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VISCOELASTIC RESPONSE OF A FLAT LAYER IN CARTESIAN GEOMETRY

Master Thesis

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Prague 2002

I hereby declare that I have elaborated this master thesis on my own and that the references include all sources of information I have exploited. I agree with lending of this master thesis.

Prague, August 30, 2002

Petr Toman

Abstrakt

Název práce : Viskoelastická odezva rovinné vrstvy v Kartézské geometrii
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Viskoelastická odezva Země, jakožto geofyzikální úloha, pracuje na přechodných časových škálách, tedy od čistě elastického po čistě viskózní modelování. V této práci jsme provedli kompletní geometrickou reformulaci takovéto úlohy ze sférické geometrie do geometrie Kartézské, tedy takové, která se využívá k lokálnímu popisu. Skalární a vektorové harmonické funkce byly nahrazeny Fourierovskými bázovými funkcemi. Stejným postupem jako v metodě normálních modů jsme v případě elasticity dospěli k soustavě hloubkově závislých lineárních obyčejných diferenciálních rovnic prvního řádu. V případě užití viskoelastické reologie, tj. Maxwellovského modelu, jsme postupovali dle metody přímek založené na přímém řešení v čase. Dospěli jsme k soustavě lineárních parciálních diferenciálních rovnic prvního řádu. K převodu na soustavu obyčejných diferenciálních rovnic jsme aplikovali prostorovou diskretizaci. Modální rozklad nám umožnil řešit úlohu vlastních čísel. Numerická implementace nám zprostředkovala řešení úloh a ukazuje průběh důležitých veličin, jako např. horizontální a vertikální posunutí nebo přírůstkový gravitační potenciál.

Klíčová slova : ledovcový výzdvih, viskoelasticita, normální mody, metoda přímek

Abstract

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Viscoelastic response of the Earth as a geophysical problem is operating on the transient temporal scales, between domains of purely elastic and viscous modelling. In this study we present a complete geometrical reformulation of such problem from the spherical approach to the Cartesian approach, which is considered as a model of local description. The scalar and vector spherical harmonics are being replaced by the Fourier scalar and vector basis functions. Following the normal modes approach in the elastic case, we obtain a set of linear first-order differential equations with respect to the depth. For the viscoelastic rheology, i.e., using the Maxwell model of viscoelasticity, we follow method-of-lines approach based on the direct solution of the response in time and we obtain a linear first-order set of partial differential equations. To get ordinary differential equations in time, we enforce discretization in space. Modal decomposition is realized to solve the eigenvalue problem. Numerical modelling provides solution of these problems and shows evolution of the main outcoming quantities, such as vertical and horizontal displacement or the incremental gravitational potential.

Keywords : glacial rebound, viscoelasticity, normal modes, method of lines

Vtom jdouce, trefíme mezi jakési, kteříž plnou síň cifer majíce, přebírali se v nich. Někteří berouc z hromady, rozsazovali je; jiní zase přehršlím shrňujíc, na hromádky kladli; jiní opět z hromádek díl ubírali a obvzlášť sypali; jiní opět ty díly v jedno snášeli; a jiní zase to dělili roznášeli, až jsem se tomu jejich dílu podivil. Oni mezitím vypravovali, jak v celé filozofii jistšího umění nad toto jejich není, tu že nic chybiti, nic ujíti, nic nadbýti nemůž. "Nač pak to umění jest?" řekl jsem. Oni mé hlouposti se podivíc, hned jeden přes druhého divy mi vypravovati začnou. Jeden, že mi poví, kolik husí v stádě letí, nepočítaje jich; druhý, že mi poví, v kolika hodinách čisterna pěti rourkami vyteče; třetí, že mi poví, kolik v měšci grošů mám, nehledáje tam etc., až se jeden našel, kterýž se písek mořský v počet uvésti podvoloval a o tom hned knihu sepsal (Archimedes). Jiný příkladem jeho (ale větší subtýlnosti dokázati chtěje) dal se v počítání v slunci létajícího prášku (Euclides). I užásl jsem se: a oni mi k srozumění posloužiti chtíce, ukazovali své regule, trium, societatis, alligationis, falsi; kterýmž jsem se jakž takž vyrozuměl. Než když mne do nejzadnější, jenž algebra aneb cossa slove, uvésti chtěli, takových jsem tam divokých jakýchsi klik a háků hromady uhlédal, že mne o málo závrat nepopadl: a zavra já oči, prosil jsem, aby mne odtud vedli.

Jan Amos Komenský, *Labyrint světa a ráj srdce*, (1663)

Poděkování

Touto cestou bych velmi rád poděkoval vedoucímu mé diplomové práce doc. RNDr. Ctiradu Matyskovi, DrSc. za vše, co mi pomohlo tuto práci napsat a nejen to. Stal se mi oporou i vzorem ve všech situacích.

Zároveň děkuji konzultantovi mé diplomové práce RNDr. Ladislavu Hanykovi, PhD. za pomoc v nekonečné bitvě s vojsky počítačovými.

Panu doc. RNDr. Jiřímu Zahradníkovi, DrSc. děkuji za vedení katedry geofyziky takovým způsobem, jenž umožnil vzniknout tak vynikajícímu pracovnímu prostředí. Děkuji i celé katedře geofyziky, jejíž přátelská atmosféra je opravdu vyjímečná.

Tato diplmová práce byla podporována z grantu UK č. 175/2000 *Integrační metody modelování viskoelastické odezvy Země*

Mé mamince a Peti...

Contents

List of Symbols and Abbreviations

Symbol	Meaning
t	time
a	scaling length (average radius of the Earth, i.e. 6371km)
$\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$	unit base vectors (orthogonal)
$\boldsymbol{I}$	unit diagonal tensor
$\boldsymbol{n}$	outward unit normal with respect to the boundary
$\boldsymbol{r}$	horizontal position vector
$\boldsymbol{q}$	horizontal wave vector
$\boldsymbol{u}$	displacement vector
$\boldsymbol{\tau}_0$	hydrostatic Cauchy stress tensor
ρ_0	hydrostatic density
φ_0	hydrostatic gravitational potential
p_0	hydrostatic mechanical pressure
$\boldsymbol{e}$	strain tensor
$\boldsymbol{\tau}$	incremental Cauchy stress tensor
$\boldsymbol{f}$	total volume force acting on the continuum
φ_1	incremental gravitational potential
ρ_1	Eulerian incremental density
$\boldsymbol{\tau}^E$	elastic part of incremental Cauchy stress tensor
$\boldsymbol{y}^E$	elastic solution vector
$\boldsymbol{y}$	viscoelastic solution vector
λ	elastic Lammé parameter
μ	shear modulus
K	bulk modulus (isentropic incompressibility)
η	viscosity
ξ	inverse Maxwell time
γ_L	interface density (prescribed load)
Π	mean normal stress (pressure)
1-D	one-dimensional, here depth-dependent
2-D	two-dimensional, here
BV	boundary value
FL	flat layer
IV	Initial Value method
l.h.s.	left-hand side
MOL	method of lines
NM	normal modes
ODE	ordinary differential equation
PDE	partial differential equation
PREM	preliminary reference model
r.h.s	right-hand side

Úvod. Invitatio. Einleitung.

Chapter 1

Introduction

1.1 Postglacial Rebound

In the presented thesis we deal with a subject of the local response of the Earth (i.e. *flat layer*) to an arbitrary external *surface load*. Such problem is a long-standing concern in geophysics (e.g. *Peltier*, 1974; *Farrell*, 1972). Interest in the problem has been motivated in large part by studies of the long-term adjustment of the Earth as a consequence of the late Pleistocene glacial cycles.

The response to surface loading is characterized in accordance with observations by fast perturbations of an initial state during progressive changes of surface load, followed by a slower relaxation towards a new state of isostatic equilibrium after stabilization of the load. This behaviour can be well described by viscoelastic rheologies with the preference given to that of the Maxwell solid. When referring to the modelling of the viscoelasticity, we will first follow the normal-mode (NM) approach developed by Prof. W.R. Peltier and his colleagues (e.g. *Peltier*, 1974; *Wu & Peltier*, 1982; etc.) and then we will employ initial value/method of lines (IV/MOL) approach and eigenvalue analysis developed by dr. L. Hanyk (e.g. *Hanyk*, 1999; *Hanyk et al.*, 2002; etc.).

