Numerical modeling of ice-sheet dynamics

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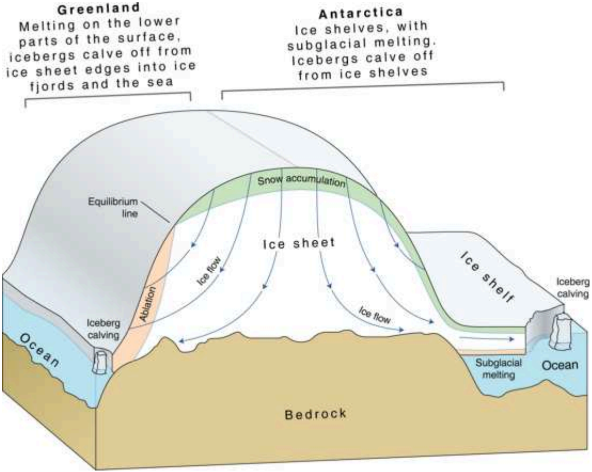
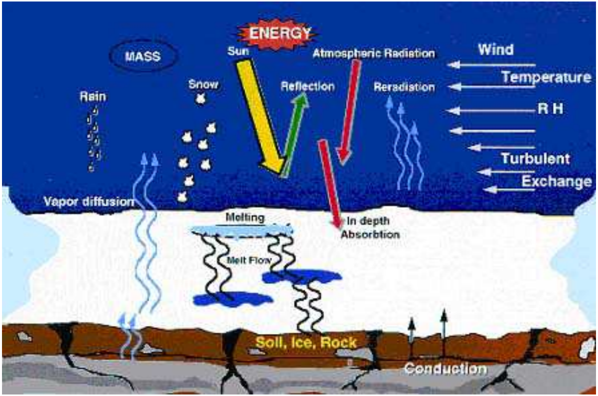
Overview

- Ice-sheets and their dynamics
- Continuum thermo-mechanical model of a glacier
- The Shallow Ice approximation
- The SIA-I iterative algorithm
- Numerical benchmarks
  - ISMIP-HOM A exp. - steady-state
  - ISMIP-HOM F exp. - prognostic
  - EISMINT - Greenland Ice Sheet models

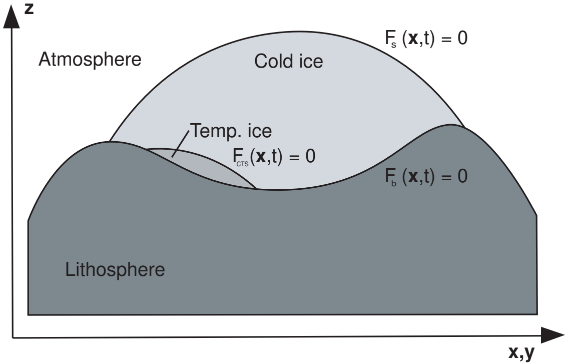
Scheme of an ice sheet



Transport processes in an ice sheet



Simplified ice sheet model



- Cold-ice zone (pure ice only)
- Temperate-ice zone (liquid water present)
- Free surface and glacier base represented by differentiable surfaces

Modelled problem - cold ice zone

- Stokes' flow problem

$$\mathrm{div}\boldsymbol{\tau} + \rho\vec{g} = \vec{0}$$

- Equation of continuity (Constant homogeneous ice density)

$$\mathrm{div}\vec{v} = 0$$

- Energy balance – Heat transport equation

$$\rho c(T)\dot{T} = \mathrm{div}(\kappa(T)\mathrm{grad}\,T) + \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}$$

