text{def}}{=} \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{\alpha}{r}\frac{\mathrm{d}R}{\mathrm{d}r} = 0$. Multiplying by r^2, it is easy to see, that this is indeed a homogeneous Euler differential equation. Let us solve the equation substituting the characteristic solution r^λ, we get

$$\mathrm{E}_\alpha\left[r^\lambda\right] = \frac{\mathrm{d}^2\left(r^\lambda\right)}{\mathrm{d}r^2} + \frac{\alpha}{r}\frac{\mathrm{d}\left(r^\lambda\right)}{\mathrm{d}r} = \lambda\left(\lambda-1\right)r^{\lambda-2} + \alpha\lambda r^{\lambda-2} = \lambda(\lambda-1+\alpha)r^{\lambda-2}. \quad \text{(B.1)}$$

Hence, for $\alpha \neq 1$, the roots of the characteristic polynomial are $\lambda = 0$ and $\lambda = 1 - \alpha$, respectively. The solution of a homogeneous ODE is, therefore,

$$R(r) = \frac{A}{r^{\alpha-1}} + Br. \quad \text{(B.2)}$$

For example, for $\alpha = 4$, the general solution is

$$R(r) = \frac{A}{r^3} + Br. \quad \text{(B.3)}$$

Another important example in an equation with the same operator E applied twice, i.e. $\mathrm{E}_\alpha^2[R] = 0$. Substituting $R = r^\lambda$, we get

$$\mathrm{E}_\alpha\left[\mathrm{E}_\alpha\left[r^\lambda\right]\right] = \lambda(\lambda-1+\alpha)\mathrm{E}_\alpha\left[r^{\lambda-2}\right] = \lambda(\lambda-1+\alpha)(\lambda-2)(\lambda-3+\alpha)r^{\lambda-4}. \quad \text{(B.4)}$$

The general solution is, therefore,

$$R(r) = \frac{A}{r^{\alpha-1}} + \frac{B}{r^{\alpha-3}} + C + Dr^2, \quad \text{(B.5)}$$

where $\alpha \neq \pm 1$ nor $\alpha \neq \pm 3$. Two other important special cases include $\alpha = 4$, for which

$$R(r) = \frac{A}{r^3} + \frac{B}{r} + C + Dr^2 \quad \text{(B.6)}$$

or $\alpha = 6$, for which

$$R(r) = \frac{A}{r^5} + \frac{B}{r^3} + C + Dr^2 \quad \text{(B.7)}$$

are the solutions.

B.1 Nonhomogeneous equation

Since $\mathrm{E}_\alpha[R] \stackrel{\text{def}}{=} R'' + \frac{\alpha}{r}R' = r^{-\alpha}\left(r^\alpha R'\right)'$, we get, integrating the solution of the nonhomogeneous equation $\mathrm{E}_\alpha[R] = f$:

$$R(r) = \int r^{-\alpha}\left(\int r^\alpha f(r)\,\mathrm{d}r\right)\mathrm{d}r. \quad \text{(B.8)}$$

Substituting $f = 0$ into (B.8), we recover the homogeneous solution (B.3)

$$R(r) = \int r^{-\alpha}\tilde{A}\,\mathrm{d}r = \frac{A}{r^{\alpha-1}} + B. \quad \text{(B.9)}$$

Special cases

Let us consider an Euler differential equation with a non-zero right-hand side $\mathrm{E}_\alpha[R(r)] = f(r)$. Special cases include when the function R and its first derivative vanish at some $r = a$ (i.e. the boundary conditions are $R(a) = R'(a) = 0$). In that scenario, note that

$$R''(a) = R''(a) + \frac{\alpha}{a}R'(a) = \mathrm{E}_\alpha[R(r)]\,|_{r=a} = f(a). \tag{B.10}$$

Moreover, we are able to express $R(r)$ in terms of $f(r)$ as a single integral. Integrating $(r^\alpha R')' = r^\alpha f(r)$ from a to r, we get

$$r^\alpha R'(r) - a^\alpha R'(a) = r^\alpha R'(r) = \int_a^r s^\alpha f(s)\mathrm{d}s. \tag{B.11}$$

Multiplying the previous relation by $r^{-\alpha}$ and integrating, we get, finally,

$$R(r) = \int_a^r t^{-\alpha} \int_a^t s^\alpha f(s)\ \mathrm{d}s\ \mathrm{d}t. \tag{B.12}$$

Green's function approach

We can express (B.12) as a single integral. Let us suppose, that the solution of $\mathrm{E}_\alpha[R] = f(r)$ with the boundary conditions $R(a) = R'(a) = 0$ may be written as

$$R(r) = \int_a^\infty G(r,s)f(s)\ \mathrm{d}s, \tag{B.13}$$

where $G(r,s)$ is the so-called **Green's function**. This solution satisfies the boundary conditions $R(a) = R'(a) = 0$ if $G(a,s) = G_{,r}(a,s) = 0$. Moreover, if $\mathrm{E}_\alpha[G(r,s)] = \delta(r-s)$, where $\delta(r-a)$ is the *Dirac delta distribution* and where $\mathrm{E}_\alpha[\,\cdot\,]$ is applied with respect to r, then, straightforwardly,

$$\mathrm{E}_\alpha[R(r)] = \int_a^\infty \mathrm{E}_\alpha[G(r,s)]\,f(s)\ \mathrm{d}s = \int_a^\infty \delta(r-s)f(s)\ \mathrm{d}s = f(r), \tag{B.14}$$

so (B.13) is indeed a solution. Let us solve the system

$$\mathrm{E}_\alpha[G(r,s)] = \delta(r-s), \tag{B.15}$$

$$G(a,s) = 0, \tag{B.16}$$

$$G_{,r}(a,s) = 0. \tag{B.17}$$

By (B.12) and in view of the properties of Dirac delta and Heaviside distributions (the formulae (A.14) and (A.16)), we get (assuming $s > a$, so $\theta(a-s) = 0$)

$$\begin{aligned} G(r,s) &= \int_a^r t^{-\alpha} \int_a^t u^\alpha \delta(u-s)\mathrm{d}u\mathrm{d}t = \int_a^r t^{-\alpha}s^\alpha\theta(t-s)\mathrm{d}t \\ &= \frac{s^\alpha}{1-\alpha}\left(r^{1-\alpha} - s^{1-\alpha}\right)\theta(r-s). \end{aligned} \tag{B.18}$$