By the term *flat layer* (FL) we are understanding non-rotating, self-gravitating, horizontally infinite layer, responding in time as an elastic, viscoelastic or viscous continuum, or a combination of these. *Surface loading*, which is expected to be large enough and not too distant in the past to initiate the observable response at present, has been achieved by glaciers during the last Pleistocene epoch. The whole processes leading to the resulting equilibrium is named *glacial isostatic adjustment* (GIA).

The basic idea of the NM approach is to decompose the response into a set of normal modes (NM). Field partial differential equation (PDEs) governing the viscoelastic response of the Earth are transformed into the Laplace domain and are subjected to spherical harmonic decomposition in the spherical geometry approach or subjected to Fourier horizontal decomposition in the approach presented in this thesis. This leads to the system of ordinary differential equations (ODEs) with respect to the depth:

$$\frac{d}{dr}\boldsymbol{y}(s,r) = \boldsymbol{A}(s,r)\boldsymbol{y}(s,r), \tag{1.1}$$

with r denoting the depth and s the Laplace variable. The solution vector $\boldsymbol{y}$ incorporates coefficients of scalar representation of the unknown physical quantities, i.e, the displacement vector, the incremental stress tensor and the incremental gravitational potential. Matrix $\boldsymbol{A}$ represents a linear operator acting on the this system. This approach is similar to that from the theory of free oscillations (e.g. *Martinec*, 1984) and the theory of Earth tides (e.g. *Novotný*, 1998). The time dependence of the PDEs in the viscoelastic cases gives for the resulting ODEs parameterization by Laplacian variable s. In the presented thesis we employ the NM approach only for the case of elastic response, i.e., we do not use the Laplace transform.

However, in the middle of the 90's was an extensive discussion of positive and negative features of the NM approach in progress. The question was how to accurately invert the complex Laplace spectra of compressible realistic Earth models back into the time domain. With these difficulties, the idea of numerical solution to the problem entirely in the time domain appeared quite naturally. The time derivatives in the constitutive relation of Maxwell viscoelasticity had been replaced by the finite-differences (FD) formulas. In the same time the complete initial-value (IV) formulation of the forward[1] GIA problem was to be appear. In this formulation of

[1] The forward problem of GIA expresses evaluation of observable response functions (i.e. vertical/horizontal displacements and gravitational anomalies) from the prescribed model, parameters (i.e. density, elastic parameters and viscosity) and from the model of a surface load.

the IV approach replacement of time derivatives by the FD formulas leads to a system of linear non-homogenous ODEs:

$$\frac{d}{dr}\boldsymbol{y}(t,r) = \boldsymbol{A}(t,r)\boldsymbol{y}(t,r) + \boldsymbol{q}(t,r) \tag{1.2}$$

$$\frac{d}{dt}\boldsymbol{q}(t,r) = \frac{\mu(r)}{\eta(r)}\left[\tilde{\boldsymbol{Q}}\boldsymbol{y}(t,r) + \tilde{\tilde{\boldsymbol{Q}}}(r)\boldsymbol{q}(t,r)\right], \tag{1.3}$$

with t time, μ the shear modulus and η the viscosity. Matrices $\tilde{\boldsymbol{Q}}$ and $\tilde{\tilde{\boldsymbol{Q}}}$ can be found in *Hanyk*, 1999, equations (7.36) and (6.39). But again, different problems[2] appeared. The new formulation of the IV approach overcame those difficulties. The system (1.2)–(1.3) was replaced by the differential system with respect to both time and the depth which allowed to make use of the method of lines (MOL):

$$\frac{d}{dt}\left[\frac{d}{dr}\boldsymbol{y}(t,r) - \boldsymbol{A}(r)\boldsymbol{y}(t,r)\right] = \frac{\mu(r)}{\eta(r)}\left[\boldsymbol{D}(r)\frac{d}{dr}\boldsymbol{y}(t,r) + \boldsymbol{E}(r)\boldsymbol{y}(t,r)\right]. \tag{1.4}$$

In harmony with *Hanyk*, 1999 we use this approach to obtain a local response of the system to an arbitrary surface load for the case of viscoelastic response.

1.2 Overview of the Thesis

The main task rests upon reformulation of the formalism developed by e.g. *Peltier*, 1974; *Wu & Peltier*, 1982 and last but not least by *Hanyk*, 1999; *Hanyk et al.*, 1998; *Hanyk et al.*, 2002. The physical quantities and the equations are being transferred and rewritten from the spherical harmonic representation (i.e. global description in spherical coordinates) into the Fourier modes representation (i.e. local description in Cartesian coordinates). We employ the approach of the method of lines to the governing PDEs, i.e., we discretize the equations in space. This results in a system of time-dependent ODEs, which is fundamentally different from the modal approach where the time dependence is dealt first.

In Chapter 2 we collect a system of field PDEs governing the response of gravitating, compressible model of flat layer to surface loading. With the elastic constitutive relation considered first, the field PDEs are converted into a system of ODEs with respect to the depth by the means of the Fourier expansion analysis. To find analytical solution of the ODE system for homogenous and incompressible model, we follow analysis presented by *Wu & Peltier*, 1982. In Chapter 3 we switch to the constitutive relation of the Maxwell viscoelasticity which introduces time evolution into the system. To derive a linear first-order system of PDEs with respect to both time and the depth, we carefully reproduce the process given for the case of the elastic rheology. In Chapter 4 we will enforce discretization in space to get ODEs in time. Then we will realize modal decomposition to solve the eigenvalue problem and show how the main Fortran codes work in the way of simplified schemes. Chapter 5 summarizes the results developed in Chapters 2–4 and shows the main outcomes. Formulation of conclusions and final remarks can be found in Chapter 6. Appendices represent a complete guide to the calculus used in both Chapters 2 and 3. In the Appendix A we show how the Fourier transform affects physical quantities through its properties. In the Appendix B we present the exact steps, which may not be clear at the first glance when going through Chapters 2 and 3.

[2] A reader interested in detailed historical and problem review should look into monograph by *Hanyk*, 1999.

Země. Terra. Die Erde.

Chapter 2

Elastic Response of a Flat Layer

In this chapter we collect a fundamental set of field partial differential equations (PDEs) of gravitational elastodynamics for incremental field variables, describing the response of the flat layer to the unit surface load. We derive a system of ordinary differential equations (ODEs) for the Fourier expansion coefficients of field variables from the field PDEs in the special case of elastic, isotropic, non-rotating model of flat layer.

2.1 Field Partial Differential Equations

A gravitating, compressible, non-rotating continuum in hydrostatic equilibrium is considered. Although the effects of rotation and non-hydrostatic pre-stress may be incorporated into the this theory, we omit them in this analysis. The reason is that they are believed to be insignificant in the context of glacial rebound (*Tromp & Mitrovica*, I., 1999). It is conventional to decompose total fields into initial and incremental parts. The incremental fields are employed for description of infinitesimal, quasi-static, gravitational-viscoelastic perturbations of the initial fields[1].

