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DIPLOMOVÁ PRÁCE

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Barotropní oceánický slapový model

Katedra geofyziky

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Název práce: Barotropní oceánický slapový model

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Abstrakt: Hlavním cílem této práce je vyvinutí numerického modelu oceánského proudění buzeného slapovým působením Slunce a Měsíce. Proudění je popsáno rovnicemi mělké vody, které jsou odvozeny ze základních zákonů zachování za předpokladu, že poměr vertikálního a horizontálního rozměru zkoumaného problému je malý, což vede k formulaci 2-D úlohy. Dále byly napsány programy pro řešení rovnic mělké vody, jejichž funkčnost je demonstrována na několika případech. Součástí programů je efemeridní slapový modul, který počítá úplný lunisolární slapový potenciál.

Klíčová slova: oceánské proudění, rovnice mělké vody, lunisolární slapový potenciál

Title: Barotropic ocean tide model

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Abstract: The main aim of this thesis is developing of a numerical model of an oceanic circulation forced by the lunisolar tidal potential. The circulation is described by the shallow water equations which are derived from the fundamental balance laws assuming that the ratio of the vertical and horizontal dimension of the investigated problem is small, which leads to the formulation of the 2-D task. Furthermore, programs for solving the shallow water equations were written. Their functionality is demonstrated on several examples. The programs include an ephemeridal tidal modul which computes the complete lunisolar tidal potential.

Keywords: oceanic flow, shallow water equations, lunisolar tidal potential

Contents

Introduction

Physical oceanography deals with studying physical properties and physical conditions within the ocean. This includes for instance waves, tides, currents, temperature and salinity structure etc. This discipline is related to many other fields of science such as geophysics, meteorology and applied mathematics.

This work is mainly focused on tides and waves in the ocean which are described by the Navier-Stokes equations completed by appropriate boundary conditions. However, as generally known, there has not been proven yet that there exists a smooth and globally defined solution in three space dimensions and time. The Clay Mathematics Institute has called this one of the seven most important open problem in mathematics (Fefferman, 2000). Moreover, the complicated nature of the equations restricts possibilities of numerical solving.

In modelling of ocean motions the moving free surface represents a crucial problem. An approximate way to overcome this problem is by so-called *shallow water approximation*. This approximation, which is common and successful in many applications, can be applied when the vertical dimension of the problem is negligible in comparison with the horizontal dimension. Under this assumption the Navier-Stokes equations will reduce to the shallow water equations for the elevation of the free surface and the horizontal components of the velocity. The derivation of the shallow water equation is the topic of Chapter 1. The derivation includes the Reynolds stresses which are usually omitted in the derivation.

The outline of this work is to develop an oceanic model based on the shallow water equations which is forced by the lunisolar tidal potential. Numerical methods are described in Chapter 2. Chapter 3 briefly reviews the theory of the tides. Unlike many similar models, which use partial tide forcing by frequency-domain simulation, our model uses the complete lunisolar tidal potential and computes the positions of the Moon and the Sun (ephemerides) in every time step.

In Chapter 4, we present examples of numerical simulations with various geometries of ocean and bathymetry which were computed by newly developed programs written in Fortran 90.

Chapter 1

Derivation of the shallow water equations

1.1 Introduction

In this chapter we provide a derivation of the shallow water equations, sometimes called Saint-Venant equations after Adhémar Jean Claude Barré de Saint-Venant, french mechanician and mathematician, who first derived them in 1871 (de Saint-Venant, 1871). This set of equations is used in modelling of many geophysical phenomena, such as oceans, atmosphere, shelf and coastal seas, rivers or even avalanches. The derivation is classical with neglected viscosity (e.g. Pedlosky, 1987, or Vallis, 2006). The derivation including the viscous term was published in several papers (Gerbeau and Perthame, 2000, Ferrari and Saleri, 2004, Marche, 2006), however with a different stress tensor than is usually used in oceanographic models. We provide the derivation in the Cartesian and the geographical coordinates including a full stress tensor with the eddy viscosity from fundamental balance laws. These fundamental laws are the continuity equation and the equation of motion (see Martinec, 2011):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1.1}$$

$$\rho \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \nabla \cdot \mathbf{t} + \vec{f}, \tag{1.2}$$

where ρ is applied density, $\vec{v}$ the velocity, $\mathbf{t}$ the stress tensor and $\vec{f}$ external body forces. Note that for the material time derivative $\frac{\mathrm{D}}{\mathrm{D}t}$ in the Eulerian representation it holds $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$.

We consider incompressible fluid which means that the continuity equation takes the form

$$\nabla \cdot \vec{v} = 0. \tag{1.3}$$

The external forces are the Coriolis force and the gravitation. For the sake of simplicity we do not consider the tidal force so far. The stress tensor takes an usual form consisting of the pressure and a symmetric tensor with null trace

$$\mathbf{t} = -p\mathbf{I} + \boldsymbol{\sigma}. \tag{1.4}$$

Under these conditions we can write the Navier-Stokes equations which are a base of the derivation

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \nabla\cdot(\vec{v}\otimes\vec{v})\right) = -\nabla p + \nabla\cdot\boldsymbol{\sigma} - 2\vec{\Omega}\times\vec{v} + \vec{g}. \tag{1.5}$$

In the Cartesian coordinates the continuity equation and the Navier-Stokes equations take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.6}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}, \tag{1.7}$$

$$\frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial(vw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}, \tag{1.8}$$

$$\frac{\partial w}{\partial t} + \frac{\partial(wu)}{\partial x} + \frac{\partial(wv)}{\partial y} + \frac{\partial w^2}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \frac{\partial\sigma_{zx}}{\partial x} + \frac{\partial\sigma_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}, \tag{1.9}$$

where f is the Coriolis parameter, $f = 2\Omega\sin\phi$, Ω is the mean angular velocity of the Earth, ϕ is the latitude and g is the gravitational acceleration.

1.2 The Reynolds tensor and the eddy viscosity

The relation between the tensor $\boldsymbol{\sigma}$ in equation (1.4) and the velocity is described by the constitutive equation. Usually we prescribe the relation for the classical Newtonian fluid

$$\boldsymbol{\sigma} = \eta(\nabla\vec{v} + \nabla^{\mathrm{T}}\vec{v}), \tag{1.10}$$

where η is the molecular viscosity of the fluid. However, in the case of oceanic circulation this term is negligible in comparison with other terms. This can be easily proven by an estimation of the Reynolds number which describes the ratio between the magnitudes of the nonlinear term and the viscous term. The Reynolds number is

$$R_e = \frac{\rho U L}{\eta}, \tag{1.11}$$

where U and L are the characteristic scales of the horizontal velocity and the horizontal dimension, respectively. When we take in account typical values for an ocean, i.e. $U \approx 10^{-1}$ m/s, $L \approx 10^6$ m, $\rho \approx 10^3$ kg/m^3 and $\eta \approx 10^{-3}$ Pa.s (Pond and Pickard, 1983), the Reynolds number is approximately 10^{11}, and therefore the term with the molecular viscosity can be neglected. Nevertheless, although the friction within the fluid is weak, the dissipative nature of the friction, completely distinct from the conservative nature of the inertial forces, requires particular attention. Indeed, we can not neglect the internal friction of the fluid. Even though its direct effect on the large-scale motion is absolutely negligible, the dissipation of the kinetic energy plays an important role.

The presence of the friction arises from the turbulent character of oceanic flow. The Reynolds number controls whether the flow is laminar or turbulent. Typical threshold value between laminar and turbulent flow is about 1000. However, as was already shown, the Reynolds number for an ocean is several orders of magnitude greater. Hence, the flow is a priori turbulent. However, there is a problem how to describe the turbulences on very short scales when we are interested in the dynamics of only large-scale motions. The problem of mathematical description of turbulences is one of the most crucial in fluid modelling and still not satisfactorily solved. In this work we follow the ideas of Joseph Valentin Boussinesq and Osborne Reynolds from the late nineteenth century who presented one of ways of how turbulences can be modeled (see Boussinesq, 1877 and Reynolds, 1895, or Schmitt, 2007).

Consider the velocity field $\vec{v}$ in the fluid split into two parts. The first one, denoted by $\langle\vec{v}\rangle$, represents the large-scale flow and the second one, denoted by $\vec{v'}$, represents the small-scale turbulences, i.e.

$$\vec{v} = \langle\vec{v}\rangle + \vec{v'}. \tag{1.12}$$

The large-scale flow is supposed to be the velocity field which is averaged over some characteristic time. This characteristic time should be long enough, so that the average of $\vec{v'}$ over this time vanishes. Mathematically this means

$$\langle\vec{v'}\rangle = 0 \qquad \langle\langle\vec{v}\rangle + \vec{v'}\rangle = \langle\vec{v}\rangle. \tag{1.13}$$

Now, let us consider the continuity equation and the Navier-Stokes equations with the decomposition (1.12). The continuity equation remains in the same form for both part, the large-scale flow and the small-scale turbulences. The equation of motion is different. For example, the x-component is (neglecting the term with the molecular viscosity)

$$\frac{\partial}{\partial t}(\langle u\rangle + u') + \frac{\partial}{\partial x}(\langle u\rangle + u')^2 + \frac{\partial}{\partial y}(\langle u\rangle + u')(\langle v\rangle + v') + \\ \frac{\partial}{\partial z}(\langle u\rangle + u')(\langle w\rangle + w') = -\frac{1}{\rho}\frac{\partial}{\partial x}(\langle p\rangle + p') + f(\langle v\rangle + v'). \tag{1.14}$$

The average of this equation yields

$$\frac{\partial\langle u\rangle}{\partial t} + \left(\frac{\partial\langle u\rangle^2}{\partial x} + \frac{\partial(\langle u\rangle\langle v\rangle)}{\partial y} + \frac{\partial(\langle u\rangle\langle w\rangle)}{\partial z}\right) + \left(\frac{\partial\langle u'u'\rangle}{\partial x} + \frac{\partial\langle u'v'\rangle}{\partial y} + \frac{\partial\langle u'w'\rangle}{\partial z}\right) = \\ = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x} + f\langle v\rangle. \tag{1.15}$$

Equation (1.15) is written in terms of the large-scale velocities, however the nonlinear term consists of two parts. The second part is written in terms of multiplies of the turbulences since there is no reason why these quadratic terms should vanish by averaging. However, we have more unknown quantities than the equations and we need to write the quadratic fluctuation velocities in terms of the large-scale velocities. Formally, we can introduce a new stress tensor which is created by the quadratic fluctuation velocities, e.g. for xy-component this means

$$-\frac{\sigma_{xy}}{\rho} \equiv \langle u'v' \rangle. \tag{1.16}$$

The stress tensor $\boldsymbol{\sigma}$, which is defined by this way, is called the Reynolds tensor and its components the Reynolds stresses. The full Reynolds tensor is given by

$$\frac{\boldsymbol{\sigma}}{\rho} = -\begin{pmatrix} \langle u'u' \rangle & \langle u'v' \rangle & \langle u'w' \rangle \\ \langle u'v' \rangle & \langle v'v' \rangle & \langle v'w' \rangle \\ \langle u'w' \rangle & \langle v'w' \rangle & \langle w'w' \rangle \end{pmatrix}. \tag{1.17}$$

Now, we would like to write the Reynolds tensor in terms of the large-scale velocities. One way is to use an analogy with the stress tensor of the classical Newtonian fluid and assume that the Reynolds stresses are dependent linearly on the spatial derivatives of the large-scale velocities. This idea is so-called Boussinesq hypothesis. In this case, we can write

$$\frac{\sigma_{ij}}{\rho} = A_j \frac{\partial v_i}{\partial x_j} + A_i \frac{\partial v_j}{\partial x_i}, \tag{1.18}$$

where we dropped the $\langle\rangle$. A_i are the coefficients of the eddy viscosity. The eddy viscosity is not the property of the fluid as the molecular viscosity, but the property of the flow. This is the reason why we are not able to compute, or empirically determine the eddy viscosity in contrast with the molecular viscosity. Equation (1.18) is only a hypothesis with no a priori justification. The values of the eddy viscosity coefficients can be roughly estimate from the Reynolds number based on the eddy viscosity which we can suppose to be in order of one, or we can estimate the values by the comparison of synthetic data and measured data. Moreover, there is no reason why A_i should be identical, indeed the horizontal and the vertical eddy viscosities are usually distinct. The eddy viscosity can be spatially dependent as well, and probably is in most cases, however we usually do not know anything about this dependency.

Equation (1.18) is written in the Cartesian coordinates, however we can write it in an invariant form as well

$$\frac{\sigma_{ij}}{\rho} = A_{iljk} (\nabla \vec{v})_{lk} + A_{jkil} (\nabla \vec{v})_{kl}, \tag{1.19}$$

where the tensor A_{iljk} is defined

$$A_{iljk} \equiv A_i \delta_{il} \delta_{jk}, \tag{1.20}$$

where δ_{il} is the Kronecker delta

In ocean modelling, we usually distinguish the horizontal eddy viscosity A_H and the vertical eddy viscosity A_V. Then the Reynolds tensor takes the form (following Pedlosky, 1987)

$$\frac{\boldsymbol{\sigma}}{\rho} = \begin{pmatrix} 2A_H\dfrac{\partial u}{\partial x} & A_H\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & A_V\dfrac{\partial u}{\partial z}+A_H\dfrac{\partial w}{\partial x} \\ A_H\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & 2A_H\dfrac{\partial v}{\partial y} & A_V\dfrac{\partial v}{\partial z}+A_H\dfrac{\partial w}{\partial x} \\ A_V\dfrac{\partial u}{\partial z}+A_H\dfrac{\partial w}{\partial x} & A_V\dfrac{\partial v}{\partial z}+A_H\dfrac{\partial w}{\partial x} & 2A_V\dfrac{\partial w}{\partial z} \end{pmatrix}. \tag{1.21}$$

As already mentioned, it is difficult to determine the values of the eddy viscosity coefficients. Moreover, the values can vary enormously. Estimates of A_V for an ocean are $10^{-5}-10^{-1}$ m^2/s and estimates of A_H are $10-10^5$ m^2/s (Pond and Pickard, 1983). When we compare these values with the kinematic viscosity of water which is $\nu=\eta/\rho\approx 10^{-6}$ m^2/s, we see that the eddy viscosity is several orders of magnitude greater.

1.3 The Navier-Stokes system and the boundary conditions

Considering the Reynolds tensor (1.21) in the Navier-Stokes equations (1.7)-(1.9) we can write

$$\nabla\cdot\vec{v}=0, \tag{1.22}$$

$$\begin{aligned}\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z} = &-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+\frac{\partial}{\partial x}\left(2A_H\frac{\partial u}{\partial x}\right)+\\ &\frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}+A_H\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(A_V\frac{\partial u}{\partial z}+A_H\frac{\partial w}{\partial x}\right),\end{aligned} \tag{1.23}$$

$$\begin{aligned}\frac{\partial v}{\partial t}+\frac{\partial (vu)}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial (vw)}{\partial z} = &-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu+\frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial y}+A_H\frac{\partial v}{\partial x}\right)+\\ &\frac{\partial}{\partial y}\left(2A_H\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(A_V\frac{\partial v}{\partial z}+A_H\frac{\partial w}{\partial y}\right),\end{aligned} \tag{1.24}$$

$$\begin{aligned}\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial x}+\frac{\partial (wv)}{\partial y}+\frac{\partial w^2}{\partial z} = &-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+\frac{\partial}{\partial x}\left(A_V\frac{\partial u}{\partial z}+A_H\frac{\partial w}{\partial x}\right)+\\ &\frac{\partial}{\partial y}\left(A_V\frac{\partial v}{\partial z}+A_H\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(2A_V\frac{\partial w}{\partial z}\right).\end{aligned} \tag{1.25}$$

The Navier-Stokes equations must be completed with the boundary conditions. In an ocean we have three boundaries, the coast, the ocean bottom and the free surface. At the coast we prescribe the normal flow condition which means that no fluid can cross the boundary

$$\vec{v}\cdot\vec{n}_c=0, \tag{1.26}$$

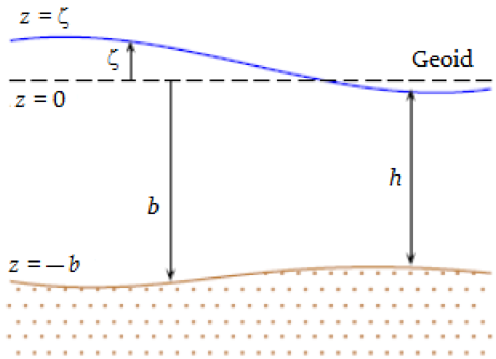


Figure 1.1: Water layer with varying bathymetry b and moving elevation ζ.

where $\vec{n}_c$ is the normal to the coast.