Hence, for $R(r)$, we get

$$\begin{aligned} R(r) &= \int_a^\infty G(r,s)f(s)\ \mathrm{d}s = \int_a^\infty \frac{s^\alpha}{1-\alpha}\left(r^{1-\alpha} - s^{1-\alpha}\right)\theta(r-s)f(s)\ \mathrm{d}s \\ &= \frac{r}{\alpha-1}\int_a^r \left(\frac{s}{r} - \left(\frac{s}{r}\right)^\alpha\right) f(s)\ \mathrm{d}s. \end{aligned} \tag{B.19}$$

From this expression, we have, for example (for $\alpha > 1$), that

$$R(\infty) = \frac{1}{\alpha-1}\int_a^\infty sf(s)\ \mathrm{d}s. \tag{B.20}$$

C Vector and tensor calculus

This thesis works solely with 3-dimensional Euclidean metric space $\mathbb{E}^3$ (throughout the thesis, the word "space" is used for $\mathbb{E}^3$). In this space, we are able to measure distances, magnitudes of vectors and angles as usual. For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with magnitudes $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$, we define the so-called **scalar product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ as a scalar $\boldsymbol{a} \cdot \boldsymbol{b}$ such that

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \gamma, \tag{C.1}$$

where γ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Similarly, we define the so-called **vector product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ as a vector $\boldsymbol{a} \times \boldsymbol{b}$ perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ such that the triple $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ is a right-handed base and

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \gamma, \tag{C.2}$$

where γ is again the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Algebraic identities

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \tag{C.3}$$
$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0} \tag{C.4}$$
$$\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = 0 \tag{C.5}$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) \tag{C.6}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) \tag{C.7}$$

Tensors

Tensors are generalisations of scalars and vectors: 0-rank tensor is a scalar, 1-rank tensor is a vector. Generally, a tensor $\mathbb{A}$ of rank n may be written as a sum

$$\mathbb{A} = \sum_{i=1}^{m} A_i \boldsymbol{a}_{i1} \boldsymbol{a}_{i2} \boldsymbol{a}_{i3} \ldots \boldsymbol{a}_{in}, \tag{C.8}$$

where A_i are scalars and $\boldsymbol{a}_{ij}$ are vectors.

Definition 4 (Trace and Cross-trace)**.** *Let* $\mathbb{A}$ *be a tensor of rank* n *given by* (C.8)*. The* ***trace*** *of* $\mathbb{A}$ *is a tensor* $\mathrm{Tr}\mathbb{A}$ *of rank* $n-2$ *defined by*

$$\mathrm{Tr}\mathbb{A} = \sum_{i=1}^{m} A_i \boldsymbol{a}_{i1} \cdot \boldsymbol{a}_{i2} \boldsymbol{a}_{i3} \ldots \boldsymbol{a}_{in}. \tag{C.9}$$

Similarly, the ***cross-trace*** *of* $\mathbb{A}$ *is a tensor* $\mathrm{Cr}\mathbb{A}$ *of rank* $n-1$ *defined by*

$$\mathrm{Cr}\mathbb{A} = \sum_{i=1}^{m} A_i \boldsymbol{a}_{i1} \times \boldsymbol{a}_{i2} \boldsymbol{a}_{i3} \ldots \boldsymbol{a}_{in}. \tag{C.10}$$

Fields

Definition 5 (Field)**.** *A field is a function which assigns a tensor of the same rank to any point in space (i.e., a tensor-valued function). If the rank is 0, the tensors are scalars and we say the field is a scalar field. If the rank is 1, the tensors are vectors and we say the field is a vector field. Generally, we say the field is a tensor field (of rank n). We say a vector field is a unit vector field if it consists only of vectors of unit length.*

Definition 6 (Isosurface)**.** *An isosurface of a scalar field f is a set of points in space for which f is constant.*

Definition 7 (Gradient)**.** *The gradient of a tensor field $\mathbb{F}$ of rank n is another tensor field $\boldsymbol{\nabla}\mathbb{F}$ of rank $n+1$ which at any point X in space satisfies*

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}\mathbb{F}(X) = \lim_{h\to 0}\frac{\mathbb{F}(X+h\boldsymbol{a})-\mathbb{F}(X)}{h} \tag{C.11}$$

for any constant vector $\boldsymbol{a}$.

This definition of the gradient satisfies all usual properties – linearity under addition and multiplication by constants and the product rule of differentiation. If the field is a scalar field, the gradient is a vector field which is perpendicular to all isosurfaces of that scalar field.

Definition 8 (Divergence, Curl and Laplacian)**.** *Let $\mathbb{F}$ be a tensor field of rank n, we define the **divergence** of $\mathbb{F}$ as a tensor $\boldsymbol{\nabla}\cdot\mathbb{F}$ of rank $n-1$ given by*

$$\boldsymbol{\nabla}\cdot\mathbb{F} = \mathrm{Tr}\boldsymbol{\nabla}\mathbb{F}. \tag{C.12}$$

*Similarly, we define the **curl** of $\mathbb{F}$ as a tensor $\boldsymbol{\nabla}\times\mathbb{F}$ of rank n given by*

$$\boldsymbol{\nabla}\times\mathbb{F} = \mathrm{Cr}\boldsymbol{\nabla}\mathbb{F}. \tag{C.13}$$

*The **Laplacian** Δ is defined as $\Delta \overset{\text{def}}{=} \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$.*