Rheology

$$\begin{aligned}
\tau_{ij} &= -p\delta_{ij} + \boldsymbol{\sigma}_{ij} \\
\boldsymbol{\sigma}_{ij} &= 2\eta\dot{\boldsymbol{\varepsilon}}_{ij} \\
\dot{\boldsymbol{\varepsilon}}_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\
\eta &= \frac{1}{2}A(T')^{-\frac{1}{3}}\dot{\varepsilon}_{II}^{-\frac{2}{3}} \\
A(T') &= A_0 \exp\left(-\frac{Q}{R(T_0 + T')}\right) \\
\dot{\varepsilon}_{II} &= \sqrt{\frac{\dot{\varepsilon}_{kl}\dot{\varepsilon}_{kl}}{2}}
\end{aligned}$$

Boundary conditions

Free surface

- **Kinematic boundary condition**

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \mathrm{grad} f_s = a^s$$

- **Dynamic boundary condition**
  - Traction-free

$$\boldsymbol{\tau} \cdot \vec{n}_s = \vec{0}$$

  - Surface temperature

$$T = T^s(\vec{x}, t)$$

  - Accumulation-ablation function

$$a^s = a^s(\vec{x}, t)$$

Glacier base

- **Kinematic boundary condition $f_b(\vec{x}, t)$ given**
- **Dynamic boundary conditions**
  - Frozen-bed conditions, $T < T_M$:

$$\vec{v} = \vec{0}$$

  - Sliding law $T = T_M$:

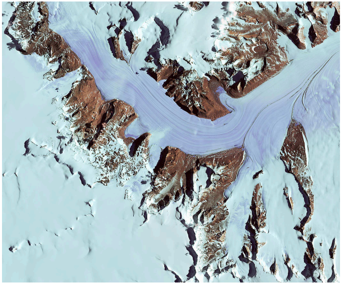
$$\vec{v} = \vec{g}(\boldsymbol{\tau} \cdot \vec{n})$$

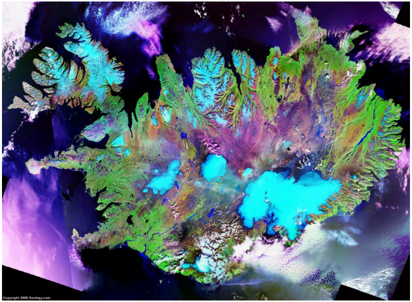
  - Geothermal heat flux

$$\vec{q}^{+} = \vec{q}^{geo}(\vec{x}, t)$$

Scaling Approximation – plausibility and motivation

- Glaciers and ice sheets are typically very flat features, with the aspect (height:horizontal) ratio less than $\frac{1}{10}$ for small valley glaciers and one order less for big ice sheets ($\frac{1}{100}$)





Scaling Approximation – plausibility and motivation

- Natural scaling may be introduced:
  kinematic quantities

$$
\begin{aligned}
(x_1, x_2, x_3) &= ([L]\tilde{x}_1, [L]\tilde{x}_2, [H]\tilde{x}_3) \\
(v_1, v_2, v_3) &= ([V_{\leftrightarrow}]\tilde{v}_1, [V_{\leftrightarrow}]\tilde{v}_2, [V_{\updownarrow}]\tilde{v}_3) \\
(f_s(x_1, x_2), f_b(x_1, x_2)) &= [H](\tilde{f}_s(\tilde{x}_1, \tilde{x}_2), \tilde{f}_b(\tilde{x}_1, \tilde{x}_2)) \\
\varepsilon &= \frac{[H]}{[L]} = \frac{[V_{\updownarrow}]}{[V_{\leftrightarrow}]}
\end{aligned}
$$

dynamic quantities -

$$
\begin{aligned}
p &= \rho g [H]\tilde{p} \\
(\sigma_{13}, \sigma_{23}) &= \rho g [H](\tilde{\sigma}_{13}, \tilde{\sigma}_{23}) \\
(\sigma_{11}, \sigma_{22}, \sigma_{12}) &= \rho g [H](\tilde{\sigma}_{11}, \tilde{\sigma}_{22}, \tilde{\sigma}_{12})
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dynamic quantities - Shallow Ice Approximation SIA

$$
\begin{aligned}
p &= \rho g [H] \tilde{p} \\
(\sigma_{13}, \sigma_{23}) &= \varepsilon \rho g [H] (\tilde{\sigma}_{13}, \tilde{\sigma}_{23}) \\
(\sigma_{11}, \sigma_{22}, \sigma_{12}) &= \varepsilon^2 \rho g [H] (\tilde{\sigma}_{11}, \tilde{\sigma}_{22}, \tilde{\sigma}_{12})
\end{aligned}
$$