At the bottom and on the free surface we assume no fluid can cross the boundaries as well. We introduce new quantities, the elevation of the surface $\zeta(x, y, t)$, measured from the geoid to the surface, the bathymetry $b(x, y)$, measured from the geoid to the bottom, and the height of water column $h(x, y, t) = \zeta + b$ (see Figure 1.1). Since no fluid crosses the free surface, the vertical velocity of the free surface is equal to the material time derivative of the surface elevation

$$w = \frac{\partial \zeta}{\partial t} + \vec{v} \cdot \nabla \zeta. \tag{1.27}$$

The second condition is the stress condition. We assume no influence of wind or atmospheric pressure which means

$$\mathbf{t} \cdot \vec{n}_s = \vec{0}, \tag{1.28}$$

where $\vec{n}_s$ is the outward normal on the free surface which is defined by

$$\vec{n}_s = \frac{1}{\sqrt{1 + |\nabla \zeta|^2}} \begin{pmatrix} -\frac{\partial \zeta}{\partial x} \\ -\frac{\partial \zeta}{\partial y} \\ 1 \end{pmatrix}. \tag{1.29}$$

The situation at the bottom is analogous. The velocity condition is

$$w = -\vec{v} \cdot \nabla b. \tag{1.30}$$

The stress condition is a little bit different than on the free surface. Only the tangent components of the stress vector are taken in account. Moreover, we shall consider the bottom friction which means that the stress condition takes the form

$$\frac{1}{\rho} (\boldsymbol{\sigma} \cdot \vec{n}_b)_{tg} + \vec{s}_{fr} = \vec{0}, \tag{1.31}$$

where $\vec{n}_b$ is the outward normal on the bottom which is

$$\vec{n}_b = \frac{1}{\sqrt{1+|\nabla b|^2}} \begin{pmatrix} -\frac{\partial b}{\partial x} \\ -\frac{\partial b}{\partial y} \\ -1 \end{pmatrix} \tag{1.32}$$

and $\vec{s}_{fr}$ is the bottom stress vector. The form of the bottom stress vector is suggested by empirical laws. A linear form is used for laminar flow (e.g. Gerbeau and Perthame, 2000), while in the case of turbulent flow a quadratic form is often used (e.g. Weis, 2006). Some authors use both models (e.g. Bresch and Desjardins, 2003, or Marche, 2006). Since the flow is turbulent in our case, we assume that the bottom stress vector has the quadratic form $\vec{s}_{fr} = k\vec{v}|\vec{v}|$ with dimensionless coefficient k.

1.4 The nondimensionalized system

In this section we introduce the dimensionless quantities to write the equations in a dimensionless form. Then we will be able to neglect small terms in the equations.

Let us consider characteristic scales of geometry and velocity: H for the vertical dimension, L for the horizontal dimension and U for the horizontal velocity. Then we introduce a "smal" parameter $\epsilon = H/L$ and the derived scales: ϵU for the vertical velocity, U^2 for the pressure over the density and L/U for the time.

Now we can define the following dimensionless quantities

$$\begin{aligned} \tilde{x} &= \frac{x}{L}, & \tilde{y} &= \frac{y}{L}, & \tilde{z} &= \frac{z}{L}, \\ \tilde{\zeta} &= \frac{\zeta}{H}, & \tilde{b} &= \frac{b}{H}, & \tilde{h} &= \frac{h}{H}, \\ \tilde{u} &= \frac{u}{U}, & \tilde{v} &= \frac{v}{U}, & \tilde{w} &= \frac{w}{\epsilon U}, \\ \tilde{t} &= \frac{tU}{L}, & \frac{\tilde{p}}{\tilde{\rho}} &= \frac{p}{\rho U^2}. \end{aligned}$$

Furthermore, we define the modified friction coefficient $k_0 = k/\epsilon$ and some dimensionless numbers, the horizontal and vertical inverse Reynolds numbers ν_H and ν_V, respectively, the Froude number F_r and the Rossby number R_o, which are defined by

$$\begin{aligned} \nu_H &= \frac{A_H}{UL}, & \nu_V &= \frac{A_V}{UL}, \\ F_r &= \frac{U}{\sqrt{gH}}, & R_o &= \frac{U}{fL}. \end{aligned}$$

For the sake of simplicity we drop the ˜ and thus we can write the dimensionless Navier-Stokes equations. The continuity equation remains unchanged

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.33}$$

The equation of motion becomes

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{v}{R_o}+\frac{\partial}{\partial x}\left(2\nu_H\frac{\partial u}{\partial x}\right)+ \\ \frac{\partial}{\partial y}\left(\nu_H\frac{\partial u}{\partial y}+\nu_H\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}\right), \tag{1.34}$$

$$\frac{\partial v}{\partial t}+\frac{\partial (vu)}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial (vw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{u}{R_o}+\frac{\partial}{\partial x}\left(\nu_H\frac{\partial u}{\partial y}+\nu_H\frac{\partial v}{\partial x}\right)+ \\ \frac{\partial}{\partial y}\left(2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}\right), \tag{1.35}$$

$$\epsilon^2\left(\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial x}+\frac{\partial (wv)}{\partial y}+\frac{\partial w^2}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{1}{F_r^2}+\frac{\partial}{\partial x}\left(\nu_V\frac{\partial u}{\partial z}+\epsilon^2\nu_H\frac{\partial w}{\partial x}\right)+ \\ \frac{\partial}{\partial y}\left(\nu_V\frac{\partial v}{\partial z}+\epsilon^2\nu_H\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(2\nu_V\frac{\partial w}{\partial z}\right). \tag{1.36}$$

We rewrite the boundary conditions in the dimensionless form as well. The velocity condition (1.27) remains in same form

$$w=\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}, \tag{1.37}$$

and the stress condition (1.28) becomes

$$-\frac{\partial \zeta}{\partial x}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial u}{\partial x}\right)-\frac{\partial \zeta}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}=0, \tag{1.38}$$

$$-\frac{\partial \zeta}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-\frac{\partial \zeta}{\partial y}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial y}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}=0, \tag{1.39}$$

$$-\epsilon^2\left(\frac{\partial \zeta}{\partial x}\left(\nu_H\frac{\partial w}{\partial x}+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}\right)+\frac{\partial \zeta}{\partial y}\left(\nu_H\frac{\partial w}{\partial y}+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}\right)\right)+2\nu_V\frac{\partial w}{\partial z}-\frac{p}{\rho}=0. \tag{1.40}$$

Analogous situation is at the bottom. The velocity condition (1.30) is

$$w=-u\frac{\partial b}{\partial x}-v\frac{\partial b}{\partial y} \tag{1.41}$$

and the stress condition (1.31) is

$$2\nu_H\frac{\partial b}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial b}{\partial y}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\nu_H\frac{\partial w}{\partial x}=Nk_0u|\vec{v}|, \tag{1.42}$$

$$\frac{\partial b}{\partial x}\nu_H\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+2\nu_H\frac{\partial b}{\partial y}\frac{\partial v}{\partial y}+\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial y}=Nk_0v|\vec{v}|, \tag{1.43}$$

(1.44)

where N is the norm of the outward normal $N = \sqrt{1 + \epsilon^2|\nabla b|^2}$ and $|\vec{v}|$ is the norm of the velocity vector $|\vec{v}| = \sqrt{u^2 + v^2 + \epsilon^2 w^2}$.

In the first step of derivation the hydrostatic approximation is applied. The terms of the order $O(\epsilon^2)$ are neglected. This includes also the vertical inverse Reynolds number since it can be roughly estimated as $\epsilon^2\nu_H$ (Pond and Pickard, 1983). However, the terms with ν_H can not be neglected since they are of the order $O(1)$. In this approximation, the vertical equation of motions (1.36) reduces to the equation of hydrostatic balance

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{F_r^2} \tag{1.45}$$

and the vertical boundary conditions on the free surface (1.40) reduces to

$$\frac{p}{\rho} = 0. \tag{1.46}$$

Likewise, the norm of the outward normal at the bottom and the norm of the velocity vector simplify as

$$N = 1, \tag{1.47}$$

$$|\vec{v}| = \sqrt{u^2 + v^2}. \tag{1.48}$$

The other equations remain unchanged.

1.5 Vertical averaging

Now we would like to get rid of the vertical dimension in the equations and simplify the 3-D system to a 2-D problem.

Let us define the vertically averaged quantity for a function $f(t, x, y, z)$

$$\bar{f}(t, x, y) = \frac{1}{h(t, x, y)}\int_{-b}^{\zeta} f(t, x, y, z)\,\mathrm{d}z. \tag{1.49}$$

Now we will integrate the nondimensionalized Navier-Stokes equations from the bottom to the surface. Then we will use the Leibniz integral rule and apply the boundary conditions. Let us integrate the continuity equation (1.33)

$$\begin{aligned}
0 = &\int_{-b}^{\zeta} \nabla\cdot\vec{v}\,\mathrm{d}z = \\
= &\int_{-b}^{\zeta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathrm{d}z + w|_{z=\zeta} - w|_{z=-b} = \\
= &\frac{\partial}{\partial x}\int_{-b}^{\zeta} u\,\mathrm{d}z + \frac{\partial}{\partial y}\int_{-b}^{\zeta} v\,\mathrm{d}z - u|_{z=\zeta}\frac{\partial \zeta}{\partial x} - u|_{z=\zeta}\frac{\partial b}{\partial x} - \\
&v|_{z=-b}\frac{\partial \zeta}{\partial y} - v|_{z=-b}\frac{\partial b}{\partial y} + w|_{z=\zeta} - w|_{z=-b}.
\end{aligned} \tag{1.50}$$

After applying the boundary conditions (1.37) and (1.41) we obtain the equation of the water elevation
$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0. \tag{1.51}$$
The horizontal components of the equation of motion (1.34) and (1.35) are modified in analogous way. Besides the velocity boundary conditions, we apply also the stress boundary conditions (1.38), (1.39), (1.42) and (1.43) which yields

$$\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial (h\overline{u^2})}{\partial x} + \frac{\partial (h\overline{uv})}{\partial y} = -\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial x} + \frac{1}{\rho}\frac{\partial b}{\partial x} p|_{z=-b} - k_0 u\sqrt{u^2+v^2}|_{z=-b} + \frac{h\bar{v}}{R_o} + 2\nu_H \frac{\partial}{\partial x}\left(h\overline{\frac{\partial u}{\partial x}}\right) + \nu_H \frac{\partial}{\partial y}\left(h\overline{\frac{\partial u}{\partial y}} + h\overline{\frac{\partial v}{\partial x}}\right), \tag{1.52}$$

$$\frac{\partial (h\bar{v})}{\partial t} + \frac{\partial (h\overline{uv})}{\partial x} + \frac{\partial (h\overline{v^2})}{\partial y} = -\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial y} + \frac{1}{\rho}\frac{\partial b}{\partial y} p|_{z=-b} - k_0 v\sqrt{u^2+v^2}|_{z=-b} - \frac{h\bar{u}}{R_o} + \nu_H \frac{\partial}{\partial x}\left(h\overline{\frac{\partial u}{\partial y}} + h\overline{\frac{\partial v}{\partial x}}\right) + 2\nu_H \frac{\partial}{\partial y}\left(h\overline{\frac{\partial v}{\partial y}}\right). \tag{1.53}$$

The pressure terms in the equations above are given by the equation of hydrostatic balance (1.45) which implies

$$p(t,x,y,z) = -\frac{\rho}{F_r^2}(z - \zeta(t,x,y,)). \tag{1.54}$$

Hence, integrating with respect to z from $-b$ to ζ, and dividing by h we obtain the vertically averaged pressure

$$\bar{p}(t,x,y) = \frac{\rho h^2}{2F_r^2}. \tag{1.55}$$

Additionally, the pressure on the bottom is

$$p(t,x,y,z=-b) = \frac{\rho h}{F_r^2}. \tag{1.56}$$

Therefore, it holds

$$-\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial x} + \frac{1}{\rho}\frac{\partial b}{\partial x} p|_{z=-b} = -\frac{h}{F_r^2}\frac{\partial \zeta}{\partial x}, \tag{1.57}$$
$$-\frac{1}{\rho}\frac{\partial (h\bar{p})}{\partial y} + \frac{1}{\rho}\frac{\partial b}{\partial y} p|_{z=-b} = -\frac{h}{F_r^2}\frac{\partial \zeta}{\partial y}. \tag{1.58}$$

In the next step we will assume that the horizontal flow is weakly dependent on the depth and the derivatives of the velocities are small quantities of the order $O(\epsilon)$. After the hydrostatic approximation this is the second assumption of the shallow water approximation:

$$\frac{\partial u}{\partial z}(t,x,y,z) = O(\epsilon), \qquad \frac{\partial v}{\partial z}(t,x,y,z) = O(\epsilon). \tag{1.59}$$

Therefore, we can write

$$u(t,x,y,z) = \bar{u}(t,x,y) + O(\epsilon), \qquad v(t,x,y,z) = \bar{v}(t,x,y) + O(\epsilon), \tag{1.60}$$

which implies that

$$\overline{u^2} = \bar{u}^2 + O(\epsilon), \qquad \overline{v^2} = \bar{v}^2 + O(\epsilon), \qquad \overline{uv} = \bar{u}\bar{v} + O(\epsilon). \tag{1.61}$$

The second implication is

$$\overline{\frac{\partial u}{\partial y}} = \frac{\partial \bar{u}}{\partial y} + O(\epsilon), \qquad \overline{\frac{\partial v}{\partial x}} = \frac{\partial \bar{v}}{\partial x} + O(\epsilon), \qquad \overline{\frac{\partial u}{\partial y}} + \overline{\frac{\partial v}{\partial x}} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} + O(\epsilon). \tag{1.62}$$

Substituting (1.60), (1.61) and (1.62) in (1.52) and (1.53) we obtain the nondimensionalized system of the shallow water equation with a precision of $O(\epsilon)$:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0, \tag{1.63}$$

$$\begin{aligned}\frac{\partial (h\bar{u})}{\partial t} + \frac{\partial (h\bar{u}^2)}{\partial x} + \frac{\partial (h\bar{u}\bar{v})}{\partial y} =& - \frac{h}{F_r^2}\frac{\partial \zeta}{\partial x} - k_0\bar{u}\sqrt{\bar{u}^2+\bar{v}^2} + \frac{h\bar{v}}{R_o} + \\ & 2\nu_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + \nu_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right),\end{aligned} \tag{1.64}$$

$$\begin{aligned}\frac{\partial (h\bar{v})}{\partial t} + \frac{\partial (h\bar{u}\bar{v})}{\partial x} + \frac{\partial (h\bar{v}^2)}{\partial y} =& - \frac{h}{F_r^2}\frac{\partial \zeta}{\partial y} - k_0\bar{v}\sqrt{\bar{u}^2+\bar{v}^2} - \frac{h\bar{u}}{R_o} + \\ & \nu_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right) + 2\nu_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right).\end{aligned} \tag{1.65}$$

The shallow water equations are now written in the dimensionless form. However, it is much more practical to use the physical quantities than the dimensionless quantities. We will use the relations from the beginning of Section 1.4 and apply them to the dimensionless shallow water equations (note that the dimensionless quantities were denoted by the ˜, however we dropped it in all equations for the sake of simplicity). Moreover, let us introduce depth mean transport $U = h\bar{u}$ and $V = h\bar{v}$ since for numerical purposes it is convenient to use these quantities than the velocity components. Then the shallow water equation in the local Cartesian coordinates are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{1.66}$$

$$\begin{aligned}\frac{\partial U}{\partial t} + \frac{\partial U\bar{u}}{\partial x} + \frac{\partial U\bar{v}}{\partial y} =& - gh\frac{\partial \zeta}{\partial x} - \frac{k}{h^2}U\sqrt{U^2+V^2} + fV + \\ & 2A_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial x}\right) + A_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right),\end{aligned} \tag{1.67}$$

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V\bar{u}}{\partial x} + \frac{\partial V\bar{v}}{\partial y} =& - gh\frac{\partial \zeta}{\partial y} - \frac{k}{h^2}V\sqrt{U^2+V^2} - fU + \\ & A_H \frac{\partial}{\partial x}\left(h\frac{\partial \bar{u}}{\partial y} + h\frac{\partial \bar{v}}{\partial x}\right) + 2A_H \frac{\partial}{\partial y}\left(h\frac{\partial \bar{v}}{\partial y}\right).\end{aligned} \tag{1.68}$$

1.6 Derivation in the geographical coordinates

1.6.1 Spherical approximation

Up to now we used the Cartesian coordinates. The shallow water equations in the local Cartesian coordinates can be used for shelf seas where the curvature of the Earth's surface can be neglected. However, when we are interested in global ocean modelling, we need to use the spherical or geographical coordinates. Unfortunately, there is no way of rewriting the shallow water equations from the local Cartesian coordinates to the geographical coordinates which would be valid on the whole sphere. Hence, we need to do the whole derivation from the beginning. The procedure is analogous as in the case of the Cartesian coordinates with one exceptions.

Before we write the equations in the geographical coordinates, we introduce first approximation. The radial coordinate r can be expressed as

$$r = a + z = a\left(1 + \frac{z}{a}\right), \tag{1.69}$$

where a is the mean radius of the Earth and z is new radial coordinate, $z \in (-b, \zeta)$. Since $z \approx H = 10^3$ m and $a \approx 6 \times 10^6$ m, the term $\frac{z}{a}$ is of the order $O(\epsilon)$ and can be neglected if compared with terms $O(1)$. Therefore we shall use the so-called spherical approximation

$$r = a. \tag{1.70}$$

Equation (1.69) implies

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial z}. \tag{1.71}$$

With this approximation we can write following relations where ϕ is the latitude, λ the longitude, u the longitudinal velocity, v the latitudinal velocity and w the radial velocity.

1.6.2 The Navier-Stokes system

We start with the Navier-Stokes equations in the invariant form (1.3 and 1.5). The expressions for the invariant differential operators in the geographical coordinates can be found in Appendix A. The formulae below are written already in spherical approximation (1.69) and (1.71).