The equation of motion and the Poisson equation for the initial state of the above described continuum consist the terms of the initial (Cauchy) stress tensor $\boldsymbol{\tau}_0$, the initial gravitational potential φ_0, the initial density distribution ρ_0 and the forcing term $\boldsymbol{f}_0$,

$$\nabla \cdot \boldsymbol{\tau}_0 + \boldsymbol{f}_0 = 0, \tag{2.1}$$

$$\Delta \varphi_0 - 4\pi G \rho_0 = 0, \tag{2.2}$$

where G is the Newton gravitational constant. The boundary conditions at the surface and all internal boundaries require the continuity of the normal initial stress,

$$[\boldsymbol{n} \cdot \boldsymbol{\tau}_0]_-^+ = 0,$$

of the initial gravitational potential,

$$[\varphi_0]_-^+ = 0,$$

and normal component of its gradient,

$$[\boldsymbol{n} \cdot \nabla \varphi_0]_-^+ = 0,$$

where $\boldsymbol{n}$ is the outward unit normal with respect to the boundary. Moreover, the tangential stress should vanish at liquid boundaries and at the surface, i.e.

$$\boldsymbol{n} \cdot \boldsymbol{\tau}_0 = (\boldsymbol{n} \cdot \boldsymbol{\tau}_0 \cdot \boldsymbol{n})\boldsymbol{n}.$$

The assumption of the hydrostatic initial stress requires no deviatoric stresses,

$$\boldsymbol{\tau}_0 = -p_0 \boldsymbol{I}, \tag{2.3}$$

where p_0 is the mechanical pressure and $\boldsymbol{I}$ is the unit diagonal tensor. The force f_0 is taken to be equal to the gravity force per unit volume,

$$f_0 = -\rho_0 \nabla \varphi_0. \tag{2.4}$$

Because of the depth-dependent distribution of density, $\rho_0(z)$, where z is the depth, all initial fields become only depth-dependent. We introduce the gravitational acceleration $g_0(z)$ by the relation

$$g_0 \boldsymbol{e}_z = \nabla \varphi_0. \tag{2.5}$$

Hence the equations (2.1)–(2.2) reduce into the form,

$$p_0' + \rho_0 g_0 = 0, \tag{2.6}$$

$$g_0' - 4\pi G \rho_0 = 0, \tag{2.7}$$

[1]An inquisitive reader should look into e.g. *Martinec*, 1984.

where (B.3) and (B.4) have been used. The prime $'$ stands as a symbol for differentiation with respect to z.

The incremental fields include the displacement vector $\boldsymbol{u}$, the incremental Cauchy stress tensor $\boldsymbol{\tau}$, the incremental gravitational potential φ_1 and the incremental density ρ_1. Let us recall that the incremental gravitational potential φ_1 has its origin in both the internal mass redistribution and the gravitational forcing of the applied load. We also must not forget, that it is necessary to adopt the incremental fields for the concept of Lagrangian and Eulerian formulations. Let us mention that the Lagrangian description relates the current value of a field at the material point to the initial position of that point and that the Eulerian description relates the field to the current, local position[2]. With this conventional casting (the material-local form) and with the state that $\boldsymbol{\tau}$ is in Lagrangian description and φ_1 and ρ_1 are in Eulerian description, the equations for the incremental fields (i.e., for infinitesimal, quasistatic perturbations) take the form as follows[3] :

$$\nabla \cdot \boldsymbol{\tau} + \boldsymbol{f} = \boldsymbol{0}, \tag{2.8}$$

$$\Delta\varphi_1 - 4\pi G\rho_1 = 0, \tag{2.9}$$

where the forcing term $\boldsymbol{f}$ and the Eulerian incremental density ρ_1 are

$$\boldsymbol{f} = -\rho_0\nabla\varphi_1 - \rho_1\nabla\varphi_0 - \nabla(\rho_0\boldsymbol{u}\cdot\nabla\varphi_0), \tag{2.10}$$

$$\rho_1 = -\nabla\cdot(\rho_0\boldsymbol{u}). \tag{2.11}$$

The individual terms on the right-hand side of the expression for the force $\boldsymbol{f}$ represent in sequence: **selfgravitation**, **buoyancy** and **prestress**. Introducing the strain tensor $\boldsymbol{e}$, cf. (B.28),

$$\boldsymbol{e} = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right], \tag{2.12}$$

we can write the linearized constitutive relation in the form

$$\boldsymbol{\tau}^E = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + 2\mu\boldsymbol{e} = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + \mu\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right], \tag{2.13}$$

with λ and μ the elastic Lammé parameters. They are both related to the bulk modulus or the isentropic incompressibility K through the relation

$$K = \lambda + \tfrac{2}{3}\mu. \tag{2.14}$$

The superscript ^{E} is applied for the "elastic part" of any tensor, which could be written in the form of (2.13). According to *Peltier*, 1974 we rewrite the constitutive relation of Maxwell viscoelasticity as

$$\dot{\boldsymbol{\tau}} + \frac{\mu}{\eta}(\boldsymbol{\tau} - K\nabla\cdot\boldsymbol{u}\boldsymbol{I}) = \dot{\boldsymbol{\tau}}^E, \tag{2.15}$$

where η is the dynamic viscosity and the dots denote differentiation with respect to time t.

The internal boundary conditions for the incremental fields require continuity of the displacement, the incremental stress, the incremental gravitational potential and also its gradient, i.e.,

$$[\boldsymbol{u}]_-^+ = \boldsymbol{0}, \qquad [\boldsymbol{\tau}]_-^+ = \boldsymbol{0}, \qquad [\varphi_1]_-^+ = 0, \qquad [\nabla\varphi_1]_-^+ = \boldsymbol{0},$$

[2] More about theoretical aspects of continuum mechanics can be found in *Brdička*, 1999 or *Martinec*, 2000.

[3] The exact steps leading to the equations (2.8)–(2.9) can be found e.g. in monograph *Wolf*, 1997.

and zero tangential stress at liquid boundaries,

$$\boldsymbol{n} \cdot \boldsymbol{\tau}_0 = (\boldsymbol{n} \cdot \boldsymbol{\tau}_0 \cdot \boldsymbol{n})\boldsymbol{n}.$$

Under the presence of the load prescribed by the interface density γ_L, the surface boundary conditions for the incremental stress and gradient of the incremental gravitational potential balance the applied load take the form

$$\begin{aligned} [\boldsymbol{n} \cdot \boldsymbol{\tau}]_-^+ &= -g_0 \gamma_L \boldsymbol{n}, \\ [\boldsymbol{n} \cdot (\nabla \varphi_1 + 4\pi G \rho_0 \boldsymbol{u})]_-^+ &= -4\pi G \gamma_L. \end{aligned}$$

Let us now summarize the equations. The fundamental system of the field PDEs for the incremental fields $\boldsymbol{u}, \varphi_1$ and $\boldsymbol{\tau}$ in the continuum with the Maxwell viscoelastic rheology and depth-dependent distribution of density ρ_0 takes the form

$$\nabla \cdot \boldsymbol{\tau}^E + \boldsymbol{f} = 0, \tag{2.16}$$

$$\boldsymbol{f} = -\rho_0 \nabla \varphi_1 + \nabla \cdot (\rho_0 \boldsymbol{u})\, g_0 \boldsymbol{e}_z - \nabla (\rho_0 g_0 \boldsymbol{e}_z \cdot \boldsymbol{u}),$$

$$\nabla \cdot (\nabla \varphi_1 + 4\pi G \rho_0 \boldsymbol{u}) = 0, \tag{2.17}$$

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}^E - \frac{\mu}{\eta}(\boldsymbol{\tau} - K \nabla \cdot \boldsymbol{u} \boldsymbol{I}). \tag{2.18}$$

In the following chapters we elaborate this system of PDEs further. Depth-dependent distribution of density and elastic Lammé parameters is considered. For this case we apply in the following section the Fourier decomposition to (2.16)–(2.17) together with the elastic constitutive relation, i.e. $\boldsymbol{\tau} = \boldsymbol{\tau}^E$.

2.2 Fourier Decomposition

Let $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ be the unit basis vectors of a Cartesian system x, y and z, denoting the two horizontal coordinates and the depth, respectively. Let us assume, that the direction of vector $\boldsymbol{e}_z$ is thought to be opposite to the direction of gravitational acceleration vector $\boldsymbol{g}_0 = -\nabla \varphi_0$, i.e., the vector $\boldsymbol{e}_z$ is pointing upwards. We introduce a scalar basis function, which allows expansion of scalar fields:

$$B_{kl} = e^{i\boldsymbol{q} \cdot \boldsymbol{r}}, \tag{2.19}$$

where $\boldsymbol{q} = (k, l)$ denotes a horizontal wave vector and $\boldsymbol{r} = (x, y)$ denotes a horizontal position vector. As we will see in the following chapters, it is very useful to define the dimensionless magnitude N of the wave vector $\boldsymbol{q}$,

$$|\boldsymbol{q}|^2 = k^2 + l^2 \equiv \frac{N}{a^2}, \tag{2.20}$$

where a is a scaling length, e.g., the average radius of the Earth ($a = 6371$km). We also introduce the vector expansion functions:

$$\boldsymbol{G}_{kl}^{(-1)} \equiv B_{kl} \boldsymbol{e}_z = e^{i\boldsymbol{q} \cdot \boldsymbol{r}} \boldsymbol{e}_z, \tag{2.21}$$

$$\boldsymbol{G}_{kl}^{(1)} \equiv a\,(\nabla B_{kl}) = iake^{i\boldsymbol{q} \cdot \boldsymbol{r}} \boldsymbol{e}_x + iale^{i\boldsymbol{q} \cdot \boldsymbol{r}} \boldsymbol{e}_y, \tag{2.22}$$

$$\boldsymbol{G}_{kl}^{(0)} \equiv a\,(\boldsymbol{e}_z \times \nabla B_{kl}) = -iale^{i\boldsymbol{q} \cdot \boldsymbol{r}} \boldsymbol{e}_x + iake^{i\boldsymbol{q} \cdot \boldsymbol{r}} \boldsymbol{e}_y. \tag{2.23}$$

This orthogonal set of vector functions allows expansions of vector fields into the spheroidal and toroidal components. As we can see, the scalar basis function (2.19) and vector basis function (2.21) are automatically dimensionless. On the other hand, we had to rescale expressions for vector basis function (2.22)–(2.23), i.e., multiply by the scaling factor a to keep them dimensionless.