Differential identities

The following identities hold for any sufficiently smooth vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\Delta\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\Delta\boldsymbol{a} \tag{C.14}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{a} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{a} - \Delta\boldsymbol{a} \tag{C.15}$$

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{a} = \frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{a}\cdot\boldsymbol{a}\right) - \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) \tag{C.16}$$

$$\boldsymbol{\nabla}\left(\boldsymbol{a}\cdot\boldsymbol{b}\right) = \boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{a} + \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) \tag{C.17}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\boldsymbol{\nabla}\cdot\boldsymbol{b} - \boldsymbol{b}\boldsymbol{\nabla}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{a} \tag{C.18}$$

These identities are not independent. For example, if we write $\boldsymbol{a}\to\boldsymbol{a}+\gamma\boldsymbol{b}$ for γ real and substitute this into (C.16), the left-hand side simplifies:

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{a} + \gamma\left[\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{a}\right] + \mathcal{O}\left(\gamma^2\right).$$

Similarly, the right-hand side turns out to be

$$\frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{a}\cdot\boldsymbol{a}\right) - \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{a}) + \gamma\left[\boldsymbol{\nabla}\left(\boldsymbol{a}\cdot\boldsymbol{b}\right) - \boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b}) - \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{a})\right] + \mathcal{O}\left(\gamma^2\right).$$

Comparing the coefficients at γ, we get (C.17). Similarly, putting $\boldsymbol{a}=\boldsymbol{b}$, we get (C.16) from (C.17).

D Coordinate systems

Definition 9 (Coordinate system)**.** *A coordinate system in $\mathbb{E}^3$ is a set of any three sufficiently smooth distinct scalar fields, the so-called* ***coordinates****. A coordinate system is said to be orthogonal if the isosurfaces of coordinates are perpendicular to each other at any point in space (except at a finite number of points).*

In this thesis, six orthogonal coordinate systems are used: **Cartesian** (x, y, z), **cylindrical** (ρ, φ, z), **spherical** (r, θ, φ), **rotated Cartesian** (w, y, s), **rotated cylindrical** (σ, ϕ, s) and **rotated spherical** (r, ϑ, ϕ). These systems are constructed out of in total 12 scalar fields $x, y, z, \rho, \varphi, r, \theta, w, \sigma, \phi, s, \vartheta$. We will refer to $x, y, z, \rho, \varphi, r, \theta$ as the **non-rotated coordinates**, $w, y, s, \sigma, \phi, r, \vartheta$ are the **rotated coordinates** (note that y and r are both rotated and non-rotated at the same time). The fields, with respect to a particular point X, correspond to either distances $(x, y, z, w, s, \sigma, \rho, r)$ or angles $(\varphi, \phi, \vartheta, \theta)$ – see Figure D.1. In this figure, the point X' is the orthogonal projection of the point X to the xy-plane, whereas the point X'' is the orthogonal projection to the inclined wy-plane. The angle between the xy- and wy-planes is β. Also, in view of Figure D.1, the coordinates are related by relations given by Table D.1. Note that these relations *are invariant* under replacement of the non-rotated coordinates by the rotated coordinates and vice versa, in addition with $\beta \leftrightarrow -\beta$. Table D.2 summarizes the types of isosurfaces of the coordinates.

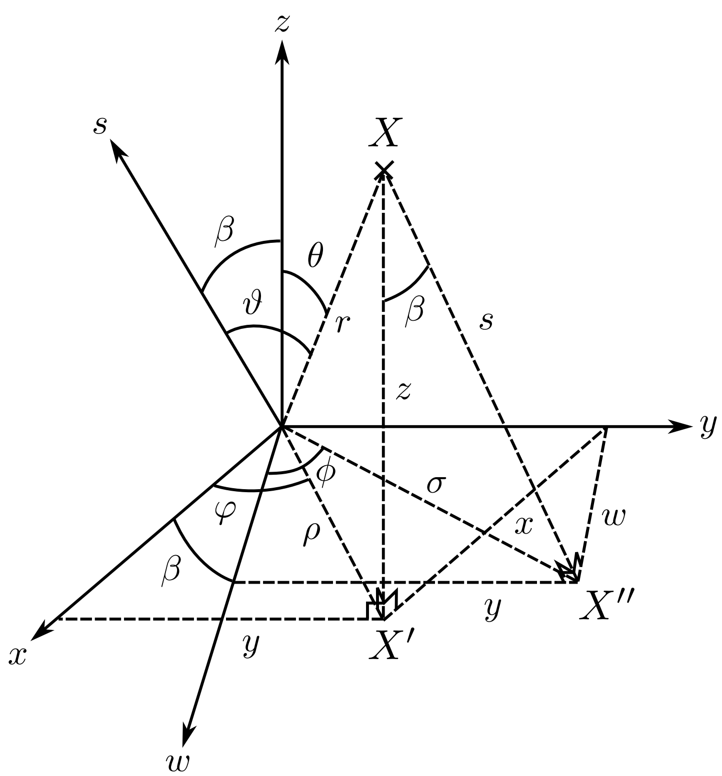

Figure D.1: Coordinates

Table D.1: Relations between coordinates

Non-rotated	Rotated	Mixed
$x = \rho \cos \varphi$	$w = \sigma \cos \phi$	$z = s \cos \beta - w \sin \beta$
$y = \rho \sin \varphi$	$y = \sigma \sin \phi$	$x = s \sin \beta + w \cos \beta$
$z = r \cos \theta$	$s = r \cos \vartheta$	$s = z \cos \beta + x \sin \beta$
$\rho = r \sin \theta$	$\sigma = r \sin \vartheta$	$w = -z \sin \beta + x \cos \beta$

Table D.2: Isosurfaces of coordinates

Coordinate	Isosurfaces
x, y, z, s	parallel planes
ρ, σ	coaxial cylinders
r	concentric spheres
φ, ϕ	coaxial half-planes
θ, ϑ	concentric cones

D.1 Coordinate unit vectors

Definition 10. *A coordinate unit vector is a unit vector field perpendicular to isosurfaces of its corresponding coordinate pointing in the direction in which the coordinate is increasing.*

The coordinate unit vectors corresponding to the coordinates given by Figure D.1 are $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{\theta}}$ (non-rotated), $\hat{\boldsymbol{w}}, \hat{\boldsymbol{s}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\vartheta}}$ (rotated) and $\hat{\boldsymbol{y}}, \hat{\boldsymbol{r}}$ (both being rotated and non-rotated). The non-rotated coordinate unit vectors are visualised at particular z and φ isosurfaces in Figure D.2.