The continuity equation in geographical coordinates reads

$$\begin{aligned} 0 &= \nabla \cdot \vec{v} = \\ &= \frac{\partial w}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi v)}{\partial\phi} + \frac{\partial u}{\partial\lambda}\right). \end{aligned} \tag{1.72}$$

The advection term in the momentum equation reads

$$\nabla \cdot (\vec{v} \otimes \vec{v}) = \left[\frac{\partial w^2}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, vw)}{\partial\phi} + \frac{\partial(uw)}{\partial\lambda}\right) - \frac{1}{a}\left(v^2 + u^2\right)\right]\vec{e}_z +$$
$$\left[\frac{\partial(vw)}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, v^2)}{\partial\phi} + \frac{\partial(uv)}{\partial\lambda}\right) + \frac{1}{a}\left(vw + \tan\phi\, u^2\right)\right]\vec{e}_\phi +$$
$$\left[\frac{\partial(uw)}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, uv)}{\partial\phi} + \frac{\partial v^2}{\partial\lambda}\right) + \frac{1}{a}\left(uw - \tan\phi\, uv\right)\right]\vec{e}_\lambda. \tag{1.73}$$

The gradient of the pressure is

$$\nabla p = \frac{\partial p}{\partial z}\vec{e}_z + \frac{1}{a}\frac{\partial p}{\partial\phi}\vec{e}_\phi + \frac{1}{a\cos\phi}\frac{\partial p}{\partial\lambda}\vec{e}_\lambda. \tag{1.74}$$

The Reynolds tensor in the geographical coordinates can be formulated by the definition (1.19)

$$\begin{aligned}\frac{\boldsymbol{\sigma}}{\rho} = {} & 2A_V\frac{\partial w}{\partial z}\vec{e}_z \otimes \vec{e}_z + \\
& 2\frac{A_H}{a}\left(\frac{\partial v}{\partial\phi} + w\right)\vec{e}_\phi \otimes \vec{e}_\phi + \\
& 2\frac{A_H}{a}\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda} + w - \tan\phi\, v\right)\vec{e}_\lambda \otimes \vec{e}_\lambda + \\
& \left[A_V\frac{\partial v}{\partial z} + \frac{A_H}{a}\left(\frac{\partial w}{\partial\phi} - v\right)\right](\vec{e}_z \otimes \vec{e}_\phi + \vec{e}_\phi \otimes \vec{e}_z) + \\
& \left[A_V\frac{\partial u}{\partial z} + \frac{A_H}{a}\left(\frac{1}{\cos\phi}\frac{\partial w}{\partial\lambda} - u\right)\right](\vec{e}_z \otimes \vec{e}_\lambda + \vec{e}_\lambda \otimes \vec{e}_z) + \\
& \frac{A_H}{a}\left(\frac{\partial u}{\partial\phi} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \tan\phi\, u\right)(\vec{e}_\phi \otimes \vec{e}_\lambda + \vec{e}_\lambda \otimes \vec{e}_\phi).\end{aligned} \tag{1.75}$$

Finally, the divergence of the Reynolds tensor is

$$\nabla \cdot \boldsymbol{\sigma} = \left[\frac{\partial\sigma_{zz}}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi z})}{\partial\phi} + \frac{\partial\sigma_{\lambda z}}{\partial\lambda}\right) - \frac{1}{a}\left(\sigma_{\phi\phi} + \sigma_{\lambda\lambda}\right)\right]\vec{e}_z +$$
$$\left[\frac{\partial\sigma_{z\phi}}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi\phi})}{\partial\phi} + \frac{\partial\sigma_{\lambda\phi}}{\partial\lambda}\right) + \frac{1}{a}\left(\sigma_{\phi z} + \tan\phi\,\sigma_{\lambda\lambda}\right)\right]\vec{e}_\phi +$$
$$\left[\frac{\partial\sigma_{z\lambda}}{\partial z} + \frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi\lambda})}{\partial\phi} + \frac{\partial\sigma_{\lambda\lambda}}{\partial\lambda}\right) + \frac{1}{a}\left(\sigma_{\lambda z} - \tan\phi\,\sigma_{\lambda\phi}\right)\right]\vec{e}_\lambda. \tag{1.76}$$

Therefore, the momentum equation in the longitudinal direction is

$$\frac{\partial u}{\partial t}+\frac{\partial(uw)}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, uv)}{\partial\phi}+\frac{\partial v^2}{\partial\lambda}\right)+\frac{1}{a}\left(uw-\tan\phi\, uv\right)=fv+$$
$$\frac{1}{\rho}\left[-\frac{1}{a\cos\phi}\frac{\partial p}{\partial\lambda}+\frac{\partial\sigma_{z\lambda}}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi\lambda})}{\partial\phi}+\frac{\partial\sigma_{\lambda\lambda}}{\partial\lambda}\right)+\frac{1}{a}\left(\sigma_{\lambda z}-\tan\phi\,\sigma_{\lambda\phi}\right)\right], \tag{1.77}$$

the momentum equation in the latitudinal direction is

$$\frac{\partial v}{\partial t}+\frac{\partial(vw)}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, v^2)}{\partial\phi}+\frac{\partial(uv)}{\partial\lambda}\right)+\frac{1}{a}\left(vw+\tan\phi\, u^2\right)=-fu+$$
$$\frac{1}{\rho}\left[-\frac{1}{a}\frac{\partial p}{\partial\phi}+\frac{\partial\sigma_{z\phi}}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi\phi})}{\partial\phi}+\frac{\partial\sigma_{\lambda\phi}}{\partial\lambda}\right)+\frac{1}{a}\left(\sigma_{\phi z}+\tan\phi\,\sigma_{\lambda\lambda}\right)\right] \tag{1.78}$$

and the momentum equation in the radial direction is

$$\frac{\partial w}{\partial t}+\frac{\partial w^2}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\, vw)}{\partial\phi}+\frac{\partial(uw)}{\partial\lambda}\right)-\frac{1}{a}\left(v^2+u^2\right)=-g+$$
$$\frac{1}{\rho}\left[-\frac{\partial p}{\partial z}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{a\cos\phi}\left(\frac{\partial(\cos\phi\,\sigma_{\phi z})}{\partial\phi}+\frac{\partial\sigma_{\lambda z}}{\partial\lambda}\right)-\frac{1}{a}\left(\sigma_{\phi\phi}+\sigma_{\lambda\lambda}\right)\right]. \tag{1.79}$$

We should also mention the boundary conditions in the geographical coordinates. The velocity boundary conditions (1.27) and (1.30) read

$$w|_{z=\zeta}=\frac{\partial\zeta}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial\zeta}{\partial\lambda}+\frac{v}{a}\frac{\partial\zeta}{\partial\phi}, \tag{1.80}$$
$$w|_{z=-b}=-\frac{u}{a\cos\phi}\frac{\partial b}{\partial\lambda}-\frac{v}{a}\frac{\partial b}{\partial\phi}. \tag{1.81}$$

The outward normals on the boundaries (1.29) and (1.32) read

$$\vec{n}_s=\frac{1}{\sqrt{1+|\nabla\zeta|^2}}\left(-\frac{u}{a\cos\phi}\frac{\partial\zeta}{\partial\lambda}\vec{e}_\lambda-\frac{v}{a}\frac{\partial\zeta}{\partial\phi}\vec{e}_\phi+\vec{e}_z\right), \tag{1.82}$$
$$\vec{n}_b=\frac{1}{\sqrt{1+|\nabla b|^2}}\left(-\frac{u}{a\cos\phi}\frac{\partial b}{\partial\lambda}\vec{e}_\lambda-\frac{v}{a}\frac{\partial b}{\partial\phi}\vec{e}_\phi-\vec{e}_z\right). \tag{1.83}$$

1.6.3 The nondimensionalized system

Now, we introduce the same dimensionless quantities as in Section 1.4. Furthermore, we introduce new "nondimensionalized" latitude $\tilde{\phi}=a\phi/L$ and longitude $\tilde{\lambda}=a\lambda/L$ and a parameter $\gamma=L/a$. The reason for introducing the new scaled coordinates, which might look senseless at first sight, is simplifying of forms of equations. The parameter γ is of the order $O(1)$ since we consider L to be approximately 10^6 m. We will write the dimensionless Navier-Stokes equations in the geographical coordinates and drop the ˜ for the sake of simplicity. However, we will distinguish between the original latitude ϕ, which we use as the argument of the trigonometric functions, and the scaled latitude $\tilde{\phi}$, which we use for scaling of derivatives. Then we will neglect the small terms which are of the order $O(\epsilon)$.

The continuity equation (1.72) in the dimensionless quantities is

$$\frac{\partial w}{\partial z} + \frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, v)}{\partial\tilde{\phi}} + \frac{\partial u}{\partial\lambda}\right) = 0. \tag{1.84}$$

The longitudinal momentum equation (1.77) reads

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial(uw)}{\partial z} + \frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, uv)}{\partial\tilde{\phi}} + \frac{\partial v^2}{\partial\lambda}\right) + \gamma u\,(\epsilon w - \tan\phi\, v) = \\
&= -\frac{1}{\rho\cos\phi}\frac{\partial p}{\partial\lambda} + \frac{v}{R_o} + \frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z} + \frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial\lambda} - \frac{\nu_H\gamma}{\epsilon}u\right) + \\
&\quad \frac{\nu_H}{\cos\phi}\left[\frac{\partial}{\partial\tilde{\phi}}\left(\cos\phi\left(\frac{\partial u}{\partial\tilde{\phi}} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \gamma\tan\phi\, u\right)\right) + \right.\\
&\quad \left. 2\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda} + \epsilon\gamma w - \gamma\tan\phi\, v\right)\right] + \\
&\quad \frac{\nu_V\gamma}{\epsilon}\frac{\partial u}{\partial z} + \nu_H\gamma\left(\frac{\epsilon}{\cos\phi}\frac{\partial w}{\partial\lambda} - \gamma u - \tan\phi\left(\frac{\partial u}{\partial\tilde{\phi}} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \gamma\tan\phi\, u\right)\right),
\end{aligned} \tag{1.85}$$

after neglecting the $O(\epsilon)$ terms

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial(uw)}{\partial z} + \frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, uv)}{\partial\tilde{\phi}} + \frac{\partial v^2}{\partial\lambda}\right) - \gamma\tan\phi\, uv = \\
&= -\frac{1}{\rho\cos\phi}\frac{\partial p}{\partial\lambda} + \frac{v}{R_o} + \frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z} + \frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial\lambda} - \frac{\nu_H\gamma}{\epsilon}u\right) + \\
&\quad \frac{\nu_H}{\cos\phi}\left[\frac{\partial}{\partial\tilde{\phi}}\left(\cos\phi\left(\frac{\partial u}{\partial\tilde{\phi}} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \gamma\tan\phi\, u\right)\right) + 2\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda} - \gamma\tan\phi\, v\right)\right] - \\
&\quad \nu_H\gamma\left(\gamma u + \tan\phi\left(\frac{\partial u}{\partial\tilde{\phi}} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \gamma\tan\phi\, u\right)\right).
\end{aligned} \tag{1.86}$$

The latitudinal momentum equation (1.78) reads

$$\begin{aligned}
&\frac{\partial v}{\partial t} + \frac{\partial(vw)}{\partial z} + \frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, v^2)}{\partial\tilde{\phi}} + \frac{\partial(uv)}{\partial\lambda}\right) + \gamma(\epsilon vw + \tan\phi\, u^2) = \\
&= -\frac{1}{\rho}\frac{\partial p}{\partial\tilde{\phi}} - \frac{u}{R_o} + \frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z} + \nu_H\frac{\partial w}{\partial\tilde{\phi}} - \frac{\nu_H\gamma}{\epsilon}v\right) + \\
&\quad \frac{\nu_H}{\cos\phi}\left[2\frac{\partial}{\partial\tilde{\phi}}\left(\cos\phi\left(\frac{\partial v}{\partial\tilde{\phi}} + \epsilon\gamma w\right)\right) + \frac{\partial}{\partial\lambda}\left(\frac{\partial u}{\partial\tilde{\phi}} + \frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda} + \gamma\tan\phi\, u\right)\right] + \\
&\quad \frac{\nu_V\gamma}{\epsilon}\frac{\partial v}{\partial z} + \nu_H\gamma\left(\epsilon\frac{\partial w}{\partial\tilde{\phi}} - \gamma v + 2\tan\phi\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda} + \epsilon\gamma w + \gamma\tan\phi\, v\right)\right),
\end{aligned} \tag{1.87}$$

after neglecting the $O(\epsilon)$ terms

$$\frac{\partial v}{\partial t}+\frac{\partial (vw)}{\partial z}+\frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, v^2)}{\partial\tilde{\phi}}+\frac{\partial(uv)}{\partial\lambda}\right)+\gamma\tan\phi\, u^2=$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial\tilde{\phi}}-\frac{u}{R_o}+\frac{\partial}{\partial z}\left(\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial\tilde{\phi}}-\frac{\nu_H\gamma}{\epsilon}v\right)+$$
$$\frac{\nu_H}{\cos\phi}\left[2\frac{\partial}{\partial\tilde{\phi}}\left(\cos\phi\frac{\partial v}{\partial\tilde{\phi}}\right)+\frac{\partial}{\partial\lambda}\left(\frac{\partial u}{\partial\tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda}+\gamma\tan\phi\, u\right)\right]+$$
$$\nu_H\gamma\left(-\gamma v+2\tan\phi\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda}+\gamma\tan\phi\, v\right)\right). \tag{1.88}$$

And the radial momentum equation (1.79) reads

$$\epsilon^2\left(\frac{\partial w}{\partial t}+\frac{\partial w^2}{\partial z}+\frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, vw)}{\partial\tilde{\phi}}+\frac{\partial(uw)}{\partial\lambda}\right)\right)-\epsilon\gamma(v^2+u^2)=$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{1}{F_r^2}+\frac{\partial}{\partial z}\left(2\nu_V\frac{\partial w}{\partial z}\right)+$$
$$\frac{1}{\cos\phi}\left[\frac{\partial}{\partial\tilde{\phi}}\left(\cos\phi\left(\nu_V\frac{\partial v}{\partial z}+\nu_H\epsilon^2\frac{\partial w}{\partial\tilde{\phi}}-\nu_H\epsilon\gamma v\right)\right)+\right.$$
$$\left.\frac{\partial}{\partial\lambda}\left(\nu_V\frac{\partial u}{\partial z}+\frac{\nu_H\epsilon^2}{\cos\phi}\frac{\partial w}{\partial\lambda}-\nu_H\epsilon\gamma u\right)\right]-$$
$$2\nu_H\epsilon\gamma\left(\frac{\partial v}{\partial\tilde{\phi}}+\epsilon\gamma w+\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda}+\epsilon\gamma w-\gamma\tan\phi\, v\right), \tag{1.89}$$

after neglecting the $O(\epsilon)$ terms we obtain the equation of hydrostatic balance (1.45)

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-\frac{1}{F_r^2}, \tag{1.90}$$

which implies that

$$\bar{p}(t,x,y)=\frac{\rho h^2}{2F_r^2}, \tag{1.91}$$

$$p(t,x,y,z=-b)=\frac{\rho h}{F_r^2}. \tag{1.92}$$

We write the dimensionless boundary conditions as well. The dimensionless velocity boundary conditions (1.27) and (1.30) read

$$w|_{z=\zeta}=\frac{\partial\zeta}{\partial t}+\frac{u}{\cos\phi}\frac{\partial\zeta}{\partial\lambda}+v\frac{\partial\zeta}{\partial\tilde{\phi}}, \tag{1.93}$$

$$w|_{z=-b}=-\frac{u}{\cos\phi}\frac{\partial b}{\partial\lambda}-v\frac{\partial b}{\partial\tilde{\phi}}. \tag{1.94}$$

The longitudinal component of the surface stress condition (1.28) is

$$\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial\lambda}-\frac{\nu_H\gamma}{\epsilon}u-\nu_H\frac{\partial\zeta}{\partial\tilde{\phi}}\left(\frac{\partial u}{\partial\tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda}+\gamma\tan\phi\, u\right)-$$
$$\frac{1}{\cos\phi}\frac{\partial\zeta}{\partial\lambda}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial\lambda}+\epsilon\gamma w-\gamma\tan\phi\, v\right)\right)=0, \tag{1.95}$$

after neglecting the $O(\epsilon)$ term

$$\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial \lambda}-\frac{\nu_H\gamma}{\epsilon}u-\nu_H\frac{\partial \zeta}{\partial \tilde{\phi}}\left(\frac{\partial u}{\partial \tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial \lambda}+\gamma\tan\phi\, u\right)- \\ \frac{1}{\cos\phi}\frac{\partial \zeta}{\partial \lambda}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial \lambda}-\gamma\tan\phi\, v\right)\right)=0. \tag{1.96}$$

The latitudinal component of the surface stress condition is

$$\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial \tilde{\phi}}-\frac{\nu_H\gamma}{\epsilon}v-\frac{\partial \zeta}{\partial \tilde{\phi}}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{\partial v}{\partial \tilde{\phi}}+\epsilon\gamma w\right)\right)- \\ \frac{\nu_H}{\cos\phi}\frac{\partial \zeta}{\partial \lambda}\left(\frac{\partial u}{\partial \tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial \lambda}+\gamma\tan\phi\, u\right)=0, \tag{1.97}$$

after neglecting the $O(\epsilon)$ term

$$\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial \tilde{\phi}}-\frac{\nu_H\gamma}{\epsilon}v-\frac{\partial \zeta}{\partial \tilde{\phi}}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial \tilde{\phi}}\right)- \\ \frac{\nu_H}{\cos\phi}\frac{\partial \zeta}{\partial \lambda}\left(\frac{\partial u}{\partial \tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial \lambda}+\gamma\tan\phi\, u\right)=0. \tag{1.98}$$