Since we assume 1-D depth-dependent distributions of density and Lamé parameters, $\rho_0(z), \lambda(z)$ and $\mu(z)$, the reference gravitational acceleration is also only depth-dependent, $g_0 = g_0(z)$. We can write the following expansions of the scalar and vector functions φ_1 and $\boldsymbol{u}$,

$$\boldsymbol{u} = \sum_{kl} \left[U_{kl} \boldsymbol{G}_{kl}^{(-1)} + V_{kl} \boldsymbol{G}_{kl}^{(1)} + W_{kl} \boldsymbol{G}_{kl}^{(0)} \right], \tag{2.24}$$

$$\varphi_1 = \sum_{kl} F_{kl} B_{kl}. \tag{2.25}$$

According to the expansions and expressions shown in Appendix A, the following expansions of $\nabla \cdot \boldsymbol{u}, \boldsymbol{e}_z \cdot \boldsymbol{\tau}^E, \nabla \cdot \boldsymbol{\tau}^E, \boldsymbol{f}$ and $\nabla \cdot (\nabla\varphi_1 \ + \ 4\pi G \rho_0 \boldsymbol{u})$ can be found :

$$\nabla \cdot \boldsymbol{u} = \sum_{kl} X_{kl} B_{kl}, \tag{2.26}$$

$$\boldsymbol{e}_z \cdot \boldsymbol{\tau}^E = \sum_{kl} \left[T_{z,kl}^{E(-1)} \boldsymbol{G}_{kl}^{(-1)} + T_{z,kl}^{E(1)} \boldsymbol{G}_{kl}^{(1)} + T_{z,kl}^{E(0)} \boldsymbol{G}_{kl}^{(0)} \right], \tag{2.27}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{\tau}^E = \sum_{kl} \Big[& \left(T_{z,kl}^{E(-1)\prime} - \tfrac{N}{a} T_{z,kl}^{E(1)} \right) \boldsymbol{G}_{kl}^{(-1)} \\ + & \left(T_{z,kl}^{E(1)\prime} + \tfrac{\lambda}{a\beta} T_{z,kl}^{E(-1)} - \tfrac{N}{a^2}(\gamma + \mu) V_{kl} \right) \boldsymbol{G}_{kl}^{(1)} \\ + & \left(T_{z,kl}^{E(0)\prime} - \mu \tfrac{N}{a^2} W_{kl} \right) \boldsymbol{G}_{kl}^{(0)} \Big], \end{aligned} \tag{2.28}$$

$$\begin{aligned} \boldsymbol{f} = \sum_{kl} \Big[& \left(-\rho_0 g_0 \tfrac{N}{a} V_{kl} - \rho_0 Q_{kl} \right) \boldsymbol{G}_{kl}^{(-1)} \\ + & \tfrac{1}{a} \Big(-\rho_0 F_{kl} - \rho_0 g_0 U_{kl} \Big) \boldsymbol{G}_{kl}^{(1)} \Big], \end{aligned} \tag{2.29}$$

$$\nabla \cdot (\nabla\varphi_1 \ + \ 4\pi G \rho_0 \boldsymbol{u}) = \sum_{kl} \left[Q_{kl}' - \tfrac{N}{a^2} F_{kl} - 4\pi G \rho_0 \tfrac{N}{a} V_{kl} \right] B_{kl}, \tag{2.30}$$

where the following relations hold,

$$X_{kl} = U_{kl}' - \tfrac{N}{a} V_{kl}, \tag{2.31}$$

$$T_{z,kl}^{E(-1)} = 2\mu U_{kl}' + \lambda X_{kl} \ = \ \beta U_{kl}' - \lambda \tfrac{N}{a} V_{kl}, \tag{2.32}$$

$$T_{z,kl}^{E(1)} = \mu \left(V_{kl}' + \tfrac{1}{a} U_{kl} \right), \tag{2.33}$$

$$T_{z,kl}^{E(0)} = \mu W_{kl}', \tag{2.34}$$

$$Q_{kl} = F_{kl}' + 4\pi G \rho_0 U_{kl}, \tag{2.35}$$

with $\beta = \lambda + 2\mu, \gamma = \mu(3\lambda + 2\mu)/\beta$ and $N = a^2(k^2 + l^2)$. As we can see, none of these expressions contains second or higher-order derivatives of the coefficients in (2.24)–(2.25).

It is also useful to have Fourier expansion for the forcing term $\boldsymbol{f}$ without selfgravitation, cf. (2.10) and (2.16),

$$\boldsymbol{f} = \sum_{kl} \left[\left(-4\pi G \rho_0^2 U_{kl} - \rho_0 g_0 \tfrac{N}{a} V_{kl} \right) \boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{a} \left(-\rho g_0 U_{kl} \right) \boldsymbol{G}_{kl}^{(1)} \right]. \tag{2.36}$$

2.3 Ordinary Differential Equations for an Elastic Flat Layer

We introduce vector $\boldsymbol{y}_{kl}^E$ with 8 elements,

$$\boldsymbol{y}_{kl}^E = \left(U_{kl}, V_{kl}, T_{z,kl}^{E(-1)}, T_{z,kl}^{E(1)}, F_{kl}, Q_{kl}, W_{kl}, T_{z,kl}^{E(0)} \right)^T, \tag{2.37}$$

and an 8×8 matrix $\boldsymbol{A}_N \equiv (a_{kl,ij})$ to be expressed later. As all the expansions are decoupled with respect to both wavevector components k and l, we suppress these subscripts throughout this section when referring to the elements of $y^E_{kl,i}$ and $a_{kl,ij}$ of $\boldsymbol{y}^E_{kl}$ and $\boldsymbol{A}_N$, respectively.

Substituting $\boldsymbol{y}^E_{kl}$ into (2.31)–(2.35) we arrive at,

$$y_1^{E\prime} = \sum_j a_{1j} y_j^E, \qquad a_{1,1..8} = \left(0, \frac{\lambda N}{\beta a}, \frac{1}{\beta}, 0, 0, 0, 0, 0\right), \tag{2.38}$$

$$y_2^{E\prime} = \sum_j a_{2j} y_j^E, \qquad a_{2,1..8} = \left(-\frac{1}{a}, 0, 0, \frac{1}{\mu}, 0, 0, 0, 0\right), \tag{2.39}$$

$$y_5^{E\prime} = \sum_j a_{5j} y_j^E, \qquad a_{5,1..8} = \left(-4\pi G\rho_0, 0, 0, 0, 1, 0, 0\right), \tag{2.40}$$

$$y_7^{E\prime} = \sum_j a_{7j} y_j^E, \qquad a_{7,1..8} = \left(0, 0, 0, 0, 0, 0, 0, \frac{1}{\mu}\right), \tag{2.41}$$

where $\sum_j$ denotes $\sum_{j=1}^8$. In the next step we rewrite equations (2.28)–(2.29) into the form:

$$\begin{aligned}\nabla \cdot \boldsymbol{\tau}^E = \sum\nolimits_{kl} \Big[& \Big(y_3^{E\prime} - \sum\nolimits_j b_{3j} y_j^E\Big) \boldsymbol{G}_{kl}^{(-1)} \\ + & \Big(y_4^{E\prime} - \sum\nolimits_j b_{4j} y_j^E\Big) \boldsymbol{G}_{kl}^{(1)} \\ + & \Big(y_8^{E\prime} - \sum\nolimits_j b_{8j} y_j^E\Big) \boldsymbol{G}_{kl}^{(0)} \Big], \end{aligned} \tag{2.42}$$

$$\boldsymbol{f} = \sum\nolimits_{kl} \Big[-\sum\nolimits_j c_{3j} y_j^E \boldsymbol{G}_{kl}^{(-1)} - \sum\nolimits_j c_{4j} y_j^E \boldsymbol{G}_{kl}^{(1)} \Big] \tag{2.43}$$

with the auxiliary coefficients, b_{ij} and c_{ij}, given by,

$$b_{3,1..8} = \left(0, 0, 0, \frac{N}{a}, 0, 0, 0, 0\right), \tag{2.44}$$

$$b_{4,1..8} = \left(0, \frac{N}{a^2}(\gamma + \mu), -\frac{\lambda}{a\beta}, 0, 0, 0, 0, 0\right), \tag{2.45}$$

$$b_{8,1..8} = \left(0, 0, 0, 0, 0, 0, \mu\frac{N}{a^2}, 0\right), \tag{2.46}$$

$$c_{3,1..8} = \left(0, \rho_0 g_0 \frac{N}{a}, 0, 0, 0, \rho_0, 0, 0\right), \tag{2.47}$$

$$c_{4,1..8} = \left(\frac{1}{a}\rho_0 g_0, 0, 0, 0, \frac{\rho_0}{a}, 0, 0, 0\right). \tag{2.48}$$

According to the elastic momentum equation (2.16) together with (2.42)–(2.43) we can convert these equations into:

$$y_3^{E\prime} = \sum_j a_{3j} y_j^E, \qquad a_{3,1..8} = b_{3,1..8} + c_{3,1..8}, \tag{2.49}$$

$$y_4^{E\prime} = \sum_j a_{4j} y_j^E, \qquad a_{4,1..8} = b_{4,1..8} + c_{4,1..8}, \tag{2.50}$$

$$y_8^{E\prime} = \sum_j a_{8j} y_j^E, \qquad a_{8,1..8} = b_{8,1..8}. \tag{2.51}$$

Finally, the Poisson equation (2.17) combined with (2.30) yields:

$$y_6^{E\prime} = \sum_j a_{6j} y_j^E, \qquad a_{6,1..8} = \left(0, 4\pi G\rho_0 \frac{N}{a}, 0, 0, \frac{N}{a^2}, 0, 0, 0\right). \tag{2.52}$$

We have arrived at the set of linear first-order ODEs with respect to the z for the elastic solution vector assembled from (2.38)–(2.41) and (2.49)–(2.52):

$$\boxed{\boldsymbol{y}^E_{kl}(z)' = \boldsymbol{A}_N(z)\boldsymbol{y}^E_{kl}(z),} \tag{2.53}$$

where the matrix $\boldsymbol{A}_N$ holds

$$\boldsymbol{A}_N = \begin{pmatrix} 0 & \frac{\lambda N}{\beta a} & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{a} & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\ 0 & \rho_0 g_0 \frac{N}{a} & 0 & \frac{N}{a} & 0 & \rho_0 & 0 & 0 \\ \frac{1}{a}\rho_0 g_0 & \frac{N}{a^2}(\gamma+\mu) & -\frac{\lambda}{a\beta} & 0 & \frac{\rho_0}{a} & 0 & 0 & 0 \\ -4\pi G\rho_0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4\pi G\rho_0 \frac{N}{a} & 0 & 0 & \frac{N}{a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu\frac{N}{a^2} & 0 \end{pmatrix}, \tag{2.54}$$

and, let us recall, $\beta = \lambda + 2\mu, \gamma = \mu(3\lambda + 2\mu)/\beta = 3\mu K/\beta$ and $N = a^2(k^2 + l^2)$. For the case of material incompressibility, $K \to \infty$, we arrive at:

$$1/\beta \to 0, \quad \lambda/\beta \to 1, \quad \text{and} \quad \gamma \to 3\mu. \tag{2.55}$$

The system (2.53) consists of two independent systems, one with 6×6 matrix $(a_{1..6,1..6})$ containing the "spheroidal" coefficients of $\boldsymbol{u}$ and $\boldsymbol{\tau}^E$, and the second with 2×2 matrix $(a_{7..8,7..8})$, connecting the "toroidal" coefficients. A comparison of the Cartesian representation and the spherical harmonic representation of matrix $\boldsymbol{A}$ can be found in Section 5.1.

2.4 Boundary Conditions

Now we summarize the valid boundary conditions at the surface, $z = a$, and at the bottom boundary, $z = b$. Responses to an arbitrary surface loads can be deduced from the response to a surface point mass load with the Fourier representation expressed in the form, which follows *Farrell*, 1972,

$$\gamma_L = \sum\nolimits_{kl} B_{kl}. \tag{2.56}$$

This expression represents an elastic approximation[4]. The surface boundary conditions can be then written as

$$\begin{aligned} T_z^{(-1)}(a) &= -g_0, \\ T_z^{(1)}(a) &= 0, \\ Q(a) &= -4\pi G, \\ T_z^{(0)}(a) &= 0. \end{aligned} \tag{2.57}$$

[4] The viscoelastic formulation can be found in Section 3.3

Conditions at the bottom of the layer, $z = b$, are considered to have dual nature. First, we consider a layer "bonded" at its bottom side (denoted by symbol $\mathcal{A}$). And second, we consider a "laid" layer (denoted by symbol $\mathcal{B}$). Let us now write the exact formulation of these prescribed properties. The models $\mathcal{A}$ are similar to those Earth models with no liquid core. They require zero values for both the displacement $\boldsymbol{u}$ and the incremental gravitational potential φ_1,

$$\begin{aligned} U(b) &= 0, \\ V(b) &= 0, \\ W(b) &= 0, \\ F(b) &= 0. \end{aligned} \tag{2.58}$$

On the other hand, models $\mathcal{B}$ require zero values for the stress vector of traction $\boldsymbol{T}_z^E$, for the vertical component of displacement $\boldsymbol{u}$, and again for the incremental gravitational potential φ_1,

$$\begin{aligned} U(b) &= 0, \\ T_z^{(1)}(b) &= 0, \\ F(b) &= 0, \\ T_z^{(0)}(b) &= 0. \end{aligned} \tag{2.59}$$

2.5 Analytical Solution for Homogeneous and Incompressible Model

Considering all physical properties of the model constant, we may obtain solutions to (2.53) in terms of simple functions. We will focus upon one of important versions of the homogenous model which shows fundamental features of more general problem.

Let us now deal with the homogenous and incompressible model, i.e., model with constant values of density, viscosity and elastic parameters. Since the model is incompressible, $\nabla \cdot \boldsymbol{u} \equiv 0$, and since has constant density, $\frac{\partial \rho_0}{\partial z} = 0$, the incremental density (2.11) becomes zero-valued. Hence the equations (2.16) and (2.17) decouple. Equation (2.17) takes the the form

$$\Delta \varphi_1 = 0. \tag{2.60}$$

In an incompressible medium the dilatation $\delta = \nabla \cdot \boldsymbol{u}$ goes to zero and the Lammé parameter λ goes to infinity. However, their product has a finite limit (*Wu & Peltier*, 1982), i.e.,

$$\lim_{\substack{\lambda \to \infty \\ \delta \to 0}} (\lambda \delta) = \Pi, \tag{2.61}$$

which has the meaning of a mean normal stress. Rewriting constitutive equation (2.13) using (2.61) we obtain its incompressible form

$$\boldsymbol{\tau}^E = \Pi \boldsymbol{I} + 2\mu \boldsymbol{e}. \tag{2.62}$$

The momentum equation (2.16) may be rewritten into the form

$$-\nabla(\rho_0 \varphi_1 + \rho_0 g_0 \boldsymbol{u} \cdot \boldsymbol{e}_z - \Pi) - \mu \nabla \times \nabla \times \boldsymbol{u} = 0, \tag{2.63}$$

where we used definition of strain tensor (2.12) and identities (B.8)–(B.9). The divergence of (2.63) is

$$\frac{\rho_0}{\mu} \Delta(\varphi_1 + g_0 \boldsymbol{u} \cdot \boldsymbol{e}_z - \Pi/\rho_0) = 0. \tag{2.64}$$