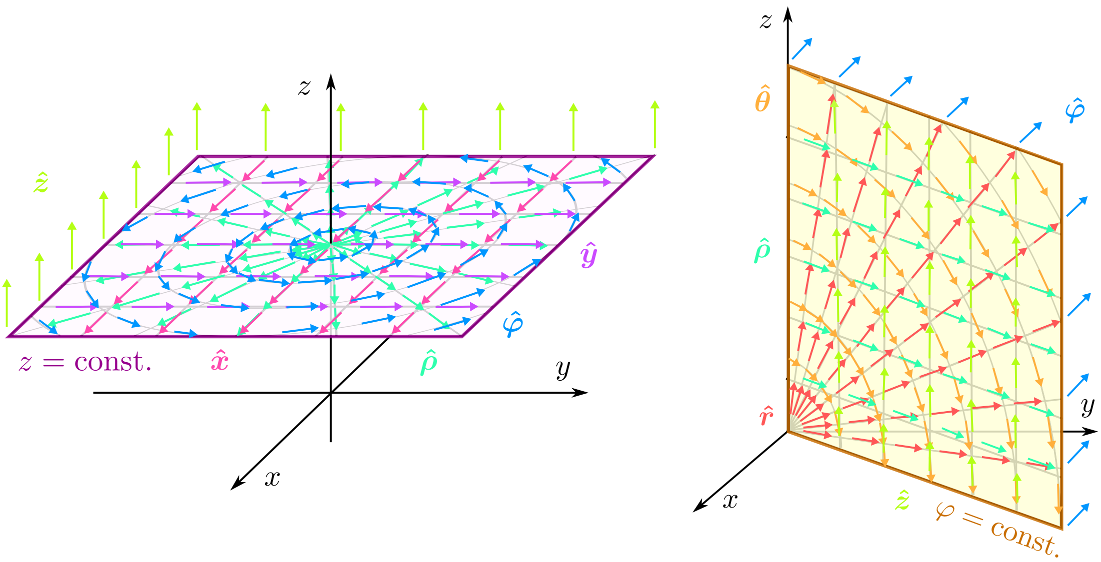

Figure D.2: Coordinate unit vector fields – z- and φ-isosurfaces

The triples $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$, $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{z}})$, $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\right)$, $(\hat{\boldsymbol{w}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{s}})$, $\left(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{s}}\right)$, $\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\phi}}\right)$ form orthonormal right-handed bases. That means, the scalar product of two different unit vectors from the same base vanishes (for example $\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}} = 0$, $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\varphi}} = 0$, $\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\rho}} = 0$ etc.). Similarly, the cross product of two different unit vectors from the same base gives the third unit vector with plus or minus according to the orientation (for example $\hat{\boldsymbol{\varphi}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\varphi}} = -\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{y}} \times \hat{\boldsymbol{s}} = \hat{\boldsymbol{w}}$ etc.). In order to find the scalar and vector products of two unit vectors from different bases, we first, in view of Figure D.2, obtain the diagrams given by Figure D.3 (the third diagram is obtained from Figure D.1 as a y-isosurface). The standard notation for vectors pointing in $(\hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{y}})$ or out $(\hat{\boldsymbol{z}})$ of a plane is used in Figure D.3.

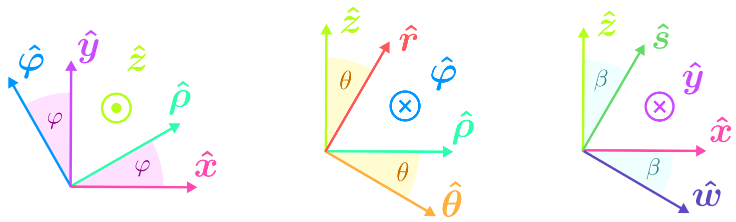

Figure D.3: Coordinate unit vector fields – Diagrams

These diagrams show how the unit vector fields $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ are related to each other at a point. In view of the definition of the scalar and cross products (see (C.1) and (C.2)), we obtain Table D.3.

Table D.3: Scalar and vector products of coordinate unit vectors

Scalar product

$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\rho}} = \cos\varphi$	$\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = \cos\theta$	$\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{s}} = \cos\beta$
$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{\varphi}} = -\sin\varphi$	$\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -\sin\theta$	$\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{w}} = -\sin\beta$
$\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\rho}} = \sin\varphi$	$\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{r}} = \sin\theta$	$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{s}} = \sin\beta$
$\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\varphi}} = \cos\varphi$	$\hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\theta}} = \cos\theta$	$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{w}} = \cos\beta$

Vector product

$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{\rho}} = \sin\varphi\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \sin\theta\hat{\boldsymbol{\varphi}}$	$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}} = \sin\beta\hat{\boldsymbol{y}}$
$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{\varphi}} = \cos\varphi$	$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\varphi}}$	$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{w}} = \cos\beta\hat{\boldsymbol{y}}$
$\hat{\boldsymbol{y}} \times \hat{\boldsymbol{\rho}} = -\cos\varphi\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{r}} = -\cos\theta\hat{\boldsymbol{\varphi}}$	$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{s}} = -\cos\beta\hat{\boldsymbol{y}}$
$\hat{\boldsymbol{y}} \times \hat{\boldsymbol{\varphi}} = \sin\varphi\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\theta}} = \sin\theta\hat{\boldsymbol{\varphi}}$	$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{w}} = \sin\beta\hat{\boldsymbol{y}}$

Other identities are obtained from Table D.3 replacing non-rotated coordinate unit vectors by rotated coordinate unit vectors and vice versa, in addition with replacement $\beta \leftrightarrow -\beta$.