And the radial component of the surface stress condition is

$$-\frac{p}{\rho}+2\nu_V\frac{\partial w}{\partial z}-\frac{\partial \zeta}{\partial \tilde{\phi}}\left(\nu_V\frac{\partial v}{\partial z}+\nu_H\epsilon^2\frac{\partial w}{\partial \tilde{\phi}}-\nu_H\epsilon\gamma v\right)- \\ \frac{1}{\cos\phi}\frac{\partial \zeta}{\partial \lambda}\left(\nu_V\frac{\partial u}{\partial z}+\frac{\nu_H\epsilon^2}{\cos\phi}\frac{\partial w}{\partial \lambda}-\nu_H\epsilon\gamma u\right)=0, \tag{1.99}$$

after neglecting the $O(\epsilon)$ terms

$$-\frac{p}{\rho}=0. \tag{1.100}$$

The treatment of the boundary conditions at the bottom is analogous. The longitudinal component of the bottom stress condition (1.31) is

$$\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial \lambda}-\frac{\nu_H\gamma}{\epsilon}u+\nu_H\frac{\partial b}{\partial \tilde{\phi}}\left(\frac{\partial u}{\partial \tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial \lambda}+\gamma\tan\phi\, u\right)+ \\ \frac{1}{\cos\phi}\frac{\partial b}{\partial \lambda}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial \lambda}+\epsilon\gamma w-\gamma\tan\phi\, v\right)\right)=Nk_0u\sqrt{u^2+v^2+\epsilon^2w^2}, \tag{1.101}$$

where N is the norm of the outward normal $N=\sqrt{1+\epsilon^2|\nabla b|^2}$. After neglecting the $O(\epsilon)$ terms the equation reads

$$\frac{\nu_V}{\epsilon^2}\frac{\partial u}{\partial z}+\frac{\nu_H}{\cos\phi}\frac{\partial w}{\partial \lambda}-\frac{\nu_H\gamma}{\epsilon}u+\nu_H\frac{\partial b}{\partial \tilde{\phi}}\left(\frac{\partial u}{\partial \tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial \lambda}+\gamma\tan\phi\, u\right)+ \\ \frac{1}{\cos\phi}\frac{\partial b}{\partial \lambda}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{1}{\cos\phi}\frac{\partial u}{\partial \lambda}-\gamma\tan\phi\, v\right)\right)=k_0u\sqrt{u^2+v^2}. \tag{1.102}$$

The latitudinal component of the bottom stress condition is

$$\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial\tilde{\phi}}-\frac{\nu_H\gamma}{\epsilon}v+\frac{\partial b}{\partial\tilde{\phi}}\left(-\frac{p}{\rho}+2\nu_H\left(\frac{\partial v}{\partial\tilde{\phi}}+\epsilon\gamma w\right)\right)+$$
$$\frac{\nu_H}{\cos\phi}\frac{\partial b}{\partial\lambda}\left(\frac{\partial u}{\partial\tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda}+\gamma\tan\phi\, u\right)=Nk_0v\sqrt{u^2+v^2+\epsilon^2w^2}, \quad (1.103)$$

after neglecting the $O(\epsilon)$ terms

$$\frac{\nu_V}{\epsilon^2}\frac{\partial v}{\partial z}+\nu_H\frac{\partial w}{\partial\tilde{\phi}}-\frac{\nu_H\gamma}{\epsilon}v+\frac{\partial b}{\partial\tilde{\phi}}\left(-\frac{p}{\rho}+2\nu_H\frac{\partial v}{\partial\tilde{\phi}}\right)+$$
$$\frac{\nu_H}{\cos\phi}\frac{\partial b}{\partial\lambda}\left(\frac{\partial u}{\partial\tilde{\phi}}+\frac{1}{\cos\phi}\frac{\partial v}{\partial\lambda}+\gamma\tan\phi\, u\right)=k_0v\sqrt{u^2+v^2}. \quad (1.104)$$

1.6.4 Vertical averaging

Now we will integrate the nondimensionalized equations in the geographical coordinates over the radial component from $-b$ to ζ. The procedure is the same as in the case of the Cartesian coordinates (see Section 1.5).

From the continuity equation (1.84) we obtain the equation of the water elevation

$$\frac{\partial\zeta}{\partial t}+\frac{1}{\cos\phi}\left(\frac{\partial(h\bar{u})}{\partial\lambda}+\frac{\partial(\cos\phi\, h\bar{v})}{\partial\tilde{\phi}}\right)=0. \quad (1.105)$$

From the longitudinal equation of motion (1.86) we obtain

$$\frac{\partial(h\bar{u})}{\partial t}+\frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, h\overline{uv})}{\partial\tilde{\phi}}+\frac{\partial h\overline{v^2}}{\partial\lambda}\right)-\gamma\tan\phi\, h\overline{uv}=-\frac{h}{F_r^2\cos\phi}\frac{\partial\zeta}{\partial\lambda}+\frac{h\bar{v}}{R_o}-$$
$$k_0u\sqrt{u^2+v^2}|_{z=-b}+\frac{\nu_H}{\cos\phi}\left[\frac{\partial}{\partial\tilde{\phi}}\left(h\cos\phi\left(\overline{\frac{\partial u}{\partial\tilde{\phi}}}+\frac{1}{\cos\phi}\overline{\frac{\partial v}{\partial\lambda}}+\gamma\tan\phi\,\bar{u}\right)\right)+\right.$$
$$\left.2\frac{\partial}{\partial\lambda}\left(\frac{h}{\cos\phi}\overline{\frac{\partial u}{\partial\lambda}}-\gamma\tan\phi\, h\bar{v}\right)\right]-\nu_H\gamma h\left(\gamma\bar{u}+\tan\phi\left(\overline{\frac{\partial u}{\partial\tilde{\phi}}}+\frac{1}{\cos\phi}\overline{\frac{\partial v}{\partial\lambda}}+\gamma\tan\phi\,\bar{u}\right)\right).$$
$$(1.106)$$

And from the latitudinal equation of motion (1.88) we obtain

$$\frac{\partial(h\bar{v})}{\partial t}+\frac{1}{\cos\phi}\left(\frac{\partial(\cos\phi\, h\overline{v^2})}{\partial\tilde{\phi}}+\frac{\partial(h\overline{uv})}{\partial\lambda}\right)+\gamma\tan\phi\, h\overline{u^2}=-\frac{h}{F_r^2}\frac{\partial\zeta}{\partial\tilde{\phi}}-\frac{h\bar{u}}{R_o}-$$
$$k_0v\sqrt{u^2+v^2}|_{z=-b}+\frac{\nu_H}{\cos\phi}\left[2\frac{\partial}{\partial\tilde{\phi}}\left(h\cos\phi\overline{\frac{\partial v}{\partial\tilde{\phi}}}\right)+\frac{\partial}{\partial\lambda}\left(h\overline{\frac{\partial u}{\partial\tilde{\phi}}}+\frac{h}{\cos\phi}\overline{\frac{\partial v}{\partial\lambda}}+\gamma\tan\phi\, h\bar{u}\right)\right]+$$
$$\nu_H\gamma h\left(-\gamma\bar{v}+2\tan\phi\left(\frac{1}{\cos\phi}\overline{\frac{\partial u}{\partial\lambda}}+\gamma\tan\phi\,\bar{v}\right)\right). \quad (1.107)$$

We assume that the radial derivatives of the horizontal velocities are small quantities of the order $O(\epsilon)$ and can be neglected, see (1.59). This implies

$$\overline{u^2} = \bar{u}^2 + O(\epsilon), \qquad \overline{v^2} = \bar{v}^2 + O(\epsilon), \qquad \overline{uv} = \bar{u}\bar{v} + O(\epsilon), \tag{1.108}$$

$$\overline{\frac{\partial u}{\partial \tilde{\phi}}} = \frac{\partial \bar{u}}{\partial \tilde{\phi}} + O(\epsilon), \quad \overline{\frac{\partial u}{\partial \lambda}} = \frac{\partial \bar{u}}{\partial \lambda} + O(\epsilon), \quad \overline{\frac{\partial v}{\partial \tilde{\phi}}} = \frac{\partial \bar{v}}{\partial \tilde{\phi}} + O(\epsilon), \quad \overline{\frac{\partial v}{\partial \lambda}} = \frac{\partial \bar{v}}{\partial \lambda} + O(\epsilon). \tag{1.109}$$

When we use these assumptions (1.108) and (1.109) and the relations between the dimensionless and physical quantities from the beginning of Sections 1.4 and 1.6.3, we obtain the shallow water equations in the geographical coordinates

$$\frac{\partial \zeta}{\partial t} + \frac{1}{a\cos\phi}\left(\frac{\partial U}{\partial \lambda} + \frac{\partial(\cos\phi\, V)}{\partial \phi}\right) = 0, \tag{1.110}$$

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{1}{a}\frac{\partial (U\bar{v})}{\partial \phi} + \frac{1}{a\cos\phi}\frac{\partial (U\bar{u})}{\partial \lambda} - 2\frac{\tan\phi}{a}U\bar{v} = -\frac{gh}{a\cos\phi}\frac{\partial \zeta}{\partial \lambda} + fV - \\
&\frac{k}{h^2}U\sqrt{U^2+V^2} + \frac{A_H}{a^2}\left[\frac{\partial}{\partial \phi}\left(h\frac{\partial \bar{u}}{\partial \phi} + \frac{h}{\cos\phi}\frac{\partial \bar{v}}{\partial \lambda} + \tan\phi\, U\right) + \right. \\
&\left.\frac{2}{\cos\phi}\frac{\partial}{\partial \lambda}\left(\frac{h}{\cos\phi}\frac{\partial \bar{u}}{\partial \lambda} - \tan\phi\, V\right) - U + \tan\phi\left(h\frac{\partial \bar{u}}{\partial \phi} + \frac{h}{\cos\phi}\frac{\partial \bar{v}}{\partial \lambda} - 2\tan\phi\, U\right)\right],
\end{aligned} \tag{1.111}$$

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{a}\frac{\partial (V\bar{v})}{\partial \phi} + \frac{1}{a\cos\phi}\frac{\partial (V\bar{u})}{\partial \lambda} + \frac{\tan\phi}{a}(U\bar{u} - V\bar{v}) = -\frac{gh}{a}\frac{\partial \zeta}{\partial \phi} - fU - \\
&\frac{k}{h^2}V\sqrt{U^2+V^2} + \frac{A_H}{a^2}\left[2\frac{\partial}{\partial \phi}\left(h\frac{\partial \bar{v}}{\partial \phi}\right) + \frac{1}{\cos\phi}\frac{\partial}{\partial \lambda}\left(h\frac{\partial \bar{u}}{\partial \phi} + \frac{h}{\cos\phi}\frac{\partial \bar{v}}{\partial \lambda} + \tan\phi\, U\right) - \right. \\
&\left. V + 2\tan\phi\left(\frac{h}{\cos\phi}\frac{\partial \bar{u}}{\partial \lambda} - h\frac{\partial \bar{v}}{\partial \phi} + \tan\phi\, V\right)\right],
\end{aligned} \tag{1.112}$$

where $U = h\bar{u}$ and $V = h\bar{v}$ denote the depth mean transports.

Chapter 2

Numerical methods

2.1 Introduction

In this chapter we will describe a way of how the shallow water equations can be numerically solved. Many models based on simple explicit schemes have been developed, however the explicit schemes are restricted by Courant-Fridrichs-Lewy (CFL) stability criterion. On the other hand, fully implicit schemes result in a damping of amplitudes. Therefore, we shall follow the ideas of Backhaus (1983) and Backhaus (1985) where a semi-implicit scheme based on classical Crank-Nicolson approach was proposed and the spatial derivatives were approximated by finite differences on a staggered grid. In this case the scheme is free of damping and does not depend on CFL restriction. This method is successfully implemented e.g. in the Hamburg Shelf Ocean Model (Ham-SOM).

2.2 Finite differences

The shallow water equations are approximated in space by finite differences on a staggered grid. We use the Arakawa C-grid (Arakawa and Lamb, 1977), see Figure 2.1. This lattice is very common in shallow water modelling.

Land areas can be implemented in the lattice by defining “dry cells” where the elevation and the bathymetry are set to be zero. For the sake of the normal flow boundary condition (1.26) we have to keep the velocities on the edges of the dry cells to be zero as well.

2.3 Semi-implicit scheme

Let n and $n+1$ denote present and future time levels, respectively. The index $n+1/2$ denotes an average of future and present time level, e.g.

$$\zeta_{i,j}^{n+1/2} = (\zeta_{i,j}^{n+1} + \zeta_{i,j}^{n})/2.$$

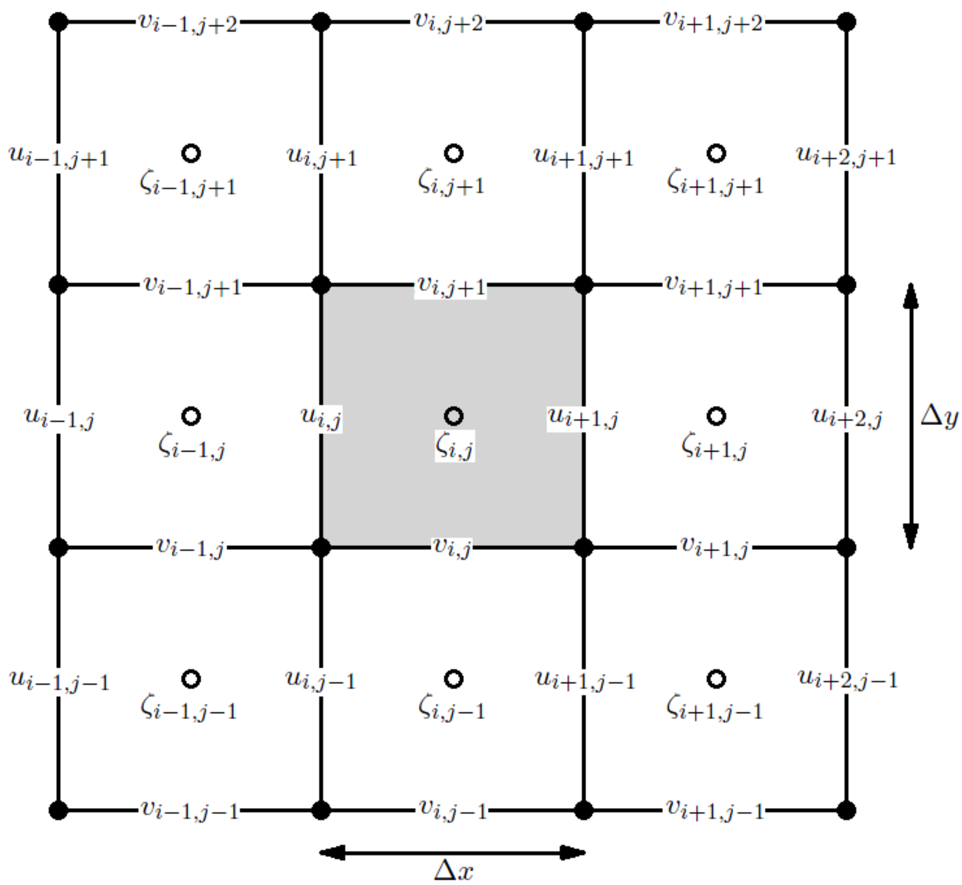


Figure 2.1: The Arakawa C-grid. The water elevation ζ is evaluated in the centers of the cells whereas the horizontal velocities u and v are evaluated at the east/west (u-points) and north/south (v-points) edges of the cells, respectively. If grey color denotes the "dry cell", then the velocity points $u_{i,j}$, $u_{i+1,j}$, $v_{i,j}$ and $v_{i,j+1}$ are kept to be zero in every time-step.

Then the equation of water elevation (1.66) can be approximated as

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^{n} - \Delta t \left(\frac{U_{i+1,j}^{n+1/2} - U_{i,j}^{n+1/2}}{\Delta x} + \frac{V_{i,j+1}^{n+1/2} - V_{i,j}^{n+1/2}}{\Delta y} \right). \tag{2.1}$$

The horizontal equations of motion (1.67) and (1.68) are approximated as

$$U_{i,j}^{n+1} = F_{i,j}^{x} \left[\alpha U_{i,j}^{n} + \beta \bar{V}_{i,j}^{n} - g h_{i,j}^{x} \Delta t \frac{\zeta_{i,j}^{n+1/2} - \zeta_{i-1,j}^{n+1/2}}{\Delta x} + X_{i,j}^{n} \Delta t \right], \tag{2.2}$$

$$V_{i,j}^{n+1} = F_{i,j}^{y} \left[-\beta \bar{U}_{i,j}^{n} + \alpha V_{i,j}^{n} - g h_{i,j}^{y} \Delta t \frac{\zeta_{i,j}^{n+1/2} - \zeta_{i,j-1}^{n+1/2}}{\Delta y} + Y_{i,j}^{n} \Delta t \right], \tag{2.3}$$

where $F_{i,j}^{x}$ and $F_{i,j}^{y}$ are the dimensionless bottom friction functions, $h_{i,j}^{x}$ and $h_{i,j}^{y}$ the spatial averaged heights of water column, $\bar{V}_{i,j}^{n}$ and $\bar{U}_{i,j}^{n}$ the spatial averaged velocities, and $X_{i,j}^{n}$ and $Y_{i,j}^{n}$ further terms including the advective and viscous terms at the u-points and v-points, respectively.