Scaling Approximation – plausibility and motivation

Standard scaling perturbation series constructed: all field unknowns $(\tilde{v}_i, \tilde{\sigma}_{ij}, \tilde{f}_s, \tilde{f}_b)$ are expressed by a power series in the aspect ratio ε

$$\tilde{q} = \tilde{q}^{(0)} + \varepsilon \tilde{q}^{(1)} + \varepsilon^2 \tilde{q}^{(2)} + \ldots$$

and inserting these expressions into governing equations (eq. of motion, continuity, rheology) leads to separation of these equations according to the order of ε. (Implicit assumption: not only $\tilde{q}$ is now scaled to unity but also its gradient (???))

Scaling Approximation – plausibility and motivation

Keeping only lowest-order terms, we arrive at the Shallow Ice Approximation, which can be explicitly solved

$$
\begin{aligned}
p^{(0)}(\cdot, x_3) &= f_s^{(0)}(\cdot) - x_3 \\
\sigma_{13}^{(0)}(\cdot, x_3) &= -\frac{\partial f_s^{(0)}(\cdot)}{\partial x_1}\left(f_s^{(0)}(\cdot) - x_3\right) \\
\sigma_{23}^{(0)}(\cdot, x_3) &= -\frac{\partial f_s^{(0)}(\cdot)}{\partial x_2}\left(f_s^{(0)}(\cdot) - x_3\right)
\end{aligned}
$$

where $(\cdot) \equiv (x_1, x_2)$

Scaling Approximation – plausibility and motivation

Also the velocities may be expressed semi-analytically as

$$v_1^{(0)}(\cdot, x_3) = v_1^{(0)}(\cdot)_{sl} + \int_{f_b^{(0)}(\cdot)}^{x_3} A(T'(\cdot, x_3')) \left({\sigma_{13}^{(0)}}^2 + {\sigma_{23}^{(0)}}^2 \right) \sigma_{13}^{(0)}(\cdot, x_3') dx_3'$$

$$v_2^{(0)}(\cdot, x_3) = v_2^{(0)}(\cdot)_{sl} + \int_{f_b^{(0)}(\cdot)}^{x_3} A(T'(\cdot, x_3')) \left({\sigma_{13}^{(0)}}^2 + {\sigma_{23}^{(0)}}^2 \right) \sigma_{23}^{(0)}(\cdot, x_3') dx_3'$$

v_3 from equation of continuity

$$v_3^{(0)}(\cdot, x_3) = v_3^{(0)}(\cdot)_{sl} - \int_{f_b^{(0)}(\cdot)}^{x_3} \left(\frac{\partial v_1^{(0)}}{\partial x_1}(\cdot, x_3') + \frac{\partial v_2^{(0)}}{\partial x_2}(\cdot, x_3') \right) dx_3'$$

Problems of SIA

- Zeroth order model – looses validity in regions where higher-order terms become important:
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Problems of SIA

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  - Regions of high curvature of the surface
  - Regions where a-priori dynamic scaling assumptions are violated (floating ice – SSA, ice streams)

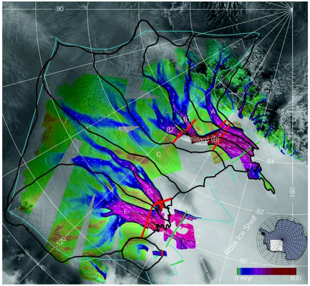


Figure: RADARSAT Antarctic Mapping Project

Solution?