Miscellaneous

In this thesis, the following identities are used:

$$-\frac{\rho s}{r}\hat{\boldsymbol{\theta}} + \frac{\sigma z}{r}\hat{\boldsymbol{\vartheta}} = s\hat{\boldsymbol{z}} - z\hat{\boldsymbol{s}}, \tag{D.1}$$

$$\sigma r \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\vartheta}} = \rho r \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{\theta}} = -\rho\sigma \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\vartheta}} = zs - r^2 \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{z}}, \tag{D.2}$$

$$zr \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\sigma}} = -sz \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\theta}} = -sr \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\rho}} = -sz \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\vartheta}} = \frac{szr^2}{\rho\sigma} (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}) \cdot \hat{\boldsymbol{r}}. \tag{D.3}$$

To prove them, we first decompose $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\theta}}$ into base $\hat{\boldsymbol{r}}, \hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\vartheta}}$ into base $\hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}$, respectively. Using Table D.1 and Table D.3, we get:

$$\rho \hat{\boldsymbol{\varphi}} = r\, \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} \tag{D.4}$$

$$\rho \hat{\boldsymbol{\theta}} = z \hat{\boldsymbol{r}} - r \hat{\boldsymbol{z}} \tag{D.5}$$

$$\rho \hat{\boldsymbol{\rho}} = r \hat{\boldsymbol{r}} - z \hat{\boldsymbol{z}} \tag{D.6}$$

$$\sigma \hat{\boldsymbol{\phi}} = r\, \hat{\boldsymbol{s}} \times \hat{\boldsymbol{r}} \tag{D.7}$$

$$\sigma \hat{\boldsymbol{\vartheta}} = s \hat{\boldsymbol{r}} - r \hat{\boldsymbol{s}} \tag{D.8}$$

$$\sigma \hat{\boldsymbol{\sigma}} = r \hat{\boldsymbol{r}} - s \hat{\boldsymbol{s}} \tag{D.9}$$

The proof of (D.1) is straightforward, by using (D.5) and (D.8):

$$-\frac{\rho s}{r} \hat{\boldsymbol{\theta}} + \frac{\sigma z}{r} \hat{\boldsymbol{\vartheta}} = -\frac{s}{r} (z \hat{\boldsymbol{r}} - r \hat{\boldsymbol{z}}) + \frac{z}{r} (s \hat{\boldsymbol{r}} - r \hat{\boldsymbol{s}}) = s \hat{\boldsymbol{z}} - z \hat{\boldsymbol{s}}.$$

To prove (D.2), by using (D.8), we get

$$\sigma \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\vartheta}} = \hat{\boldsymbol{z}} \cdot (s \hat{\boldsymbol{r}} - r \hat{\boldsymbol{s}}) = \frac{zs}{r} - r \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{s}},$$

where $\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = \cos\theta = \frac{z}{r}$ by Table D.3 and Table D.1. The other equal signs in (D.2) are proved in the same way by using (D.4) – (D.9). Similarly, by using the formulae (C.5) and (C.7), we get

$$\rho\sigma \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\sigma}} = (r\, \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}) \cdot (r \hat{\boldsymbol{r}} - s \hat{\boldsymbol{s}}) = -sr\, (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}) \cdot \hat{\boldsymbol{s}} = sr \hat{\boldsymbol{r}} \cdot (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{s}}),$$

the rest in proving (D.3) is obvious.

D.2 Differential identities

Table D.4: Differential identities for coordinates and coordinate unit vectors

Scalar gradient	Vector gradient	Divergence
$\boldsymbol{\nabla} r = \hat{\boldsymbol{r}}$	$\boldsymbol{\nabla}\hat{\boldsymbol{r}} = \frac{1}{r}\left(\mathbb{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right)$	$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{r}} = \frac{2}{r}$
$\boldsymbol{\nabla}\rho = \hat{\boldsymbol{\rho}}$	$\boldsymbol{\nabla}\hat{\boldsymbol{\rho}} = \frac{1}{\rho}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}$	$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}} = \frac{1}{\rho}$
$\boldsymbol{\nabla}\varphi = \frac{1}{\rho}\hat{\boldsymbol{\varphi}}$	$\boldsymbol{\nabla}\hat{\boldsymbol{\varphi}} = -\frac{1}{\rho}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}$	$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\varphi}} = 0$
$\boldsymbol{\nabla}\theta = \frac{1}{r}\hat{\boldsymbol{\theta}}$	$\boldsymbol{\nabla}\hat{\boldsymbol{\theta}} = \frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}$	$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}} = \frac{z}{\rho r}$
Curl	**Scalar Laplacian**	**Vector Laplacian**
$\boldsymbol{\nabla}\times\hat{\boldsymbol{r}} = \boldsymbol{0}$	$\Delta r = \frac{2}{r}$	$\Delta\hat{\boldsymbol{r}} = -\frac{2}{r^2}\hat{\boldsymbol{r}}$
$\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}} = \boldsymbol{0}$	$\Delta\rho = \frac{1}{\rho}$	$\Delta\hat{\boldsymbol{\rho}} = -\frac{1}{\rho^2}\hat{\boldsymbol{\rho}}$
$\boldsymbol{\nabla}\times\hat{\boldsymbol{\varphi}} = \frac{1}{\rho}\hat{\boldsymbol{z}}$	$\Delta\varphi = 0$	$\Delta\hat{\boldsymbol{\varphi}} = -\frac{1}{\rho^2}\hat{\boldsymbol{\varphi}}$
$\boldsymbol{\nabla}\times\hat{\boldsymbol{\theta}} = \frac{1}{r}\hat{\boldsymbol{\varphi}}$	$\Delta\theta = \frac{z}{\rho r^2}$	$\Delta\hat{\boldsymbol{\theta}} = -\frac{z}{\rho r^2}\hat{\boldsymbol{r}} - \frac{1}{\rho^2}\hat{\boldsymbol{\theta}}$

Similarly, as in case of Table D.3, other identities are obtained from Table D.4 replacing non-rotated coordinates by rotated coordinates.