We shall discuss two ways of how the bottom friction can be approximated. Let the x-component of the equation of motion has a simple form

$$U^{n+1} = U^{n} + \Delta t (X^{n} - \tau^{x}), \tag{2.4}$$

where τ^{x} is the bottom friction and X^{n} again denotes further terms. We discuss two formulations of the approximation of the bottom friction

1. explicit formulation: $\tau^{x} = r U^{n} (\sqrt{U^2 + \bar{V}^2} / h_x^2)^n$
2. semi-implicit formulation: $\tau^{x} = r U^{n+1} (\sqrt{U^2 + \bar{V}^2} / h_x^2)^n$.

In first case equation (2.4) takes the form

$$U^{n+1} = U^{n} (1 - r \Delta t (\sqrt{U^2 + \bar{V}^2} / h_x^2)^n) + \Delta t X^{n}. \tag{2.5}$$

However, in the case of shallow areas and strong currents the second term in the brackets in 2.5 can exceed unity which would imply a reversal of the flow and numerical instability would be produced. Hence, the explicit formulation of the bottom friction is not suitable.

In second case equation (2.4) takes the form

$$U^{n+1} (1 + r \Delta t (\sqrt{U^2 + \bar{V}^2} / h_x^2)^n) = U^{n} + \Delta t X^{n}, \tag{2.6}$$

which leads to the expression for the dimensionless bottom friction function

$$F^{x} = \frac{1}{(1 + r \Delta t (\sqrt{U^2 + \bar{V}^2} / h_x^2)^n)}. \tag{2.7}$$

The bottom friction function, which is defined by this way, can never change the sign and thus produce instability, and therefore is more suitable than the explicit formulation. F^y is obtained in an analogous way.

The coefficients α and β arise from the approximation of the Coriolis term. The explicit formulation does not conserve the energy and produce inaccuracy as we can see on the example below. Let us consider a forward-in-time approximation of the inertia equation

$$U^{n+1} = U^n + fV^n, \tag{2.8}$$
$$V^{n+1} = V^n - fU^n. \tag{2.9}$$

The norm of the velocity vector (U^2+V^2) must be constant, however we can easily find out that $(U^{n+1})^2 + (V^{n+1})^2 = ((U^n)^2 + (V^n)^2)(1 + (f\Delta t)^2)$ which means that the velocity is increasing and the system gains the energy without real physical justification. This problem, which was already described in Fischer (1959), can be overcome by the following scheme, proposed by Backhaus (1985):

$$\begin{pmatrix} U^{n+1} \\ V^{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} U^n \\ V^n \end{pmatrix}, \tag{2.10}$$

where $\alpha = \cos(f\Delta t)$ and $\beta = \sin(f\Delta t)$. It is easily verified that now $(U^2 + V^2)$ is constant and thus the scheme conserves the energy.

Note that in Backhaus (1985) a similar matrix operator is also applied to the pressure gradient terms since it was thought that when the Coriolis term is rotated, the pressure term should be rotated as well to ensure geostrophy. However, this is not needed, the rotation applied to the Coriolis term is sufficient (Backhaus, personal communication).

In order to solve the set of the equations in finite differences (2.1)-(2.3) we will separate the terms on time levels $n+1$ and n which means that we obtain the set of the equations

$$\zeta_{i,j}^{n+1} = -\frac{\Delta t}{2}\left(\frac{U_{i+1,j}^{n+1} - U_{i,j}^{n+1}}{\Delta x} + \frac{V_{i,j+1}^{n+1} - V_{i,j}^{n+1}}{\Delta y}\right) + A_{i,j}^{\zeta}, \tag{2.11}$$
$$U_{i,j}^{n+1} = -\frac{gF_{i,j}^{x}h_{i,j}^{x}\Delta t}{2\Delta x}\left(\zeta_{i,j}^{n+1} - \zeta_{i-1,j}^{n+1}\right) + A_{i,j}^{x}, \tag{2.12}$$
$$V_{i,j}^{n+1} = -\frac{gF_{i,j}^{y}h_{i,j}^{y}\Delta t}{2\Delta y}\left(\zeta_{i,j}^{n+1} - \zeta_{i,j-1}^{n+1}\right) + A_{i,j}^{y}, \tag{2.13}$$

where $A_{i,j}^{\zeta}$, $A_{i,j}^{x}$ and $A_{i,j}^{y}$ denote further terms which are expressed on time level n and thus known. Now we will substitute equations (2.12) and (2.13) to equation (2.11) to obtain the linear system of equations for unknown water elevation on time level $n+1$

$$c_1\zeta_{i+1,j}^{n+1} + c_2\zeta_{i-1,j}^{n+1} + c_3\zeta_{i,j}^{n+1} + c_4\zeta_{i,j+1}^{n+1} + c_5\zeta_{i,j-1}^{n+1} = R_{i,j}^{n}, \tag{2.14}$$

where

$$c_1 = -\frac{(\Delta t)^2 g F^x_{i+1,j} h^x_{i+1,j}}{4(\Delta x)^2}, \tag{2.15}$$

$$c_2 = -\frac{(\Delta t)^2 g F^x_{i,j} h^x_{i,j}}{4(\Delta x)^2}, \tag{2.16}$$

$$c_4 = -\frac{(\Delta t)^2 g F^y_{i,j+1} h^y_{i,j+1}}{4(\Delta y)^2}, \tag{2.17}$$

$$c_5 = -\frac{(\Delta t)^2 g F^y_{i,j} h^y_{i,j}}{4(\Delta y)^2}, \tag{2.18}$$

$$c_3 = 1 - c_1 - c_2 - c_4 - c_5 \tag{2.19}$$

and the right side of the system of equations reads

$$R^n_{i,j} = A^\zeta_{i,j} - \frac{\Delta t}{2}\left(\frac{A^x_{i+1,j} - A^x_{i,j}}{\Delta x} + \frac{A^y_{i,j+1} - A^y_{i,j}}{\Delta y}\right). \tag{2.20}$$

2.4 Iteration scheme

The matrix of the linear system of equations (2.14) is usually very large and must be solved by an iteration scheme. We have chosen the method of successive over-relaxation (SOR, Young, 1950). General algorithm for solving a given square system of m linear equations $A\cdot\vec{x} = \vec{b}$, where $\vec{x}$ is the vector of unknown variables, is

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j>i} a_{ij} x_j^{(k)} - \sum_{j<i} a_{ij} x_j^{(k+1)}\right) \quad \text{for } i = 1, 2, \ldots, m, \tag{2.21}$$

where k is the iteration index and ω is a relaxation parameter. The choice of the relaxation parameter is not easy since it depends upon the properties of the coefficient matrix. Hence, the relaxation parameter must be find out by numerical experiments to achieve the fastest iteration. Note that for $\omega = 1$ SOR reduces to the Gauss-Seidel iteration.

The method of SOR can be improved by dividing the cells into black and white cells like a chessboard (e.g. Larsgård, 2007), see Figure 2.2. The water elevation at a black point is dependent only on adjoining white points and vice versa as we can easily verified. This means that we can perform iteration first at black points, which are evaluated from white points, and then at white points, which are evaluated from already updated black points. Therefore, chessboard SOR is faster and much easier to parallelize than original SOR.

In our problem, equation (2.21) for the "black elevations" reads

$$\zeta_{i,j}^{(k+1)} = (1-\omega)\zeta_{i,j}^{(k)} + \frac{\omega}{c_3}\left(R^n_{i,j} - c_1\zeta^{(k)}_{i+1,j} - c_2\zeta^{(k)}_{i-1,j} - c_4\zeta^{(k)}_{i,j+1} - c_5\zeta^{(k)}_{i,j-1}\right) \tag{2.22}$$

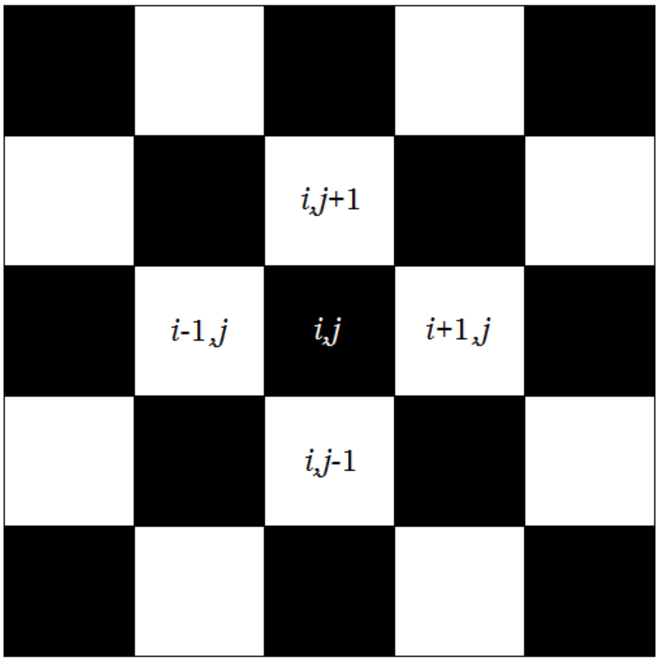


Figure 2.2: The scheme of chessboard SOR. The back point (i, j) is dependent only on white adjoining points.

and for the "white elevations" it holds

$$\zeta_{i,j}^{(k+1)} = (1-\omega)\zeta_{i,j}^{(k)} + \frac{\omega}{c_3}\left(R_{i,j}^n - c_1\zeta_{i+1,j}^{(k+1)} - c_2\zeta_{i-1,j}^{(k+1)} - c_4\zeta_{i,j+1}^{(k+1)} - c_5\zeta_{i,j-1}^{(k+1)}\right). \quad (2.23)$$

2.5 The geographical coordinates

The numerical method was described in the Cartesian coordinates for the sake of simplicity. However, rewriting to the geographical coordinates is straightforward. In this section we write the finite difference equations without any detailed description.

The shallow water equations in the geographical coordinates (1.110)-(1.112) can be approximated as

$$\zeta_{i,j}^{n+1} = \zeta_{i,j}^n - \frac{\Delta t}{a\cos\phi_j}\left(\frac{U_{i+1,j}^{n+1/2} - U_{i,j}^{n+1/2}}{\Delta\lambda} + \frac{\cos\phi_{j+1/2}V_{i,j+1}^{n+1/2} - \cos\phi_{j-1/2}V_{i,j}^{n+1/2}}{\Delta\phi}\right), \quad (2.24)$$

$$U_{i,j}^{n+1} = F_{i,j}^x \left[\alpha U_{i,j}^n + \beta \bar{V}_{i,j}^n - g h_{i,j}^x \Delta t \frac{\zeta_{i,j}^{n+1/2} - \zeta_{i-1,j}^{n+1/2}}{a \cos\phi_j \Delta\lambda} + X_{i,j}^n \Delta t \right], \tag{2.25}$$

$$V_{i,j}^{n+1} = F_{i,j}^y \left[-\beta \bar{U}_{i,j}^n + \alpha V_{i,j}^n - g h_{i,j}^y \Delta t \frac{\zeta_{i,j}^{n+1/2} - \zeta_{i,j-1}^{n+1/2}}{a \Delta\phi} + Y_{i,j}^n \Delta t \right]. \tag{2.26}$$

After separating time levels we obtain

$$\zeta_{i,j}^{n+1} = -\frac{\Delta t}{2a\cos\phi_j} \left(\frac{U_{i+1,j}^{n+1} - U_{i,j}^{n+1}}{\Delta\lambda} + \frac{\cos\phi_{j+1/2} V_{i,j+1}^{n+1} - \cos\phi_{j-1} V_{i,j}^{n+1}}{\Delta\phi} \right) + A_{i,j}^{\zeta}, \tag{2.27}$$

$$U_{i,j}^{n+1} = -\frac{g F_{i,j}^x h_{i,j}^x \Delta t}{2a\cos\phi_j \Delta\lambda} \left(\zeta_{i,j}^{n+1} - \zeta_{i-1,j}^{n+1} \right) + A_{i,j}^x, \tag{2.28}$$

$$V_{i,j}^{n+1} = -\frac{g F_{i,j}^y h_{i,j}^y \Delta t}{2a\Delta\phi} \left(\zeta_{i,j}^{n+1} - \zeta_{i,j-1}^{n+1} \right) + A_{i,j}^y. \tag{2.29}$$

And after substituting (2.28) and (2.29) to (2.27) we again obtain the linear system of equations, which can be solved by chessboard SOR

$$c_1 \zeta_{i+1,j}^{n+1} + c_2 \zeta_{i-1,j}^{n+1} + c_3 \zeta_{i,j}^{n+1} + c_4 \zeta_{i,j+1}^{n+1} + c_5 \zeta_{i,j-1}^{n+1} = R_{i,j}^n, \tag{2.30}$$

where

$$c_1 = -\frac{(\Delta t)^2 g F_{i+1,j}^x h_{i+1,j}^x}{4(a\cos\phi_j \Delta\lambda)^2}, \tag{2.31}$$

$$c_2 = -\frac{(\Delta t)^2 g F_{i,j}^x h_{i,j}^x}{4(a\cos\phi_j \Delta\lambda)^2}, \tag{2.32}$$

$$c_4 = -\frac{(\Delta t)^2 g F_{i,j+1}^y h_{i,j+1}^y}{4a^2 \cos\phi_j (\Delta\phi)^2} \cos\phi_{j+1/2}, \tag{2.33}$$

$$c_5 = -\frac{(\Delta t)^2 g F_{i,j}^y h_{i,j}^y}{4a^2 \cos\phi_j (\Delta\phi)^2} \cos\phi_{j-1/2}, \tag{2.34}$$

$$c_3 = 1 - c_1 - c_2 - c_4 - c_5, \tag{2.35}$$

$$R_{i,j}^n = A_{i,j}^{\zeta} - \frac{\Delta t}{2a\cos\phi_j} \left(\frac{A_{i+1,j}^x - A_{i,j}^x}{\Delta\lambda} + \frac{\cos\phi_{j+1/2} A_{i,j+1}^y - \cos\phi_{j-1/2} A_{i,j}^y}{\Delta\phi} \right). \tag{2.36}$$

Chapter 3

Lunisolar tidal forcing

3.1 Tidal potential

Ocean circulation in our model is forced by tides which are consequence of the gravitational force of the Moon and the Sun. The tides can be formulated in terms of the second degree astronomical tidal potential V (see e.g. Melchior, 1983)

$$V_2 = \frac{3}{4}\frac{GMa^2}{d^3}\Big[\cos^2\phi\cos^2\delta\cos(2\tau)+ \\ \sin(2\phi)\sin(2\delta)\cos\tau+ \\ 3\left(\sin^2\phi-\frac{1}{3}\right)\left(\sin^2\delta-\frac{1}{3}\right)\Big], \tag{3.1}$$

where G is the gravitational constant, M the mass of a celestial body, d is the geocentric distance, δ the declination of the celestial body and τ the local hour angle, which is the longitude where the imaginary plane containing the Sun or Moon and the Earth's rotation axis crosses the equator and is related to the right ascension α through

$$\tau = \Omega T_{Gr} + \lambda - \alpha, \tag{3.2}$$

where Ω is the mean angular velocity of the Earth and T_{Gr} is the Greenwich sidereal time.

The right ascension α and declination δ are the celestial equivalents of the terrestrial longitude λ and latitude ϕ. When we use the equatorial coordinate system, we define the celestial poles and the celestial equator as the projection of the Earth's poles and equator onto the celestial sphere, respectively. Then the declination describes the angle from the celestial equator towards the celestial north pole. The right ascension is measured eastward from the first point of Aries which is the point in the sky where the Sun crosses the celestial equator at the March equinox (see Figures 3.1 and 3.2).

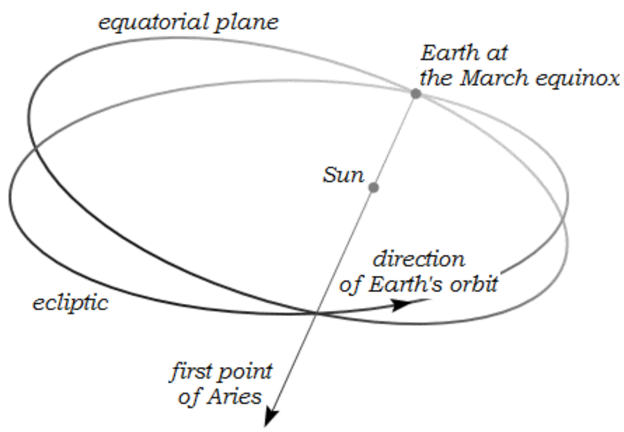


Figure 3.1: The ecliptic is defined by the mean Earth's orbit around the Sun, the equatorial plane by the Earth's equator at the equinoxes. The position of the Sun in the sky at the vernal equinox determines the first point of Aries which is the reference point for the right ascension (see Figure 3.2). From *Wikipedia, the free encyclopedia*, `http://en.wikipedia.org/wiki/File:First-point-of-aries.png`.