- Higher-order models – continuation of the expansion procedure and solving for higher-order corrections (Pattyn F., Rybak O.)
- Full Stokes Solvers – Finite Elements Methods (Elmer - Gagliardini O.), Spectral methods (Hindmarsch R.)

PROBLEM is the speed of full-Stokes and higher-order techniques, the computational demands disable usage of these techniques in large-scale evolutionary models

Any ideas?

Aim:

- Find an "intermediate" technique that would exploit the scaling assumptions of flatness of ice but would provide "better" solution than SIA:
- KEY IDEA: Apply the scaling assumptions of smallness not on the particular stress components but only on their deviations from the full-Stokes solution

SIA-I

$$\vec{u}^{n} \longrightarrow \delta\vec{u}^{n+\frac{1}{2}} \longrightarrow \vec{u}^{n+\frac{1}{2}} = \vec{u}^{n} + \epsilon_1 \delta u^{n+\frac{1}{2}}$$

$$\downarrow$$

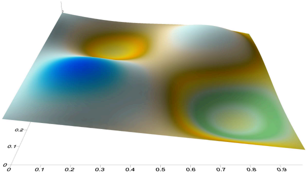
$$\vec{u}^{n+1} = (1-\epsilon_2)\vec{u}^{n+\frac{1}{2}} + \epsilon_2 \vec{u}^{*n+\frac{1}{2}} \longleftarrow \vec{u}^{*n+\frac{1}{2}} \longleftarrow \vec{v}^{n+\frac{1}{2}}$$

Ice-Sheet Model Intercomparison Project - Higher Order Models (ISMIP-HOM) - experiment A

$$f_s(x_1, x_2) = -x_1 \tan\alpha , \qquad \alpha = 0.5^\circ,$$
$$f_b(x_1, x_2) = f_s(x_1, x_2) - 1000 + 500 \sin(\omega x_1) \sin(\omega x_2)$$

with

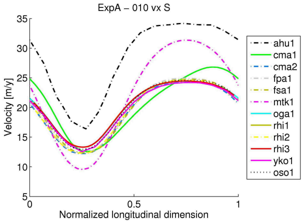
$$\omega = \frac{2\pi}{[L]} .$$

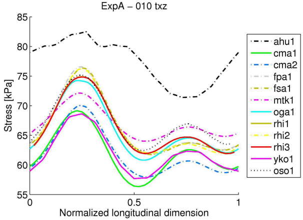


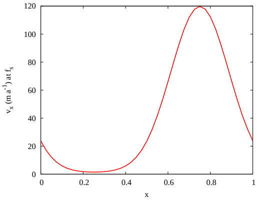
- Ice considered isothermal
- Boundary conditions
  - Free surface
  - Frozen bed
  - At the sides, periodic boundary conditions are prescribed:

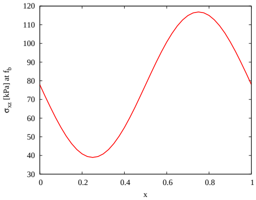
$$\vec{v}(x_1, 0, x_3) = \vec{v}(x_1, [L], x_3)$$
$$\vec{v}(0, x_2, x_3) = \vec{v}([L], x_2, x_3)$$