Derivations

We now derive the identities given by Table D.4. By definition, $\boldsymbol{\nabla} x = \hat{\boldsymbol{x}}, \boldsymbol{\nabla} y = \hat{\boldsymbol{y}}, \boldsymbol{\nabla} z = \hat{\boldsymbol{z}}$ and $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. From this, $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{x}} = x/r, \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{y}} = y/r, \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{z}} = z/r$ and also $\boldsymbol{\nabla}\boldsymbol{r} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = \mathbb{I}$. Since the identity tensor is independent on coordinate system, we have for the other two orthogonal systems $\mathbb{I} = \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}$. Comparing, we get $\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \stackrel{\text{def}}{=} \mathbb{I}_\varphi$. Taking the scalar product of the previous relation with $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{\rho}}$, we get

$$z\hat{\boldsymbol{r}} - \rho\hat{\boldsymbol{\theta}} = r\hat{\boldsymbol{z}}, \tag{D.10}$$

$$z\hat{\boldsymbol{r}} + z\hat{\boldsymbol{\theta}} = r\hat{\boldsymbol{\rho}}. \tag{D.11}$$

Another comparison gives $\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} \stackrel{\text{def}}{=} \mathbb{I}_{xy}$. Taking the scalar product with $\hat{\boldsymbol{x}}$, we get

$$\rho\hat{\boldsymbol{x}} = x\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}}. \tag{D.12}$$

Similarly, since $\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$, we get $\boldsymbol{\nabla}\boldsymbol{\rho} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} = \mathbb{I}_{xy}$. Since $r^2 = x^2 + y^2 + z^2$, in view of the chain-rule of differentiation, $2r\boldsymbol{\nabla} r = 2x\hat{\boldsymbol{x}} + 2y\hat{\boldsymbol{y}} + 2z\hat{\boldsymbol{z}} = 2\boldsymbol{r} = 2r\hat{\boldsymbol{r}}$, so $\boldsymbol{\nabla} r = \hat{\boldsymbol{r}}$. Similarly, $\rho^2 = x^2 + y^2$, so $\boldsymbol{\nabla}\rho = \hat{\boldsymbol{\rho}}$. Taking the gradient of $r\sin\theta = \rho$, we get $\frac{\rho}{r}\hat{\boldsymbol{r}} + z\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\rho}}$ in view of the chain-rule. Therefore, $\boldsymbol{\nabla}\theta = \frac{1}{r}\hat{\boldsymbol{\theta}}$. Similarly, since the gradient of $\rho\cos\varphi = x$ gives $\frac{x}{\rho}\hat{\boldsymbol{\rho}} - y\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{x}}$, we get $\boldsymbol{\nabla}\varphi = \frac{1}{\rho}\hat{\boldsymbol{\varphi}}$. The vector gradients are given, straightforwardly, as $\boldsymbol{\nabla}\hat{\boldsymbol{r}} =$

$\boldsymbol{\nabla}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r}\left(\mathbb{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right)$ since $\boldsymbol{\nabla}\boldsymbol{r} = \mathbb{I}$. Similarly, $\boldsymbol{\nabla}\left(\frac{\boldsymbol{\rho}}{\rho}\right) = \frac{1}{\rho}\left(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} - \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}}\right) = \frac{1}{\rho}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}$, since there are two ways of expressing $\mathbb{I}_{xy}$. Taking the gradient of the identity $\hat{\boldsymbol{z}}\times\boldsymbol{r} = \hat{\boldsymbol{\varphi}}r\sin\theta = \hat{\boldsymbol{\varphi}}\rho$, we get by the product rule of differentiation: $\rho\boldsymbol{\nabla}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\varphi}} = -\mathbb{I}\times\hat{\boldsymbol{z}} = -\left(\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}\right)\times\hat{\boldsymbol{z}} = \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}$, so $\boldsymbol{\nabla}\hat{\boldsymbol{\varphi}} = -\frac{1}{\rho}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}$. Similarly, since $\rho\hat{\boldsymbol{\theta}} = \rho\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{r}} = r\left(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}\right)\times\hat{\boldsymbol{r}} = z\hat{\boldsymbol{r}} - r\hat{\boldsymbol{z}}$, we recover a known identity. Taking the gradient, we get $\rho\boldsymbol{\nabla}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{z}}\hat{\boldsymbol{r}} - \hat{\boldsymbol{r}}\hat{\boldsymbol{z}} + \frac{z}{r}\left(\mathbb{I} - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\right) = \frac{1}{r}\left(r\hat{\boldsymbol{z}}\hat{\boldsymbol{r}} - r\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} + z\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + z\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}\right)$. Simplifying, using previously found identities, we get, finally, $\boldsymbol{\nabla}\hat{\boldsymbol{\theta}} = \frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}$.

The other differential identities

Actually, only the scalar gradients and vector gradients (together with vector calculus rules) suffice to derive all the other identities. The divergences, for example, are derived from the vector gradients by the relation $\boldsymbol{\nabla}\cdot\boldsymbol{v} = \mathrm{Tr}\boldsymbol{\nabla}\boldsymbol{v}$, which holds for any vector field $\boldsymbol{v}$. Similarly, $\boldsymbol{\nabla}\times\boldsymbol{v} = \mathrm{Cr}\boldsymbol{\nabla}\boldsymbol{v}$. For example, $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}} = \mathrm{Tr}\left(\frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}\right) = \frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{r}} = \frac{z}{\rho r}$ and $\boldsymbol{\nabla}\times\hat{\boldsymbol{\theta}} = \mathrm{Cr}\left(\frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}\right) = \frac{z}{\rho r}\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{\varphi}} - \frac{1}{r}\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{r}} = \frac{1}{r}\hat{\boldsymbol{\varphi}}$. For any scalar Laplacian, we use the identity $\Delta f = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f$, which holds for any suitable scalar function f. This identity also holds for any vector field $\boldsymbol{v}$. Moreover, for vector Laplacian, one can also use the identity $\Delta\boldsymbol{v} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{v} - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{v}$, which holds for any suitable vector field $\boldsymbol{v}$.