We introduce the Fourier expansion of Π

$$\Pi = \sum_{kl} P_{kl}(z) B_{kl}(x,y), \tag{2.65}$$

and then rewrite the equations (2.60) and (2.64) by means of the Fourier expansion coefficients (2.24) and (2.25)

$$\Delta_f F_{kl}(z) = 0, \tag{2.66}$$

$$\Delta_f \left(F_{kl}(z) + \zeta z U_{kl}(z) - P_{kl}(z)/\rho_0\right) = 0, \tag{2.67}$$

where ζ is the constant obtained from solution of (2.7), $\zeta = 4\pi G\rho_0$, and $\Delta_f = \frac{\partial^2}{\partial z^2} - \frac{N}{a^2}$, according to relation (B.23). The vertical component of (2.63) is

$$\frac{\partial^2}{\partial z^2}(a^2 U_{kl}) - N U_{kl} = (\rho_0/\mu) a^2 \frac{\partial}{\partial z}(F_{kl} + \zeta z U_{kl} - P_{kl}/\rho_0), \tag{2.68}$$

where we used assumption $\nabla \cdot \boldsymbol{u} \equiv 0$ to eliminate those terms with V_{kl}. The solutions of (2.66) and (2.67) are respectively

$$F_{kl} = C_1 e^{-\frac{\sqrt{N}}{a}z} + C_2 e^{\frac{\sqrt{N}}{a}z}, \tag{2.69}$$

$$F_{kl} + \zeta z U_{kl} - \frac{P_{kl}}{\rho_0} = \frac{\mu}{\rho_0}\left(C_3 e^{-\frac{\sqrt{N}}{a}z} + C_4 e^{\frac{\sqrt{N}}{a}z}\right). \tag{2.70}$$

If we substitute the solution (2.70) into (2.68), we obtain an inhomogeneous ODE for U_{kl} to which the solution is

$$U_{kl} = C_3\left[\left(\frac{a}{4N} - \frac{\sqrt{N}z}{2N}\right)e^{-\frac{\sqrt{N}}{a}z}\right] + C_4\left[\left(\frac{a}{4N} + \frac{\sqrt{N}z}{2N}\right)e^{\frac{\sqrt{N}}{a}z}\right] + C_5\left[e^{-\frac{\sqrt{N}}{a}z}\right] + C_6\left[e^{\frac{\sqrt{N}}{a}z}\right]. \tag{2.71}$$

To obtain expression for the coefficient V_{kl} we have to use relation (2.31) and the assumption of the incompressibility, $\nabla \cdot \boldsymbol{u} \equiv 0$, i.e.,

$$X_{kl} \equiv 0. \tag{2.72}$$

For evaluation of the coefficients $T^{E(-1)}_{z,kl}, T^{E(1)}_{z,kl}$ and Q_{kl} we have to follow (2.32)–(2.35), and also use (2.61), i.e.,

$$T^{E(-1)}_{z,kl} = P_{kl} + 2\mu U'_{kl}. \tag{2.73}$$

The constructed 6–vector $\boldsymbol{Y}$, which solves the incompressible equivalent of (2.53), is the superposition of six linearly independent solutions

$$\boldsymbol{Y} = \sum_{i=1}^{6} C_i \boldsymbol{y}_i, \tag{2.74}$$

where the particular elements yield

$$y_1 = \left(0, 0, \rho_0, 0, 1, -\frac{\sqrt{N}}{a}\right)^T e^{-\frac{\sqrt{N}}{a}z}, \tag{2.75}$$

$$y_2 = \left(0, 0, \rho_0, 0, 1, \frac{\sqrt{N}}{a}\right)^T e^{\frac{\sqrt{N}}{a}z}, \tag{2.76}$$

$$y_3 = \left(\frac{z}{2} + \frac{a}{4\sqrt{N}}, \frac{a}{4N} - \frac{z}{2\sqrt{N}}, -\frac{\mu}{2} - \frac{\mu z\sqrt{N}}{a} + \frac{\zeta\rho_0 z^2}{2} + \frac{\zeta z a\rho_0}{4\sqrt{N}}, -\frac{\mu}{\sqrt{N}}, 0, \frac{\zeta z}{2} + \frac{a\zeta}{4\sqrt{N}}\right)^T e^{-\frac{\sqrt{N}}{a}z}, \tag{2.77}$$

$$y_4 = \left(\frac{z}{2} - \frac{a}{4\sqrt{N}}, \frac{a}{4N} + \frac{z}{2\sqrt{N}}, -\frac{\mu}{2} + \frac{\mu z\sqrt{N}}{a} + \frac{\zeta\rho_0 z^2}{2} - \frac{\zeta z a\rho_0}{4\sqrt{N}}, \frac{\mu}{\sqrt{N}}, 0, \frac{\zeta z}{2} - \frac{a\zeta}{4\sqrt{N}}\right)^T e^{\frac{\sqrt{N}}{a}z}, \tag{2.78}$$

$$y_5 = \left(1, -\frac{1}{\sqrt{N}}, \zeta\rho_0 z - \frac{2\mu\sqrt{N}}{a}, 0, 0, \zeta\right)^T e^{-\frac{\sqrt{N}}{a}z}, \tag{2.79}$$

$$y_6 = \left(1, \frac{1}{\sqrt{N}}, \zeta\rho_0 z + \frac{2\mu\sqrt{N}}{a}, 0, 0, \zeta\right)^T e^{\frac{\sqrt{N}}{a}z}. \tag{2.80}$$

To obtain values (or exact expressions) for the constants C_i, we have to use boundary conditions (2.57)–(2.59). We construct vector of constants $\boldsymbol{c}$,

$$\boldsymbol{c} = (C_1, \ldots, C_6)^T, \tag{2.81}$$

the right-hand side vector $\boldsymbol{b}$ following from boundary conditions

$$\boldsymbol{b} = (-g_0, 0, -4\pi G, 0, 0, 0)^T. \tag{2.82}$$

The first three terms in vector $\boldsymbol{b}$ are coming from the surface boundary conditions (2.57) and the last three are coming from conditions for the bottom boundary, i.e., for both layer models $\mathcal{A}$ and $\mathcal{B}$. We also introduce matrix $\boldsymbol{M}$,

$$\boldsymbol{M} = \left(\begin{array}{cccc} \rho_0 e^{-\sqrt{N}} & \rho_0 e^{\sqrt{N}} & \left(-\frac{\mu}{2} - \mu\sqrt{N} + \frac{\zeta\rho_0 a^2}{2} + \frac{\zeta\rho_0 a^2}{4\sqrt{N}}\right) e^{-\sqrt{N}} & \cdots \\ 0 & 0 & -\frac{\mu}{\sqrt{N}} e^{-\sqrt{N}} & \cdots \\ -\frac{\sqrt{N}}{a} e^{-\sqrt{N}} & \frac{\sqrt{N}}{a} e^{\sqrt{N}} & \left(\frac{\zeta a}{2} + \frac{\zeta a}{4\sqrt{N}}\right) e^{-\sqrt{N}} & \cdots \\ 0 & 0 & \left(\frac{b}{2} + \frac{a}{4\sqrt{N}}\right) e^{-\frac{\sqrt{N}}{a} b} & \cdots \\ 0 & 0 & \left(\frac{a}{4N} - \frac{b}{2\sqrt{N}}\right) e^{-\frac{\sqrt{N}}{a} b} & \cdots \\ e^{-\frac{\sqrt{N}}{a} b} & e^{\frac{\sqrt{N}}{a} b} & 0 & \cdots \end{array}\right.$$