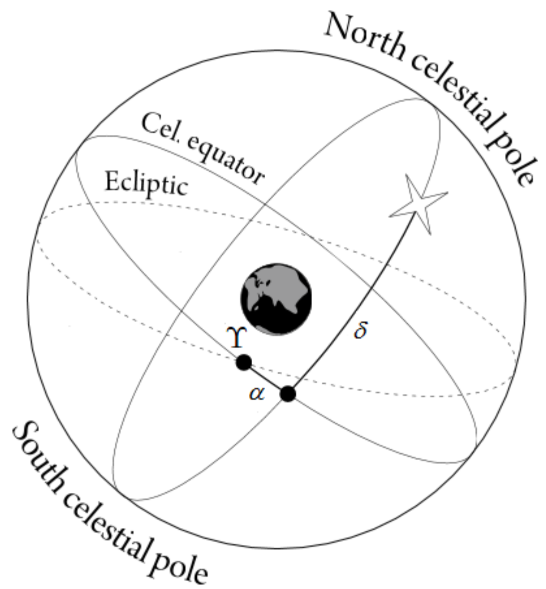
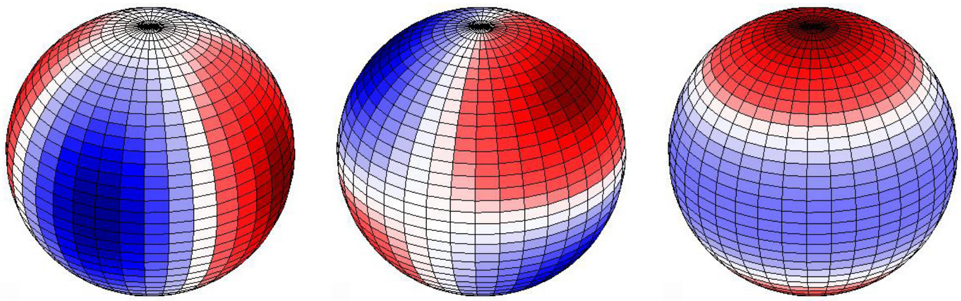


Figure 3.2: The equatorial coordinate system. The celestial poles are the projection of the Earth's poles onto the celestial sphere, the celestial equator is the projection of the Earth's equator. ♈ denotes the first point of Aries, α the right ascension and δ the declination of a celestial body. From *Wikipedia, the free encyclopedia*, `http://en.wikipedia.org/wiki/File:Equatorial_coordinate_system_%28celestial%29.svg`.



(a) Sectorial term (b) Tesseral term (c) Zonal term

Figure 3.3: The three kinds of tides. Fig. 3.3(a) shows the sectorial function. The amplitudes are zero at the poles and have a maximum at the equator when the declination of the perturbing body is zero. The tesseral function is shown in Fig. 3.3(b). The tesseral function changes the sign with the declination of the perturbing body. The amplitude is maximum at latitude 45° North and 45° South when the declination of the perturbing body is maximum. The amplitude is always zero at the equator and at the poles. The zonal function is represented in Fig. 3.3(c). Its nodal lines are the parallels 35°16' North and 35°16' South. It is a squared sine function, therefore its fundamental period is fourteen days for the Moon and six month for the Sun. From *Wikipedia, the free encyclopedia*, `http://en.wikipedia.org/wiki/Earth_tide`.

The constants in the equations (3.1) are given (Petit and Luzum, 2010):

$$\begin{aligned} GM_{☾} &= 4902.8002\,\mathrm{km^3s^{-2}}, \\ GM_{\odot} &= 1.327124421 \times 10^{11}\,\mathrm{km^3s^{-2}}, \\ a &= 6378.137\,\mathrm{km}, \\ \Omega &= 7.292115 \times 10^{-5}\,\mathrm{rads^{-1}}. \end{aligned}$$

The potential (3.1) is divided into three terms which represent the three types of spherical harmonic functions of the second order and define the tidal bands (see Figure 3.3):

1. the sectorial term corresponding to the semi-diurnal period of the tides;
2. the tesseral term corresponding to the diurnal tidal period;
3. the zonal term corresponding to the long-period tides, i.e. fortnightly to annual.

3.2 Complete lunisolar forcing

The tidal forcing $\vec{\Phi}$ is defined as the negative gradient of the potential (3.1)

$$\vec{\Phi} = -\nabla V_2, \tag{3.3}$$

which means that the East-West component reads

$$\Phi_x = -\frac{1}{a\cos\phi}\frac{\partial V_2}{\partial\lambda} \tag{3.4}$$

and the North-South component is

$$\Phi_y = -\frac{1}{a}\frac{\partial V_2}{\partial\phi}. \tag{3.5}$$

By differentiating of the tidal potential we obtain the components of the tidal acceleration

$$\begin{aligned}\Phi_x &= \frac{3}{2}\frac{GMa}{d^3}\Big(\cos\phi\cos^2\delta\sin(2\tau) + \\ &\qquad \sin\phi\sin(2\delta)\sin\tau\Big)\,, \qquad (3.6)\\ \Phi_y &= \frac{3}{4}\frac{GMa}{d^3}\Big(\sin(2\phi)\cos^2\delta\cos(2\tau) - \\ &\qquad 2\cos(2\phi)\sin(2\delta)\cos\tau - \\ &\qquad \sin(2\phi)(3\sin^2\delta - 1)\Big)\,. \qquad (3.7)\end{aligned}$$

However, we should also consider the influence of the tides of the solid Earth on the tidal potential since the response of the Earth to the tidal potential produces an additional gravitational potential and a displacement. This can be described by the combination of the Love numbers $k = 0.302$ and $h = 0.612$ which results in the factor γ (Weis, 2006)

$$\gamma = 1 + k - h \tag{3.8}$$

Then the tidal influence is described by the term $\gamma\vec{\Phi}$, which can be added to the right side of the shallow water equations (see equations (1.67) and (1.68) in the Cartesian coordinates, or equations (1.111) and (1.112) in the geographical coordinates) as the source term.

Note that until recently many models were forced by partial tides. This means that the lunisolar tidal potential was divided into partial tides, each one with characteristic frequency describing a certain aspect of the orbit of the Moon or the Earth. The decomposition was first provided by British oceanographer Arthur Thomas Doodson (Doodson, 1922), who expanded (3.1) in a Fourier series using six fundamental frequencies which corresponds to various astronomical periods (Agnew, 2007):

- lunar day, with period $T = 24\,\mathrm{h}\ 50\,\mathrm{m}\ 28.3\,\mathrm{s}$
- tropical month, $T = 27.3216\,\mathrm{d}$
- solar year, $T = 365.2422\,\mathrm{d}$
- lunar perigee, $T = 8.847\,\mathrm{yr}$

- lunar node, $T = 18.613\,\mathrm{yr}$
- solar perigee, $T = 20941\,\mathrm{yr}$

The frequencies of partial tides are then a linear combination of the six fundamental frequencies and are denoted by Doodson numbers. However, it is more usual to use a different naming system, so-called Darwin symbols. Unfortunately, this standard naming system was created in a somewhat ad hoc manner (it was begun by Thomsen for a few tides and then extended by Darwin) and therefore have no special rules. For example, M_2 denotes semi-diurnal principal lunar tide, S_2 semi-diurnal principal solar, K_1 diurnal lunisolar, Mf fortnightly, Ssa semi-annual. For more information and tables of the partial tides and their frequencies and symbols see e.g. Stewart (2008), Agnew (2007), Weis (2006).

This frequency-domain solution is easy to simulate and analyse, however, as pointed by Weis (2006), such a model can not simulate non-linear interactions between partial tides which lead to the formation of so-called shallow water tides. These tides are not direct consequence of the lunisolar tidal potential and are formed mostly in shallow water areas (e.g. coasts and shelves) where non-linear terms of the shallow water equations gain importance. Hence, in our model we use full lunisolar forcing (3.6) and (3.7) which includes all partial tides simultaneously and thus represents complete interaction dynamics.

In order to simulate the ocean circulations forced by complete lunisolar potential we need to know ephemerides of the Sun and the Moon, i.e. the coordinates α, δ and d, in every time step. For this purpose we use the set of subroutines NOVAS F3.1 by the U.S. Naval Observatory (Kaplan et al., 2011)

Chapter 4

Numerical results

4.1 Introduction

In previous chapters we formulated our problem theoretically. In this chapter we provide several examples of numerical implementation. We use the numerical methods described in Chapter 2 to develop three similar programs:

- Program `ccira.f90` uses the finite difference approximation of the shallow water equations in the local Cartesian coordinates (see equation (2.1)-(2.2)) inside a water reservoir of a rectangular shape.

- Program `gcira.f90` uses the finite difference approximation of the shallow water equations in the geographical coordinates (see equation (2.24)-(2.25)) inside a water reservoir of a rectangular shape.

- Program `gcog.f90` uses the finite difference approximation of the shallow water equations in the geographical coordinates (see equation (2.24)-(2.25)) on a sphere

All programs are written in Fortran 90 and are parallelized by OpenMP. The results of all simulations are also on the attached CD (see Appendix B).

4.2 Cartesian geometry

First, we investigate the evolution of the surface elevation in a square area 100 km $\times$ 100 km using the local Cartesian coordinates with the discretization step $\Delta x = \Delta y = 1$ km. The bathymetry and the initial setting of the elevation are drawn in Figure 4.1. The bathymetry has a shape of a Gaussian hill which leads to the formation of a circle island within the area. The initial elevation of the free surface is given by a Gaussian depression. The initial velocities were set to be zero. The bottom friction is included with the bottom friction coefficient $r = 0.003$ (Weis, 2006). The geographical coordinates of the middle of the area

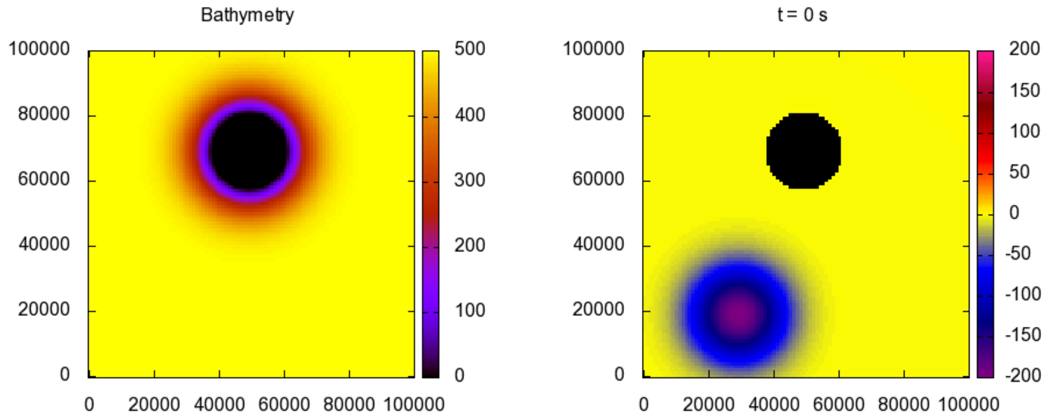


Figure 4.1: The bathymetry (left) and the initial surface elevation (right) in the square area. The black circle centered at $x = 50$ km and $y = 70$ km in the right panel is an island of radius 10 km. All quantities are in meters.

are 50° North and 14,5° East. Note that if the local Cartesian coordinates are used then the latitude and the longitude of the grid points are given by the simple relations

$$x = a\cos(\phi_0)(\lambda - \lambda_0), \tag{4.1}$$

$$y = a(\phi - \phi_0), \tag{4.2}$$

where ϕ_0 and λ_0 are the latitude and longitude of the origin of the coordinates, respectively. These relations hold locally and are valid only if the horizontal dimensions of the investigated area are small with respect to the Earth's radius and the curvature of the Earth's surface can be neglected.

Figure 4.2 shows six snapshots of the evolution of the surface elevation for the fluid without the viscous term, i.e. $A_H = 0$ m^{2}s^{-1}. We can see the formation of a wave, its propagation from the initial depression and the reflections from the edges and the island.

The effect of the eddy viscosity is shown in Figures 4.3 and 4.4 on the left panels with the snapshots of the simulation with $A_H = 10^3$ m^{2}s^{-1}. The right panels represents the differences between the simulations without and with the viscous term, respectively. It is evident that simulation which takes in account the viscous term results in smaller amplitudes of waves than the simulation with $A_H = 0$. This result is expected since the viscous term represents the inner friction of the fluid and the dissipation of the kinetic energy (see Section 1.2). The consequence of this dissipation is damping of amplitudes.

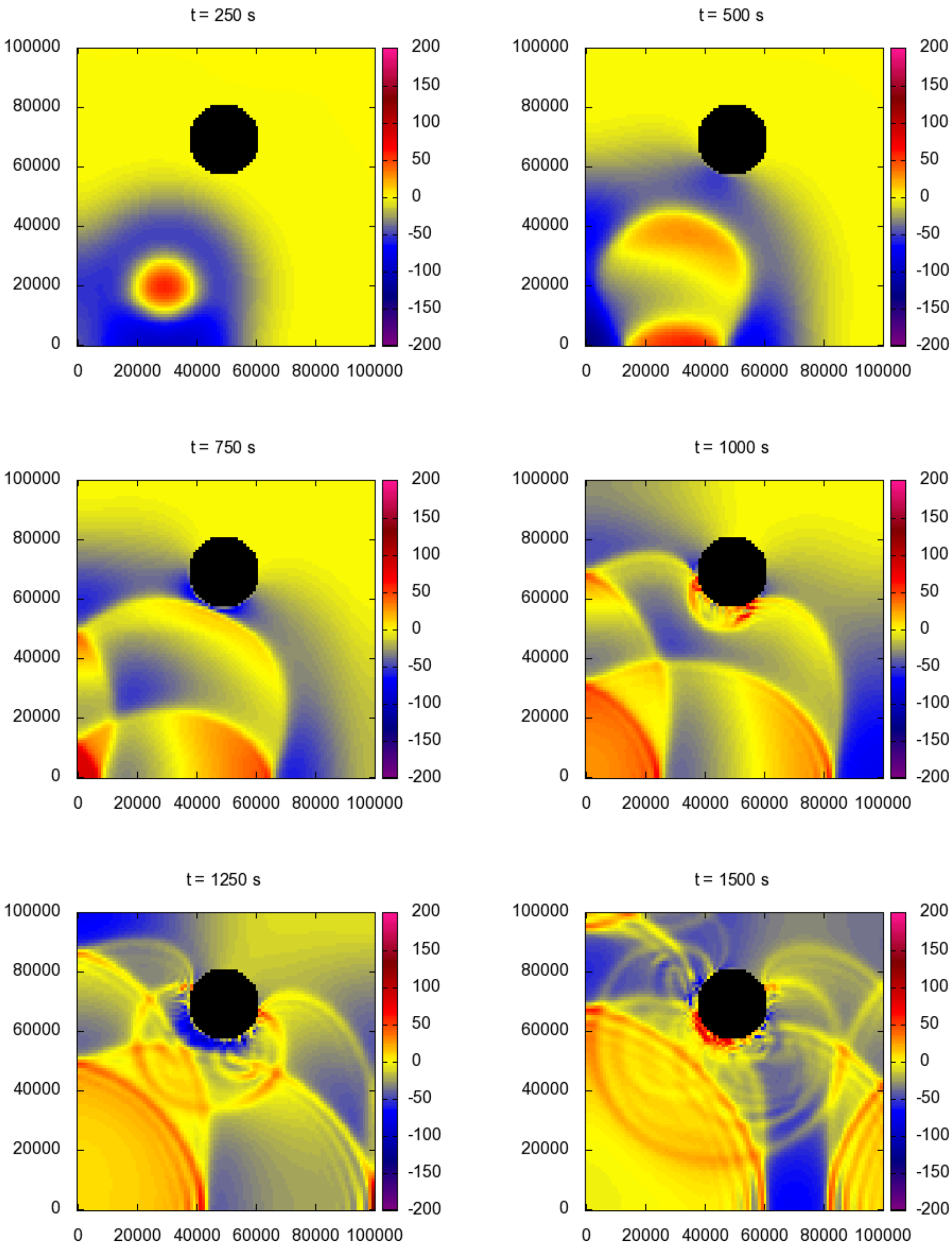


Figure 4.2: The evolution of the surface elevation from the initial state (Fig. 4.1) in the local Cartesian coordinates when the viscous term is neglected. All quantities are in meters.

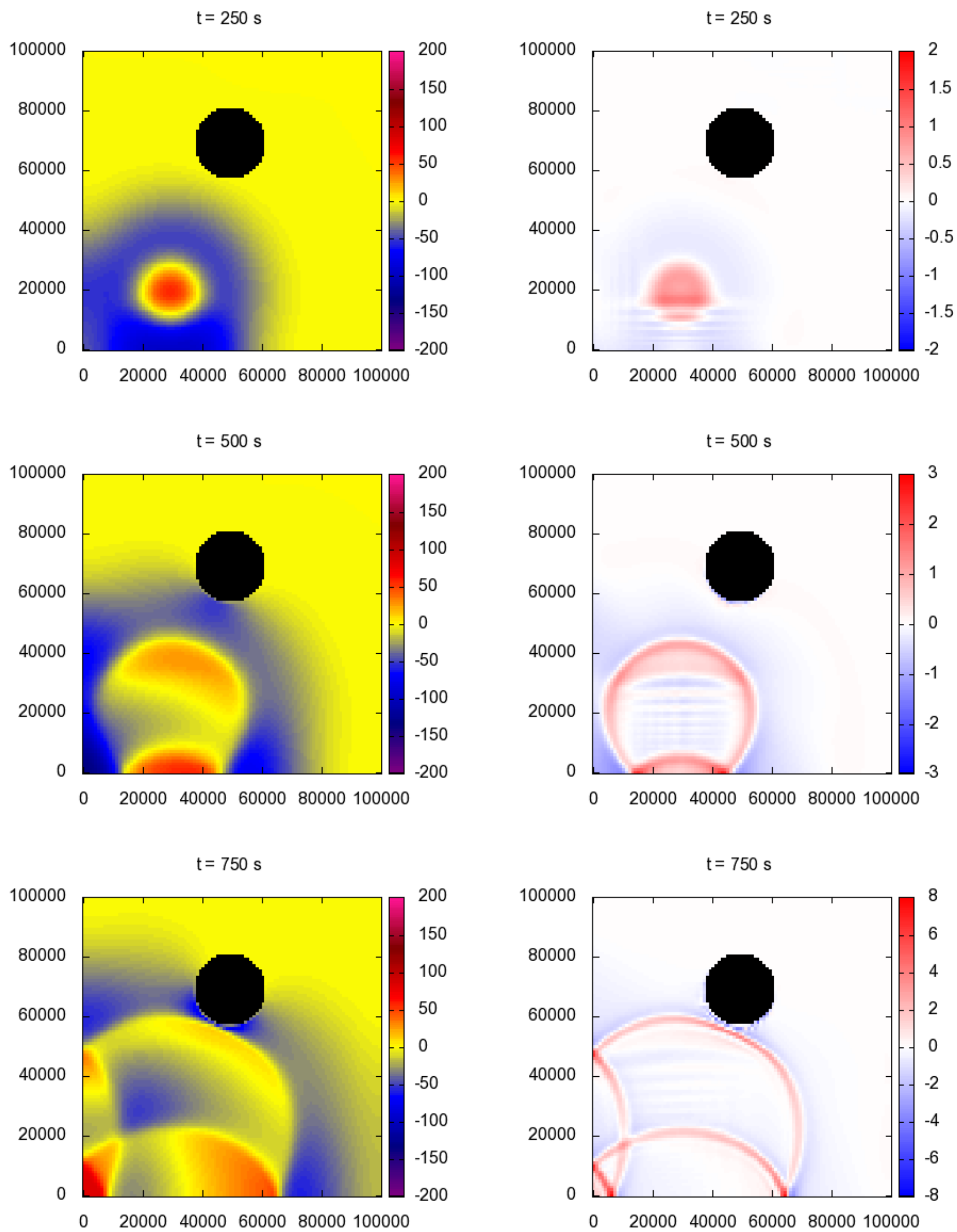


Figure 4.3: The evolution of the surface elevation from the initial state (Fig. 4.1) in the local Cartesian coordinates when the viscous term is taken in account (left) and the differences between the simulations without and with the viscous term, respectively (right). All quantities are in meters.