ISMIP - HOM experiment A









ISMIP - HOM experiment A

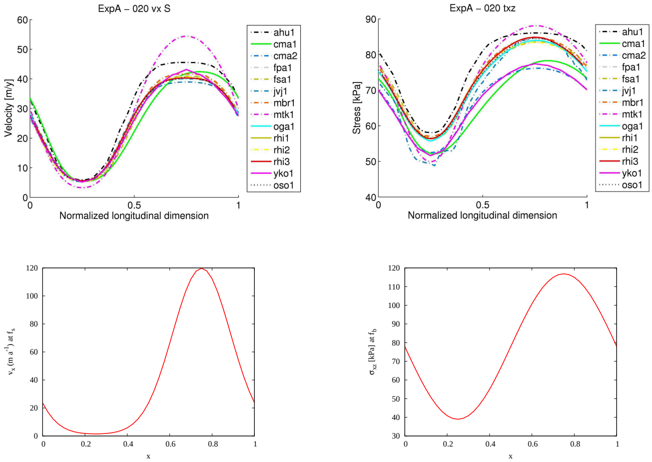


Figure: Pattyn F., ISMIP-HOM results preliminary report

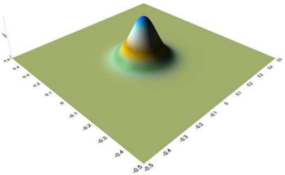
ISMIP - HOM experiment A

Computational details

- Resolution: $41 \times 41 \times 41$
- Number of iterations: $\varepsilon = \frac{1}{20} \sim 40$ iter., $\varepsilon = \frac{1}{10} \sim 100$ iter.
- Each iter. step took approximately 0.22 s at Pentium 4, 3.2.GHz
- For comparison Elmer (Gagliardini) Full-Stokes solver $\sim 10^4$ CPU s

ISMIP-HOM experiment F

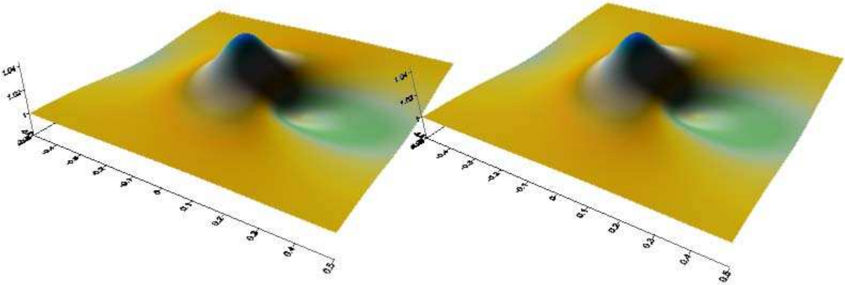
- Experiment description - ice slab flowing downslope (3°) over a Gaussian bump



- Linear rheology!
- Output: Steady state free surface profile and velocities
- Comparison with ISMIP-HOM full-Stokes finite-element solution (Olivier Gagliardini computing by Elmer)

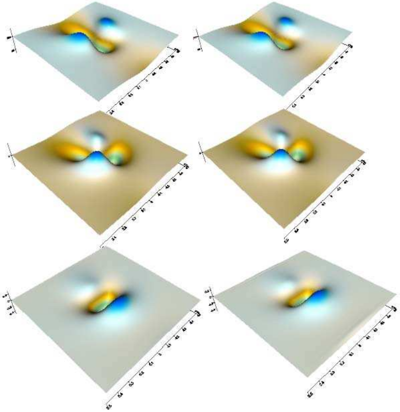
ISMIP-HOM experiment F

Free surface (left - Elmer, right- SIA-I)



ISMIP-HOM experiment F

Surface velocities



Numerical performance

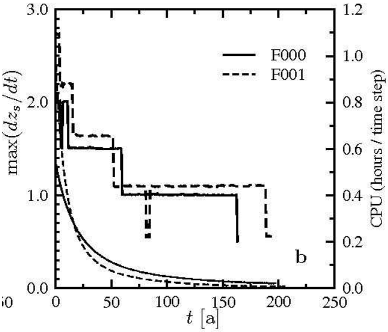
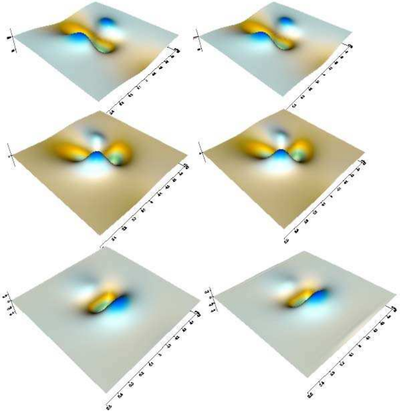