$$\left. \begin{array}{cccc} \cdots & \left(-\frac{\mu}{2} + \mu\sqrt{N} + \frac{\zeta\rho_0 a^2}{2} - \frac{\zeta\rho_0 a^2}{4\sqrt{N}}\right) e^{\sqrt{N}} & \left(\zeta\rho_0 a - \frac{2\mu\sqrt{N}}{a}\right) e^{-\sqrt{N}} & \left(\zeta\rho_0 a + \frac{2\mu\sqrt{N}}{a}\right) e^{\sqrt{N}} \\ \cdots & \frac{\mu}{\sqrt{N}} e^{\sqrt{N}} & 0 & 0 \\ \cdots & \left(\frac{\zeta a}{2} - \frac{\zeta a}{4\sqrt{N}}\right) e^{\sqrt{N}} & \zeta e^{-\sqrt{N}} & \zeta e^{\sqrt{N}} \\ \cdots & \left(\frac{b}{2} - \frac{a}{4\sqrt{N}}\right) e^{\frac{\sqrt{N}}{a} b} & e^{-\frac{\sqrt{N}}{a} b} & e^{\frac{\sqrt{N}}{a} b} \\ \cdots & \left(\frac{a}{4N} + \frac{b}{2\sqrt{N}}\right) e^{\frac{\sqrt{N}}{a} b} & -\frac{1}{\sqrt{N}} e^{-\frac{\sqrt{N}}{a} b} & \frac{1}{\sqrt{N}} e^{\frac{\sqrt{N}}{a} b} \\ \cdots & 0 & 0 & 0 \end{array} \right), \tag{2.83}$$

so that the surface boundary conditions then take the form

$$\boldsymbol{M}\boldsymbol{c} = \boldsymbol{b}. \tag{2.84}$$

The form (2.83) of the matrix $\boldsymbol{M}$ represents layer model $\mathcal{A}$. For the layer model $\mathcal{B}$ there is difference in the 5th row, according to the boundary conditions (2.59), hence the 5th row takes form

$$\boldsymbol{M}_{5,1\ldots6} = \left(0, 0, \frac{\mu}{2\sqrt{N}} e^{-\frac{\sqrt{N}}{a} b}, \frac{\mu}{2\sqrt{N}} e^{\frac{\sqrt{N}}{a} b}, 0, 0\right). \tag{2.85}$$

The constants C_i can be obtained from relation

$$\boldsymbol{c} = \boldsymbol{M}^{-1}\boldsymbol{b}. \tag{2.86}$$

The solution of such problem is to be given up the numerical implementation, in other words, we do not seek the analytical solution for the constants C_i as shown in *Wu & Peltier*, 1982.

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Chapter 3

Viscoelastic Response of a Flat Layer

At the end of Chapter 2 we arrived at the boundary-value problem (2.53) for the linear first-order ODEs. In this Chapter we will continue with the time-domain formulation of gravitational viscoelastodynamics. For the 1-D depth-dependent, Maxwell model of flat layer we derive a linear first-order system of PDEs with respect to both time and the depth. This system of PDEs is expressed for a time-dependent solution vector $\boldsymbol{y}_{kl}(t, z)$*, constructed from the expansion coefficients of field variables. The set of PDEs (3.24) becomes the main outcome of this chapter, elaborated further in Chapter 4.*

3.1 Differential Equations for a Flat Layer

Let us now recall the field PDEs (2.16)–(2.18) in the form useful for further steps. The constitutive relation of the Maxwell rheology reads

$$\dot{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}}^E - \xi\left(\boldsymbol{\tau} - K\nabla\cdot\boldsymbol{u}\boldsymbol{I}\right), \tag{3.1}$$

where we used a well-known auxiliary parameter ξ equal to the ratio of the shear modulus and viscosity, i.e., the inverse Maxwell time,

$$\xi = \frac{\mu}{\eta}. \tag{3.2}$$

If we combine the constitutive equation (3.1) with the field equations (2.16)–(2.17), we obtain a system

$$\nabla\cdot\dot{\boldsymbol{\tau}}^E + \dot{\boldsymbol{f}} = \nabla\cdot\left[\xi(\boldsymbol{\tau} - K\nabla\cdot\boldsymbol{u}\boldsymbol{I})\right], \tag{3.3}$$

$$\nabla\cdot(\nabla\varphi_1 + 4\pi G\rho_0\boldsymbol{u}) = 0. \tag{3.4}$$

where the term $\boldsymbol{\tau}^E$ denotes the elastic part of the stress tensor $\boldsymbol{\tau}$. The dots over variables denote time differentiation. The particular steps which lead to the resulting expressions are quite similar to those made in Chapter 2.

3.2 Partial Differential Equations for the Maxwell Solid

In this section we will derive a system of PDEs with respect to both time and the depth. We consider the depth-dependent distribution of these parameters:

$$\rho_0 = \rho_0(z), \quad \lambda = \lambda(z), \quad \mu = \mu(z), \quad K = K(z), \quad g_0 = g_0(z), \quad \eta = \eta(z), \quad \xi = \xi(z) \tag{3.5}$$

and spatial distribution of field variables as follows:

$$\boldsymbol{u} = \boldsymbol{u}(x,y,z), \qquad \varphi_1 = \varphi_1(x,y,z), \qquad \boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z). \tag{3.6}$$

The solution vector $\boldsymbol{y}_{kl}(t,z)$ is contructed form the coefficients of the Fourier expansion of the field variables $\boldsymbol{u}, \varphi$ and $\boldsymbol{\tau}$,

$$\boldsymbol{y}_{kl}(t,z) = \left(U_{kl}, V_{kl}, T^{(-1)}_{z,kl}, T^{(1)}_{z,kl}, F_{kl}, Q_{kl}, W_{kl}, T^{(0)}_{z,kl}\right)^T. \tag{3.7}$$

The only difference between $\boldsymbol{y}_{kl}(t,z)$ and the elastic solution vector $\boldsymbol{y}^E_{kl}(z)$ given by (2.37),

$$\boldsymbol{y}^E_{kl}(z) = \left(U_{kl}, V_{kl}, T^{E(-1)}_{z,kl}, T^{E(1)}_{z,kl}, F_{kl}, Q_{kl}, W_{kl}, T^{E(0)}_{z,kl}\right)^T. \tag{3.8}$$

is in the coefficients of the Fourier expansion of the traction $\boldsymbol{T}_z \equiv \boldsymbol{e}_z\cdot\boldsymbol{\tau}$, which is connected with the elastic traction $\boldsymbol{T}^E_z \equiv \boldsymbol{e}_z\cdot\boldsymbol{\tau}^E$ through the relation following from (3.1),

$$\dot{\boldsymbol{T}}_z = \dot{\boldsymbol{T}}^E_z - \xi(\boldsymbol{T}_z - K\nabla\cdot\boldsymbol{u}\boldsymbol{e}_z). \tag{3.9}$$

From (3.7)–(3.9) we obtain the following relations between the corresponding elements of $\boldsymbol{y}_{kl}$ and $\boldsymbol{y}^E_{kl}$,

$$\begin{array}{lllll} \dot{y}^E_1 = \dot{y}_1, & \dot{y}^E_2 = \dot{y}_2, & \dot{y}^E_5 = \dot{y}_5, & \dot{y}^E_6 = \dot{y}_6, & \dot{y}^E_7 = \dot{y}_7, \\ \dot{y}^E_3 = \dot{y}_3 + \xi(y_3 - KX_{kl}), & \dot{y}^E_4 = \dot{y}_4 + \xi y_4, & \dot{y}^E_8 = \dot{y}_8 + \xi y_8, & \text{with } X_{kl} = y_1' - \frac{N}{a}y_2. \end{array} \tag{3.10}$$

We suppress both subscripts k and l throughout this section when reffering to the elements of $y_{kl,i}$ and $a_{kl,ij}$ of $\boldsymbol{y}_{kl}$ and $\boldsymbol{A}_N$, respectively. The symbol X_{kl} means the Fourier expansion coefficient (2.31) of $\nabla \cdot \boldsymbol{u}$ and we also suppress its subscripts k and l. In this 1-D case the resulting system of PDEs remains decoupled with respect to both wave-numbers k and l which justifies this simplification of notation.