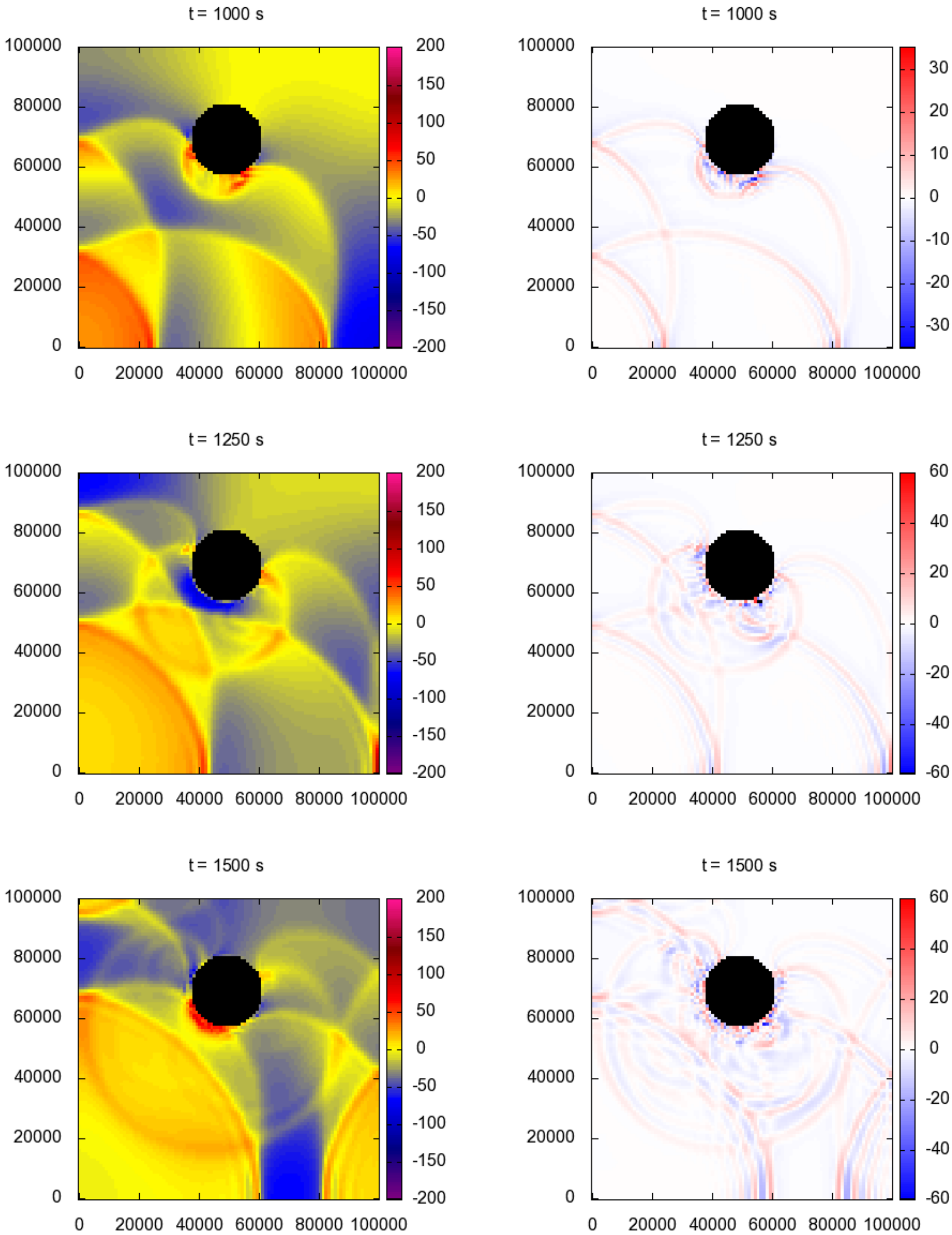


Figure 4.4: The same as Figure 4.3, but for later time instances.

4.3 Spherical geometry

The next simulation is carried out in the geographical coordinates. We use the identical settings, that is the bathymetry and initial surface elevation, as in the Cartesian case with $A_H = 0$. The results of this simulation and the comparison with the Cartesian geometry are shown in Figures 4.5 and 4.6. We can see significant differences between the two simulations when a wave is propagating along the x-axis, i.e. in the East/West direction, while the differences in wave propagation along the y-axis, i.e. in the North/South direction, are small. This is the consequence of different discretization in the longitudinal direction. While Δx is constant in the Cartesian coordinates, the longitudinal discretization element $a \cos \phi \, \Delta\lambda$ changes with the latitude and is not exactly equal to Δx. Hence, the simulations can not be compared exactly at the same nodes which results in the differences on the right panels in Figures 4.5 and 4.6. However, it does not mean that the solution either in the Cartesian or geographical coordinates is inaccurate. Both approaches are valid in this case, however an user should bear in mind the different discretization.

The last numerical simulation presented in this work describes the evolution of the surface elevation of a global ocean with an island and continents at the poles. We use the continents at the poles for the sake of simplicity since the equation in the geographical coordinates are divergent at the poles and the discretization step in the longitudinal direction decreases towards the poles since we use finite differences approximation. The continents are determined by the latitudes 70° North and 70° South. The bathymetry has the shape of a Gaussian hill and the initial elevation of the free surface is given by a Gaussian depression (see Figure 4.7).

Figure 4.8 contains four snapshots of the evolution of the surface elevation with $A_H = 10^4$ m^{2}s^{-1} and the discretization step $\Delta\phi = \Delta\lambda = 0.5°$.

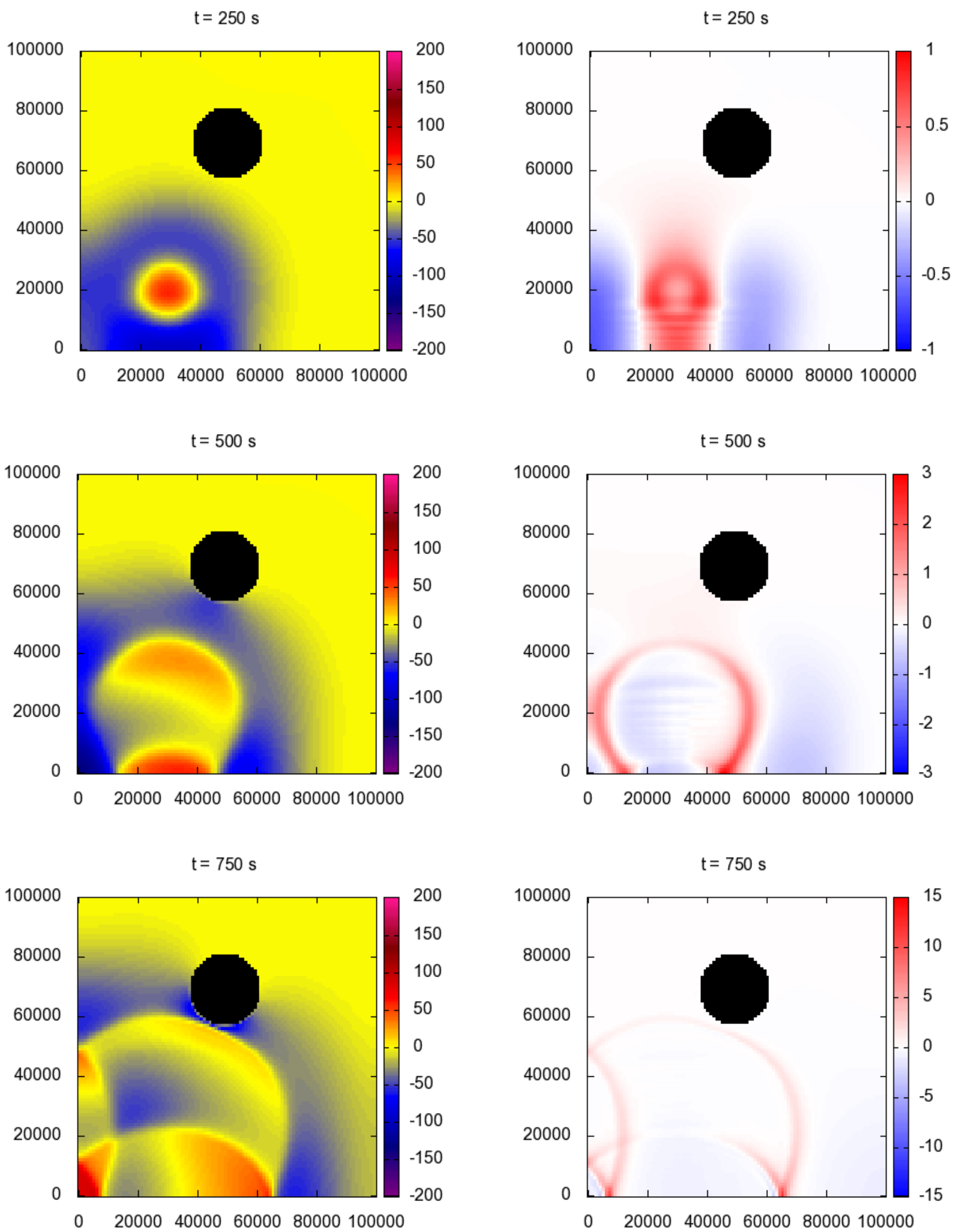


Figure 4.5: The evolution of the surface elevation from the initial state (Fig. 4.1) in the geographical coordinates when the viscous term is neglected (left) and the differences between the simulations in the local Cartesian and geographical coordinates, respectively (right). All quantities are in meters.

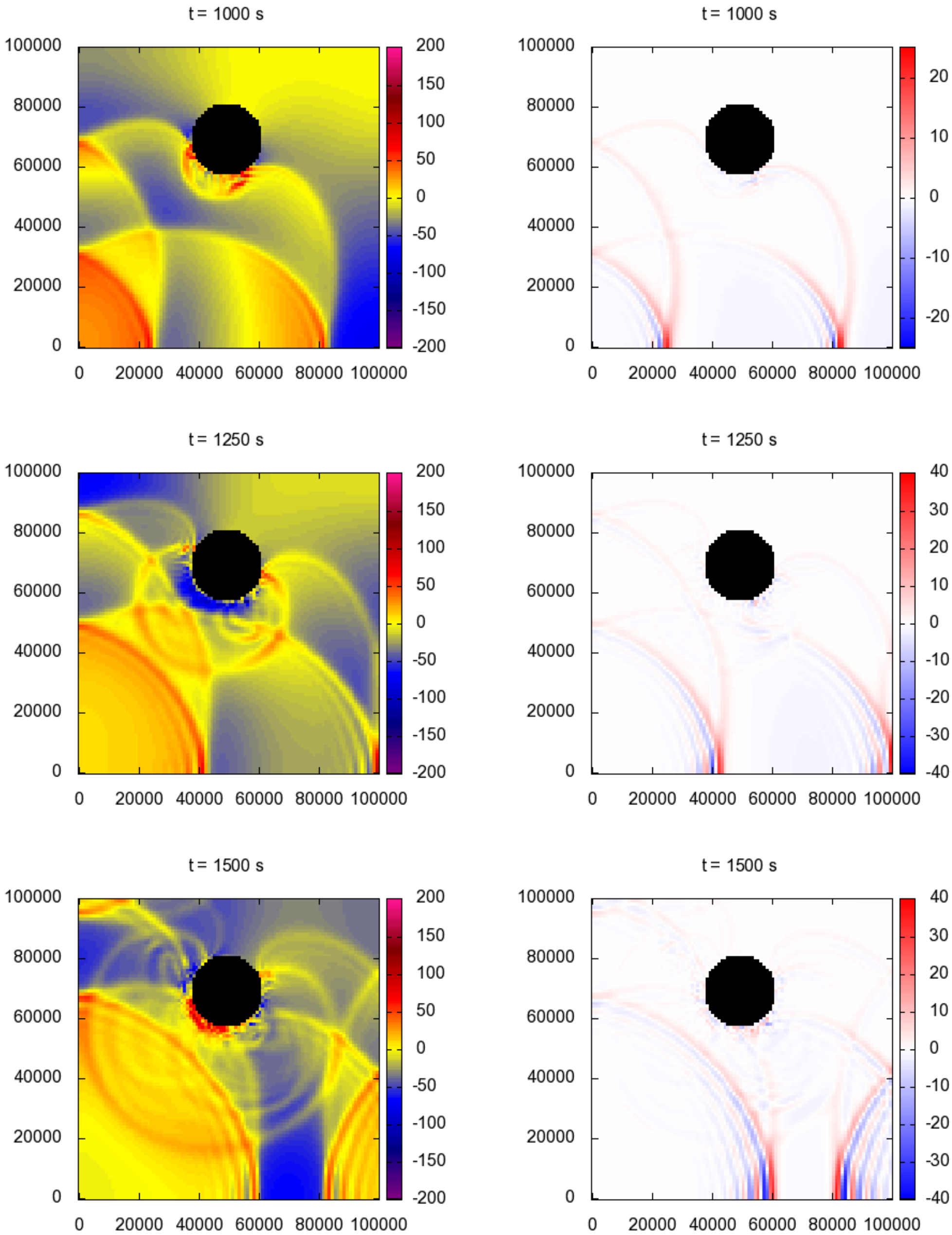


Figure 4.6: The same as Figure 4.5, but for later time instances.

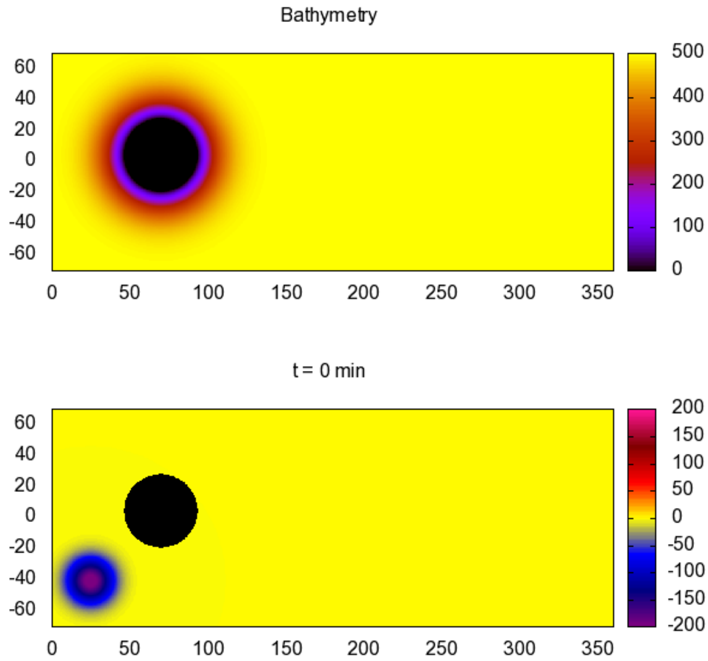


Figure 4.7: The bathymetry (top) and the initial surface elevation (below) on the globe with the continents at the poles determined by the 70° latitudes. The black circle centered at $\lambda = 70°$ and $\phi = 5°$ in the lower panel is an island of radius 45°. The latitude and the longitude are in degrees, the bathymetry and the elevation are in meters.