Figure: **Gagliardini & Zwinger, 2008**

ISMIP-HOM experiment F

Surface velocities (left-Elmer, right-SIA-I)



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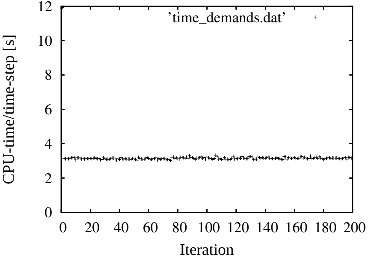
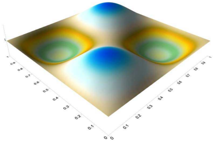


Extension for non-linear rheology

- We take setting from ISMIP-HOM experiment A (L=80) - flow of an inclined ice slab over a sinusoidal bump, periodically elongated

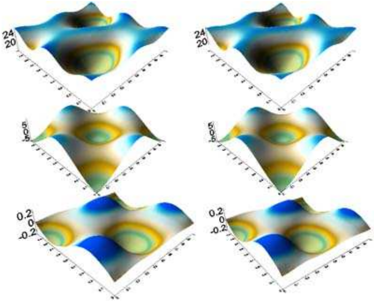


- Evolution of free surface, steady state

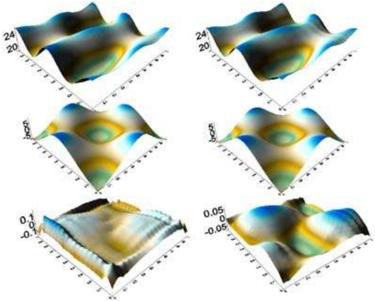
- Impossible to compare with evolution in Elmer (too CPU time demanding)
- Time demands of the SIA-I algorithm practically the same as for the linear case !!!
- Let's compare only velocities at particular time instants

Extension for non-linear rheology

$t = 50a$ (left - Elmer, right - SIA-I)



$t = 100a$ (left - Elmer, right - SIA-I)

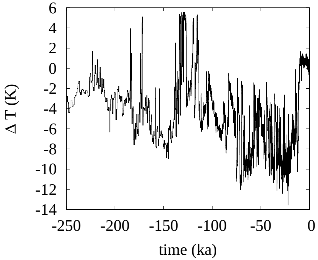


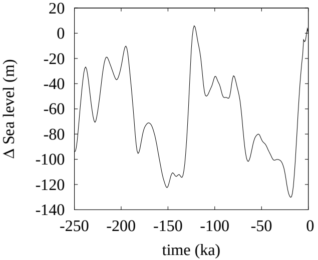
EISMINT benchmark - Greenland Ice Sheet Models

- Paleoclimatic simulation
- Prognostic experiment - global warming scenario

EISMINT Greenland - Paleoclimatic experiment

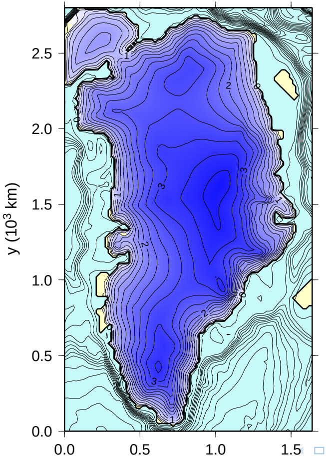
- Two glacial cycles (cca 250 ka)
- Temperature + sea-level forcing based on ice-core $\delta^{18}O$ isotope record





EISMINT Greenland - Paleoclimatic experiment

Surface topography (km), age = 150 ka



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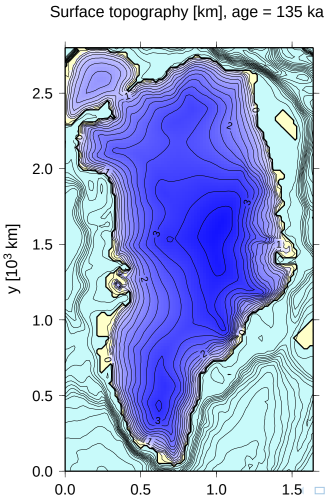