E Miscellaneous identities

In the next section, $R = R(r)$ and $S = S(r)$ are functions of r variable only, ψ will be a function of variables with unit vectors perpendicular to $\hat{\boldsymbol{\varphi}}$, i.e. $\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\nabla}\psi = 0$. Vector field $\boldsymbol{a}$ will be a constant vector field, whereas $\boldsymbol{v}$ will be any **solenoidal** field, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. The differential operator $\mathrm{E}_\alpha[\,\cdot\,]$ is defined by $\mathrm{E}_\alpha[f] \overset{\text{def}}{=} f'' + \frac{\alpha}{r} f'$.

E.1 Differential identities

$$\boldsymbol{\nabla}\cdot(\boldsymbol{r}\times\boldsymbol{a}) = 0 \tag{E.1}$$

$$\Delta(\psi\hat{\boldsymbol{\varphi}}) = \left(\Delta\psi - \frac{1}{\rho^2}\psi\right)\hat{\boldsymbol{\varphi}} \tag{E.2}$$

$$\Delta(\rho R\hat{\boldsymbol{\varphi}}) = \rho\mathrm{E}_4[R]\,\hat{\boldsymbol{\varphi}} \tag{E.3}$$

$$\boldsymbol{\nabla}(\rho R\hat{\boldsymbol{\varphi}}) = \rho R'\hat{\boldsymbol{r}}\hat{\boldsymbol{\varphi}} + R(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\varphi}} - \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\rho}}) \tag{E.4}$$

$$\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\rho}}) = \frac{z\rho}{r}R'\hat{\boldsymbol{\varphi}} \tag{E.5}$$

$$\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}}) = 2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}} \tag{E.6}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}}) = \frac{R'}{r}\left(3r\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} + \rho\hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}} - z\mathbb{I}\right) - \rho R''\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} \tag{E.7}$$

$$\Delta(z\rho S\hat{\boldsymbol{\varphi}}) = z\rho\mathrm{E}_6[S]\,\hat{\boldsymbol{\varphi}} \tag{E.8}$$

$$\boldsymbol{\nabla}\times(z\rho S\hat{\boldsymbol{\varphi}}) = 2Sz\hat{\boldsymbol{z}} - S\rho\hat{\boldsymbol{\rho}} - z\rho S'\hat{\boldsymbol{\theta}} \tag{E.9}$$

E.2 Integral identities

$$\iint_{r=a} \hat{\boldsymbol{r}}\,\mathrm{d}S = \boldsymbol{0} \tag{E.10}$$

$$\iint_{r=a} z\hat{\boldsymbol{r}}\,\mathrm{d}S = \frac{4}{3}\pi a^3\hat{\boldsymbol{z}} \tag{E.11}$$

$$\iint_{r=a} \rho\hat{\boldsymbol{\theta}}\,\mathrm{d}S = -\frac{8}{3}\pi a^3\hat{\boldsymbol{z}} \tag{E.12}$$

$$\iint_{r=a} z^2\hat{\boldsymbol{r}}\,\mathrm{d}S = \boldsymbol{0} \tag{E.13}$$

$$\iint_{r=a} \rho^2\hat{\boldsymbol{r}}\,\mathrm{d}S = \boldsymbol{0} \tag{E.14}$$

$$\iint_{r=a} z\rho\hat{\boldsymbol{r}}\,\mathrm{d}S = \boldsymbol{0} \tag{E.15}$$

$$\iint_{r=a} \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\,\mathrm{d}S = \frac{4}{3}\pi a^2\mathbb{I} \tag{E.16}$$

$$\iint_{r=a} r\hat{\boldsymbol{z}}\,\mathrm{d}S = 4\pi a^3\hat{\boldsymbol{z}} \tag{E.17}$$

Derivations

In order to improve readability, we will derive some of identities (E.1) – (E.17) successively in separate text blocks. The number of each identity will be placed at the beginning of each block in *italics*.

E.1. By the definition of the divergence and by using (C.5):

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{r}\times\boldsymbol{a}) &= \mathrm{Tr}\,[\boldsymbol{\nabla}\boldsymbol{r}\times\boldsymbol{a}] = \mathrm{Tr}\,[\mathbb{I}\times\boldsymbol{a}] = \mathrm{Tr}\,[\hat{\boldsymbol{x}}(\hat{\boldsymbol{x}}\times\boldsymbol{a}) + \hat{\boldsymbol{y}}(\hat{\boldsymbol{y}}\times\boldsymbol{a}) + \hat{\boldsymbol{z}}(\hat{\boldsymbol{z}}\times\boldsymbol{a})] \\ &= \hat{\boldsymbol{x}}\cdot(\hat{\boldsymbol{x}}\times\boldsymbol{a}) + \hat{\boldsymbol{y}}\cdot(\hat{\boldsymbol{y}}\times\boldsymbol{a}) + \hat{\boldsymbol{z}}\cdot(\hat{\boldsymbol{z}}\times\boldsymbol{a}) = 0.\end{aligned}$$