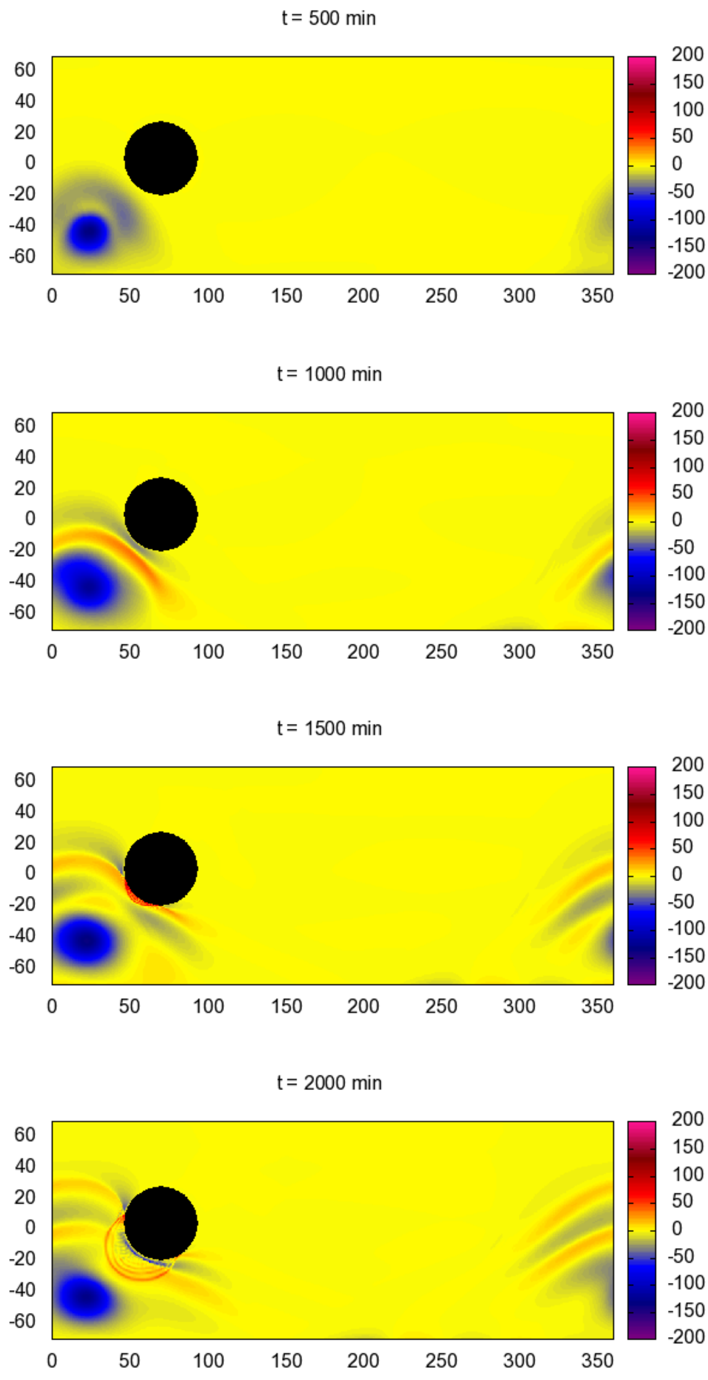


Figure 4.8: The evolution of the surface elevation from the initial state (Fig. 4.7) on the globe with the continents at the poles. The longitude and latitude are in degrees, the elevation is in meters.

Conclusions

In this work, we have dealt with shallow water modelling, both from the theoretical and numerical perspectives.

First, in Chapter 1, we derived the shallow water equations from the fundamental balance laws, i.e. the continuity equation and the equation of motion. The derivation was provided in the local Cartesian coordinates, which are appropriate for coastal and shelf sea modelling, followed by the derivation in the geographical coordinates, which is suitable for global ocean modelling. The derivation included the full stress tensor with the eddy viscosity which is usually omitted.

In Chapter 2, we introduced numerical methods for solving the shallow water equations. The spatial derivatives were approximated by finite differences on the Arakawa-C grid, the time derivative was approximated by semi-implicit scheme. The advantages of this scheme lies in the fact that it is free of damping and is not restricted by the Courant-Fridrichs-Levy stability criterion. These methods were used to develop new programs written in Fortran 90. Several examples of simulations are shown in Chapter 4.

We believe that the developed programs are a useful tool for the modelling of the ocean-tide circulation in a high resolution. The lunisolar tidal modul, that computes the complete tidal forcing based on the ephemerides, is implemented in the programs, however, not employed yet. The follow-up research with the simulations based on realistic tidal forcing can be focused e.g. on the investigation of the effect of the ocean tides on the Earth's rotation, or the Earth's magnetic field.

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Appendix A

Curvilinear orthogonal coordinates

Below we briefly introduce curvilinear orthogonal coordinates and summarize important formulae for differential operators. Moreover, we provide formulae in the geographical coordinates as a example of general curvilinear orthogonal coordinates. For more information and derivation see Martinec (2011).

A.1 Basic formulae

Given the Cartesian coordinates of a point P in the 3-D space (y_1, y_2, y_3). We can define a system of curvilinear coordinates by specifying three coordinate transformation functions x_k.

$$x_k = x_k(y_1, y_2, y_3) \quad \text{for } k = 1, 2, 3, \tag{A.1}$$

which are C^1 (continuous first partial derivatives) and the Jacobian of the transformation is nonzero almost everywhere

$$j = \det\left(\frac{\partial x_k}{\partial y_l}\right) \neq 0. \tag{A.2}$$

Then there is a unique inverse of (A.1)

$$y_k = y_k(x_1, x_2, x_3) \quad \text{for } k = 1, 2, 3. \tag{A.3}$$

The geographical coordinates $r, \phi, \lambda (x_1 \equiv r, x_2 \equiv \phi, x_3 \equiv \lambda)$ are defined by the relations to the rectangular Cartesian coordinates

$$y_1 = r\cos\phi\cos\lambda, \quad y_2 = r\cos\phi\sin\lambda, \quad y_3 = r\sin\phi, \tag{A.4}$$

or inversely

$$r = \sqrt{y_1^2 + y_2^2 + y_3^2}, \quad \phi = \arctan\left(\frac{y_3}{\sqrt{y_1^2 + y_2^2}}\right), \quad \lambda = \arctan\left(\frac{y_2}{y_1}\right). \tag{A.5}$$

The inverse of the Jacobian j is given

$$j^{-1} = r^2 \cos\phi. \tag{A.6}$$

Given the rectangular Cartesian unit base vectors $\vec{i}_1, \vec{i}_2, \vec{i}_3$ and the position vector $\vec{p}$ of the point P, which can be expressed as

$$\vec{p} = p_k \vec{i}_k. \tag{A.7}$$

Then the curvilinear coordinate unit base vectors can be defined by the relations

$$\vec{e}_k = \frac{1}{h_k}\frac{\partial \vec{p}}{\partial x_k}, \quad \text{for } k = 1, 2, 3, \tag{A.8}$$

where h_k are the scale factors, so-called the Lamé coefficients which are defined by the relation

$$h_k = \sqrt{\frac{\partial \vec{p}}{\partial x_k} \cdot \frac{\partial \vec{p}}{\partial x_k}}. \tag{A.9}$$

We assume that new curvilinear coordinates are orthogonal and the base vectors form a right-hand system, i.e.

$$\vec{e}_k \cdot \vec{e}_l = \delta_{kl}, \quad \vec{e}_k \times \vec{e}_l = \varepsilon_{klm}\vec{e}_m, \tag{A.10}$$

where δ_{kl} is the Kronecker delta and ε_{klm} is the Levi-Civita symbol

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l, \end{cases} \tag{A.11}$$

$$\varepsilon_{klm} = \begin{cases} +1, & \text{if } (k,l,m) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1, & \text{if } (k,l,m) \text{ is } (1,3,2), (3,2,1) \text{ or } (2,1,3), \\ 0, & \text{if } k = l \text{ or } l = m \text{ or } k = m. \end{cases} \tag{A.12}$$

The Lamé coefficients in the geographical coordinates are

$$h_r = 1, \quad h_\phi = r, \quad h_\lambda = r\cos\phi. \tag{A.13}$$

The unit base vectors $\vec{e}_k$ are functions of position and vary in direction. Therefore spatial derivatives of this base vectors are not zero:

$$\frac{\partial \vec{e}_k}{\partial x_l} = \sum_m \Gamma^m_{kl}\vec{e}_m, \tag{A.14}$$

where Γ^m_{kl} are the Christoffel symbols, which can be explicitly expressed by

$$\Gamma^m_{kl} = \frac{1}{h_k}\frac{\partial h_l}{\partial x_k}\delta_{lm} - \frac{1}{h_m}\frac{\partial h_k}{\partial x_m}\delta_{kl}. \tag{A.15}$$

Now we can write formulae for the partial derivatives of any vector or tensor. The partial derivative of a vector $\vec{v}$ can be formulated as

$$\frac{\partial \vec{v}}{\partial x_m} = \sum_k v_{k;l} \vec{e}_k, \tag{A.16}$$

where $v_{k;l}$ is the balanced derivative of v_k with respect to x_l

$$v_{k;l} = \frac{\partial v_k}{\partial x_l} + \sum_m \Gamma^k_{ml} v_m. \tag{A.17}$$

The partial derivative of a second-order tensor $\mathbf{T}$ can be formulated as

$$\frac{\partial \mathbf{T}}{\partial x_m} = \sum_{kl} T_{kl;m} \vec{e}_k \otimes \vec{e}_l, \tag{A.18}$$

where $T_{kl;m}$ is the balanced derivative of T_{kl} with respect to x_l

$$T_{kl;m} = \frac{\partial T_{kl}}{\partial x_m} + \sum_n \Gamma^k_{nm} T_{nl} + \sum_n \Gamma^l_{nm} T_{kn}. \tag{A.19}$$

A.2 Invariant differential operators

With use of the previous formulae we can express the invariant differential operators in the curvilinear orthogonal coordinates.

The gradient of a scalar:

$$\nabla f = \frac{1}{h_k} \frac{\partial f}{\partial x_k} \vec{e}_k \tag{A.20}$$

In the geographical coordinates:

$$\nabla f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \vec{e}_\phi + \frac{1}{r \cos \phi} \frac{\partial f}{\partial \lambda} \vec{e}_\lambda \tag{A.21}$$

The divergence of a vector:

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 v_1) + \frac{\partial}{\partial x_2} (h_1 h_3 v_2) + \frac{\partial}{\partial x_3} (h_2 h_1 v_3) \right] \tag{A.22}$$

In the geographical coordinates:

$$\nabla \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \cos \phi} \left(\frac{\partial (\cos \phi v_\phi)}{\partial \phi} + \frac{\partial v_\lambda}{\partial \lambda} \right) \tag{A.23}$$

The curl of a vector:

$$\begin{aligned}\nabla \times \vec{v} = &\frac{1}{h_2 h_3}\left[\frac{\partial(h_3 v_3)}{\partial x_2} - \frac{\partial(h_2 v_2)}{\partial x_3}\right]\vec{e}_1 + \\ &\frac{1}{h_1 h_3}\left[\frac{\partial(h_1 v_1)}{\partial x_3} - \frac{\partial(h_3 v_3)}{\partial x_1}\right]\vec{e}_2 + \\ &\frac{1}{h_1 h_2}\left[\frac{\partial(h_2 v_2)}{\partial x_1} - \frac{\partial(h_1 v_1)}{\partial x_2}\right]\vec{e}_3\end{aligned} \tag{A.24}$$

In the geographical coordinates:

$$\begin{aligned}\nabla \times \vec{v} = &\frac{1}{r\cos\phi}\left[\frac{\partial(\cos\phi v_\lambda)}{\partial\phi} - \frac{\partial v_\phi}{\partial\lambda}\right]\vec{e}_r + \\ &\left[\frac{1}{r\cos\phi}\frac{\partial v_r}{\partial\lambda} - \frac{1}{r}\frac{\partial(r v_\lambda)}{\partial r}\right]\vec{e}_\phi + \\ &\frac{1}{r}\left[\frac{\partial(r v_\phi)}{\partial r} - \frac{\partial v_r}{\partial\phi}\right]\vec{e}_\lambda\end{aligned} \tag{A.25}$$

The gradient of a vector:

$$(\nabla\vec{v})_{kl} = \begin{cases} \dfrac{1}{h_k}\left(\dfrac{\partial v_k}{\partial x_k} + \displaystyle\sum_{\substack{m \\ m\neq k}} \dfrac{1}{h_m}\dfrac{\partial h_k}{\partial x_m} v_m\right) & \text{if } l = k \\ \\ \dfrac{1}{h_k}\left(\dfrac{\partial v_l}{\partial x_k} - \dfrac{1}{h_l}\dfrac{\partial h_k}{\partial x_l} v_k\right) & \text{if } l \neq k \end{cases} \tag{A.26}$$

In the geographical coordinates:

$$\begin{aligned}\nabla\vec{v} = &\frac{\partial v_r}{\partial r}\vec{e}_r\otimes\vec{e}_r + \frac{\partial v_\phi}{\partial r}\vec{e}_r\otimes\vec{e}_\phi + \frac{\partial v_\lambda}{\partial r}\vec{e}_r\otimes\vec{e}_\lambda + \\ &\frac{1}{r}\left(\frac{\partial v_r}{\partial\phi} - v_\phi\right)\vec{e}_\phi\otimes\vec{e}_r + \frac{1}{r}\left(\frac{\partial v_\phi}{\partial\phi} + v_r\right)\vec{e}_\phi\otimes\vec{e}_\phi + \frac{1}{r}\frac{\partial v_\lambda}{\partial\phi}\vec{e}_\phi\otimes\vec{e}_\lambda + \\ &\frac{1}{r}\left(\frac{1}{\cos\phi}\frac{\partial v_r}{\partial\lambda} - v_\lambda\right)\vec{e}_\lambda\otimes\vec{e}_r + \frac{1}{r\cos\phi}\left(\frac{\partial v_\phi}{\partial\lambda} + \sin\phi\, v_\lambda\right)\vec{e}_\lambda\otimes\vec{e}_\phi + \\ &\frac{1}{r}\left(\frac{1}{\cos\phi}\frac{\partial v_\lambda}{\partial\lambda} + v_r - \tan\phi\, v_\phi\right)\vec{e}_\lambda\otimes\vec{e}_\lambda\end{aligned} \tag{A.27}$$

The divergence of a second-order tensor:

$$(\nabla \cdot \mathbf{T})_l = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{1l}) + \frac{\partial}{\partial x_2}(h_1 h_3 T_{2l}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{3l})\right] + \sum_k \frac{1}{h_k h_l}\left(\frac{\partial h_l}{\partial x_k} T_{lk} - \frac{\partial h_k}{\partial x_l} T_{kk}\right) \tag{A.28}$$

In the geographical coordinates:

$$\begin{aligned}\nabla \cdot \mathbf{T} = &\left[\frac{1}{r^2}\frac{\partial (r^2 T_{rr})}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi\, T_{\phi r})}{\partial \phi} + \frac{\partial T_{\lambda r}}{\partial \lambda}\right) - \frac{1}{r}\left(T_{\phi\phi} + T_{\lambda\lambda}\right)\right]\vec{e}_r + \\ &\left[\frac{1}{r^2}\frac{\partial (r^2 T_{r\phi})}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi\, T_{\phi\phi})}{\partial \phi} + \frac{\partial T_{\lambda\phi}}{\partial \lambda}\right) + \frac{1}{r}\left(T_{\phi r} + \tan\phi\, T_{\lambda\lambda}\right)\right]\vec{e}_\phi + \\ &\left[\frac{1}{r^2}\frac{\partial (r^2 T_{r\lambda})}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi\, T_{\phi\lambda})}{\partial \phi} + \frac{\partial T_{\lambda\lambda}}{\partial \lambda}\right) + \frac{1}{r}\left(T_{\lambda r} - \tan\phi\, T_{\lambda\phi}\right)\right]\vec{e}_\lambda\end{aligned} \tag{A.29}$$

The Laplacian of a scalar:

$$\Delta f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial x_3}\right)\right] \tag{A.30}$$

In the geographical coordinates:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial \phi}\left(\cos\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\cos^2\phi}\frac{\partial^2 f}{\partial \lambda^2} \tag{A.31}$$

The Laplacian of a vector can be best obtained by using the vector differential identity

$$\Delta \vec{v} = \nabla \times \nabla \times \vec{v} + \nabla \cdot (\nabla \vec{v}). \tag{A.32}$$

Appendix B

Contents of the attached CD

The source codes of the programs and the results of the simulations are on the attached CD. The CD contains these files:

`ccira.f90` – program which computes the shallow water equations in the local Cartesian coordinates (see equation (2.1)-(2.2)) inside a water reservoir of a rectangular shape

`gcira.f90` – program which computes the shallow water equations in the geographical coordinates (see equation (2.24)-(2.25)) inside a water reservoir of a rectangular shape

`gcog.g90` – program which computes the shallow water equations in the geographical coordinates (see equation (2.24)-(2.25)) on a sphere

`mnovas.f` – subroutines for computing ephemerides (by Kaplan et al., 2011)

`SS_EPHEM.txt` – data file which is read by `mnovas.f`

The results of the simulations are in the directories:

`globe_novis` – solution of the shallow water equations on the sphere without the viscous term

`globe_vis` – solution of the shallow water equations on the sphere with $A_H = 10^4\ \mathrm{m^2 s^{-1}}$

`square_cart_novis` – solution of the shallow water equations in the local Cartesian coordinates inside the square area without the viscous term

`square_cart_vis` – solution of the shallow water equations in the local Cartesian coordinates inside the square area with $A_H = 10^3\ \mathrm{m^2 s^{-1}}$

`square_geog_novis` – solution of the shallow water equations in the geographical coordinates inside the square area without the viscous term `square_geog_vis` – solution of the shallow water equations in the geographical coordinates inside the square area with $A_H = 10^3\ \mathrm{m^2 s^{-1}}$

Each directory contains the source codes which were used and the files:

`0.png` – figure of the initial state

`1...500.png` – 500 snapshots of the simulations

`bath.png` – figure of the bathymetry which was used

`movie.mp4` – video of the simulation