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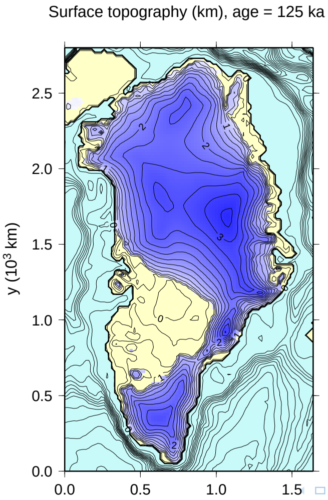

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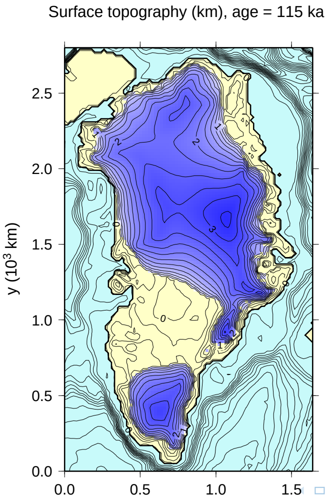
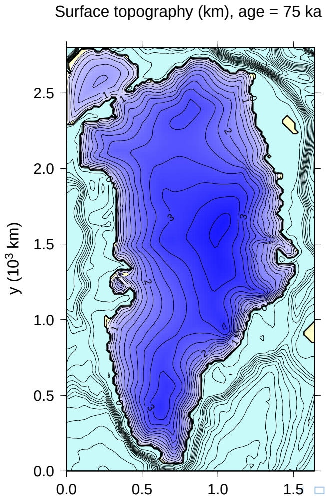


EISMINT Greenland - Paleoclimatic experiment



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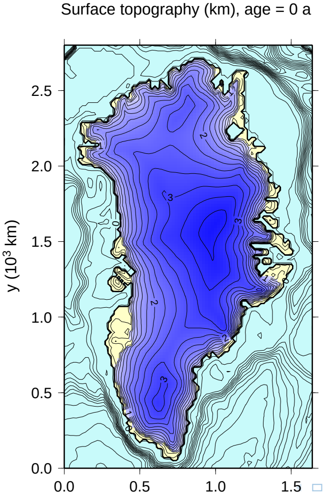

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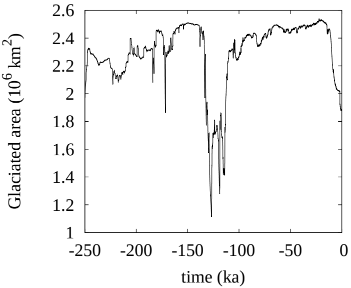
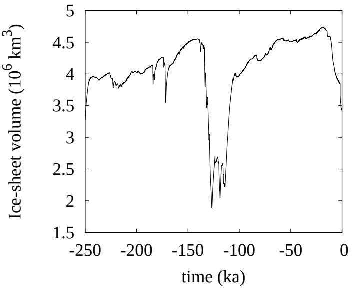
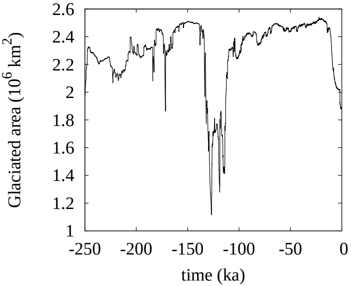
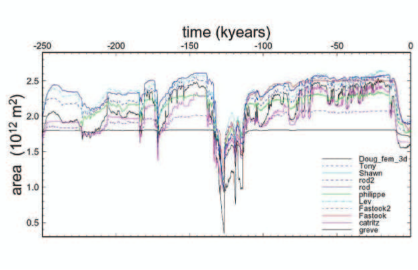