□

E.2. With help of the product rule of differentiation for Laplacian, we get

$$\Delta\left(\psi\hat{\boldsymbol{\varphi}}\right) = \Delta\psi\hat{\boldsymbol{\varphi}} + 2\boldsymbol{\nabla}\psi\boldsymbol{\cdot}\boldsymbol{\nabla}\hat{\boldsymbol{\varphi}} + \psi\Delta\hat{\boldsymbol{\varphi}} = \Delta\psi\hat{\boldsymbol{\varphi}} - \frac{\psi}{\rho^2}\hat{\boldsymbol{\varphi}} = \left(\Delta\psi - \frac{1}{\rho^2}\psi\right)\hat{\boldsymbol{\varphi}},$$

which proves the original identity. □

E.3. Since $\Delta R = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}R = \boldsymbol{\nabla}\boldsymbol{\cdot}(R'\hat{\boldsymbol{r}}) = R'' + \frac{2}{r}R'$ and $\Delta\rho = \frac{1}{\rho}$, we have in view of (E.2):

$$\Delta\left(\rho R\hat{\boldsymbol{\varphi}}\right) = \left(\Delta(\rho R) - \frac{R}{\rho}\right)\hat{\boldsymbol{\varphi}} = \left(\frac{R}{\rho} + 2\rho\Delta R + \rho\Delta R - \frac{R}{\rho}\right)\hat{\boldsymbol{\varphi}},$$

which proves the original identity. □

E.4. With help of the product rule of differentiation for the gradient, we get

$$\boldsymbol{\nabla}\left(\rho R\hat{\boldsymbol{\varphi}}\right) = \hat{\boldsymbol{\rho}}R\hat{\boldsymbol{\varphi}} + \rho R'\hat{\boldsymbol{r}}\hat{\boldsymbol{\varphi}} + \rho R\boldsymbol{\nabla}\hat{\boldsymbol{\varphi}},$$

which proves the original identity. □

E.5. With help of the product rule of differentiation for the curl,

$$\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\rho}}) = \hat{\boldsymbol{\rho}}\times R\hat{\boldsymbol{\rho}} + \rho R'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\rho}} + \rho R\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}},$$

which proves the original identity. □

E.6. With help of the product rule of differentiation for the curl, we get

$$\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}}) = \hat{\boldsymbol{\rho}}\times R\hat{\boldsymbol{\varphi}} + \rho R'\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\varphi}} + \rho R\boldsymbol{\nabla}\times\hat{\boldsymbol{\varphi}},$$

which proves the original identity. □

E.7. With help of the product rule of differentiation for the gradient and by (E.6), we get

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\times(\rho R\hat{\boldsymbol{\varphi}}) = \boldsymbol{\nabla}\left(2R\hat{\boldsymbol{z}} - \rho R'\hat{\boldsymbol{\theta}}\right) = R'\hat{\boldsymbol{r}}\hat{\boldsymbol{z}} - R'\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\theta}} - \rho R''\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} - \rho R'\boldsymbol{\nabla}\hat{\boldsymbol{\theta}},$$

which proves the original identity. □

E.8. With help of the product rule of differentiation for Laplacian and by (E.4), we get

$$\Delta\left(z\rho S\hat{\boldsymbol{\varphi}}\right) = \left(\Delta z\right)\rho S\hat{\boldsymbol{\varphi}} + 2\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{\nabla}\left(\rho S\hat{\boldsymbol{\varphi}}\right) + z\Delta\left(\rho S\hat{\boldsymbol{\varphi}}\right) = \frac{2z\rho}{r}S'\hat{\boldsymbol{\varphi}} + z\rho\mathrm{E}_4\left[S\right]\hat{\boldsymbol{\varphi}},$$

which proves the original identity. □

E.11. By rotational symmetry,

$$\iint_{r=a} z\hat{\boldsymbol{r}}\ \mathrm{d}S = A\hat{\boldsymbol{z}},$$

where A is some constant. Taking the scalar product with $\hat{\boldsymbol{z}}$ and since $\hat{\boldsymbol{z}} \bullet \hat{\boldsymbol{r}} = \frac{z}{r}$ at $r = a$, we get

$$\iint_{r=a} z^2\ \mathrm{d}S = aA.$$

Since we have not specified any preferred orientation in space for the $\hat{\boldsymbol{z}}$-axis, we have

$$aA = \iint_{r=a} x^2\ \mathrm{d}S = \iint_{r=a} y^2\ \mathrm{d}S = \iint_{r=a} z^2\ \mathrm{d}S.$$

Summing these three up, we get

$$3aA = \iint_{r=a} x^2 + y^2 + z^2\ \mathrm{d}S = a^2 \iint_{r=a} \mathrm{d}S = 4\pi a^4,$$

where we have used the value of the surface of the sphere with radius $r = a$, which is $4\pi a^2$. Dividing the previous relation by $3a$, we get the original identity. □

E.12. By rotational symmetry,

$$\iint_{r=a} \rho\hat{\boldsymbol{\theta}}\ \mathrm{d}S = A\hat{\boldsymbol{z}},$$

where A is some constant. Taking the scalar product with $\hat{\boldsymbol{z}}$, we get at $r = a$ since $\hat{\boldsymbol{z}} \bullet \hat{\boldsymbol{\theta}} = -\frac{\rho}{r}$:

$$\iint_{r=a} \rho^2\ \mathrm{d}S = -aA.$$

Since $\rho^2 = x^2 + y^2$, we have

$$-aA = \iint_{r=a} x^2 + y^2\ \mathrm{d}S = \iint_{r=a} x^2 + y^2 + z^2\ \mathrm{d}S - \iint_{r=a} z^2\ \mathrm{d}S = 4\pi a^4 - \frac{1}{3} \cdot 4\pi a^4.$$

Dividing the previous relation by a, we get the original identity. □

E.16. By symmetry in all directions,

$$\iint_{r=a} \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\ \mathrm{d}S = A\mathbb{I},$$

where A is some constant. Taking the trace of both sides,

$$3A = A\mathrm{Tr}\left[\mathbb{I}\right] = \iint_{r=a} \hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{r}}\ \mathrm{d}S = \iint_{r=a} \mathrm{d}S = 4\pi a^2.$$

Dividing by 3, we get the original identity. □

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