EISMINT Greenland - Paleoclimatic experiment (Huybrechts, 1998)

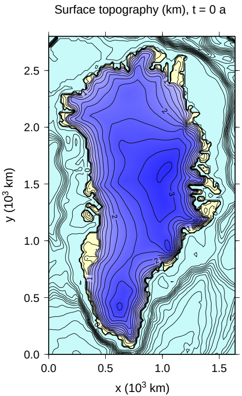




Prognostic experiment

- Model response to an artificial warming scenario
- temperature increase by 0.035° C per year for the first 80 years (total 2.8° C increase)
- by 0.0017° per year C for the remaining 420 years (0.714° C)
- In total temperature increase of 3.514° C within 500 years

Prognostic experiment



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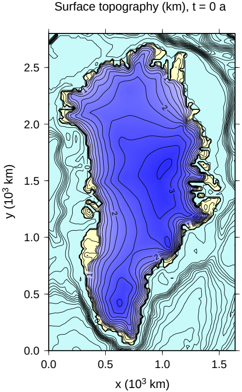

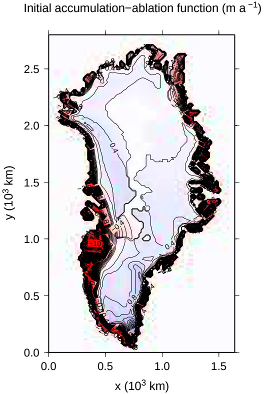

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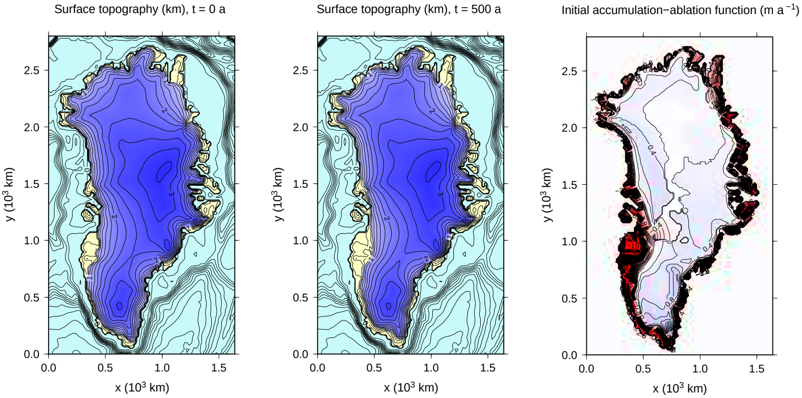

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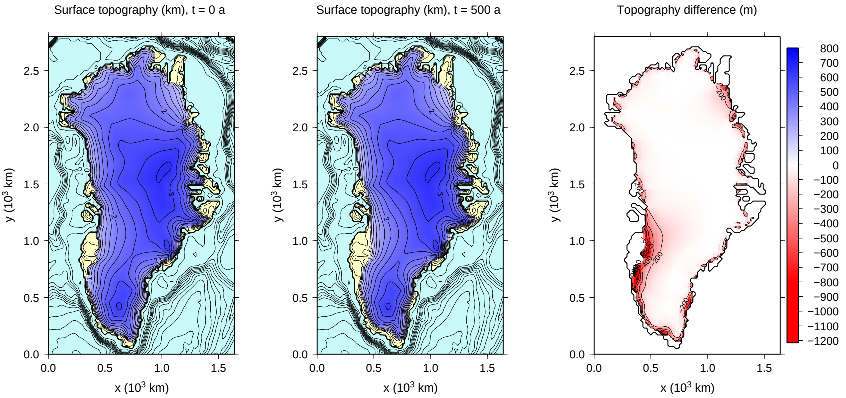

Thank you for your attention!