In (2.38)–(2.41) we collected relations for the first derivatives of the coefficients of $\boldsymbol{u}$ and φ_1. These relations remain valid in the viscoelastic case. The only arrangement is the change of variables from $\boldsymbol{y}^E_{kl}$ to $\boldsymbol{y}_{kl}$ according to (3.10). Hence we obtain the elements of $\boldsymbol{y}_{kl}$ like

$$\begin{aligned}
\dot{y_1}' - \textstyle\sum_j a_{1j}\dot{y}_j &= \xi[a_{13}(y_3 - KX) + a_{14}y_4 + a_{18}y_8] &&= \xi a_{13}(y_3 - KX), && (3.11)\\
\dot{y_2}' - \textstyle\sum_j a_{2j}\dot{y}_j &= \xi[a_{23}(y_3 - KX) + a_{24}y_4 + a_{28}y_8] &&= \xi a_{24}y_4, && (3.12)\\
\dot{y_5}' = \textstyle\sum_j a_{5j}\dot{y}_j &= \xi[a_{53}(y_3 - KX) + a_{54}y_4 + a_{58}y_8] &&= 0, && (3.13)\\
\dot{y_7}' = \textstyle\sum_j a_{7j}\dot{y}_j &= \xi[a_{73}(y_3 - KX) + a_{74}y_4 + a_{78}y_8] &&= \xi a_{78}y_8, && (3.14)
\end{aligned}$$

where $\sum_j$ is denoting summarization $\sum_{j=1}^{8}$ and the zero-valued terms (i.e., those multiplied by $a_{14}, a_{18}, a_{23}, a_{28}, a_{53}, a_{54}, a_{58}, a_{73}$ and a_{74}) have been discarded.

In the next step we express the momentum equation (3.3) in the terms of the solution vector $\boldsymbol{y}_{kl}$. It is needed to substitute $\boldsymbol{y}_{kl}$ into the left-hand side of the equation (3.3) expressed in terms of $\boldsymbol{y}^E_{kl}$ in equations (2.42)–(2.43) and to evaluate the right-hand side of (3.3). If we substitute (3.10) into (2.42)–(2.43) and differentiate with repsect to time, we obtain

$$\begin{aligned}
\nabla \cdot \dot{\boldsymbol{\tau}}^E &= \textstyle\sum_{kl} \left[\dot{y}_3' + (\xi(y_3 - KX))' - \sum_j b_{3j}\dot{y}_j - \xi b_{34}y_4\right] \boldsymbol{G}^{(-1)}_{kl}\\
&+ \textstyle\sum_{kl} \left[\dot{y}_4' + (\xi y_4)' - \sum_j b_{4j}\dot{y}_j - \xi b_{43}(y_3 - KX)\right] \boldsymbol{G}^{(1)}_{kl}\\
&+ \textstyle\sum_{kl} \left[\dot{y}_8' + (\xi y_8)' - \sum_j b_{8j}\dot{y}_j\right] \boldsymbol{G}^{(0)}_{kl}, && (3.15)\\
\dot{\boldsymbol{f}} &= \textstyle\sum_{kl} \left[-\sum_j c_{3j}\dot{y}_j \boldsymbol{G}^{(-1)}_{kl} - \sum_j c_{4j}\dot{y}_j \boldsymbol{G}^{(1)}_{kl}\right], && (3.16)
\end{aligned}$$

where the zero-valued terms (those with $b_{33}, b_{38}, b_{44}, b_{48}, b_{83}, b_{84}, b_{88}, c_{33}, c_{34}, c_{38}, c_{43}, c_{44}$ and c_{48}) have been discarded. For the right-hand side of (3.3) yields,

$$\begin{aligned}
\nabla \cdot [\xi(\boldsymbol{\tau} - K\nabla \cdot \boldsymbol{u}\boldsymbol{I})] &= \xi[\nabla \cdot \boldsymbol{\tau} - \nabla(K\nabla \cdot \boldsymbol{u})] + \nabla\xi \cdot [\boldsymbol{\tau} - K\nabla \cdot \boldsymbol{u}\boldsymbol{I}]\\
&= \xi[-\boldsymbol{f} - \nabla(K\nabla \cdot \boldsymbol{u})] + \xi'[\boldsymbol{T}_z - K\nabla \cdot \boldsymbol{u}\boldsymbol{e}_z] && (3.17)\\
&= \textstyle\sum_{kl} \left[\xi \sum_j c_{3j}y_j - \xi(KX)' + \xi'(y_3 - KX)\right] \boldsymbol{G}^{(-1)}_{kl}\\
&+ \textstyle\sum_{kl} \left[\xi \sum_j c_{4j}y_j - \xi \frac{1}{a}(KX) + \xi' y_4\right] \boldsymbol{G}^{(1)}_{kl}\\
&+ \textstyle\sum_{kl} \left[\xi' y_8\right] \boldsymbol{G}^{(0)}_{kl}, && (3.18)
\end{aligned}$$

where we use (B.19) for the representation of $\nabla(K\nabla \cdot \boldsymbol{u})$. Now we can extract the three scalar components of the equation (3.3) from (3.15)–(3.17) and removing terms appearing on both sides, we arrive at

$$\begin{aligned}
\dot{y_3}' - \textstyle\sum_j a_{3j}\dot{y}_j &= \xi[-y_3' + b_{34}y_4 + \textstyle\sum_j c3jy_j]\\
&= \xi[-y_3' + \textstyle\sum_j a_{3j}y_j] && (3.19)\\
\dot{y_4}' - \textstyle\sum_j a_{4j}\dot{y}_j &= \xi[-y_4' + b_{43}(y_3 - KX) - \tfrac{1}{a}KX + \textstyle\sum_j c_{4j}y_j]\\
&= \xi[-y_4' + \textstyle\sum_j a_{4j}y_j - b_{42}y_2 - (b_{43} + \tfrac{1}{a})KX] && (3.20)\\
\dot{y_8}' - \textstyle\sum_j a_{8j}\dot{y}_j &= -\xi y_8' && (3.21)
\end{aligned}$$

We recall that according to (2.49)–(2.51),

$$a_{3,1..8} = b_{3,1..8} + c_{3,1..8},$$
$$a_{4,1..8} = b_{4,1..8} + c_{4,1..8},$$
$$a_{8,1..8} = b_{8,1..8}.$$

We accomplish the last step by substituting the elements of $\boldsymbol{y}_{kl}^E$ into the Fourier expansion of the Poisson equation given by (2.52) and by differentiating with respect to time. But we have zero values of terms a_{63}, a_{64} and a_{68}, hence we obtain,

$$\dot{y_6}' - \textstyle\sum_j a_{6j}\dot{y_j} = 0. \tag{3.22}$$

Completing (3.11)–(3.14) together with (3.19)–(3.21) we have arrived at a linear system of PDEs with respect to time and the depth for the solution vector $\boldsymbol{y}_{kl}$,

$$\dot{\boldsymbol{y}}'_{kl}(t,z) - \boldsymbol{A_N}(z)\dot{\boldsymbol{y}}_{kl}(t,z) = \xi \begin{pmatrix} a_{13}(y_3 - KX) \\ a_{24}y_4 \\ -y'_3 + \sum_j a_{3j}y_j \\ -y'_4 + \sum_j a_{4j}y_j - b_{42}y_2 - (b_{43} + \frac{1}{a})KX \\ 0 \\ 0 \\ a_{78}y_8 \\ -y'_8 \end{pmatrix}, \tag{3.23}$$

which can be rewritten into the form

$$\boxed{\dot{\boldsymbol{y}}'_{kl}(t,z) - \boldsymbol{A}_N(z)\dot{\boldsymbol{y}}_{kl}(t,z) \;=\; \xi(z)\left[\boldsymbol{D}_N(z)\boldsymbol{y}'_{kl}(t,z) + \boldsymbol{E}_N(z)\boldsymbol{y}_{kl}(t,z)\right].} \tag{3.24}$$

Matrix $\boldsymbol{A}_N(z)$ takes the form of (2.54) and the explicit expressions of the matrices $\boldsymbol{D}_N(z)$ and $\boldsymbol{E}_N(z)$ follow from (3.23) after substitution of coefficients $a_{13}, \ldots, b_{87}$ from (2.54) and X from (2.26),

$$\boldsymbol{D}_N = \begin{pmatrix} -\dfrac{K}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\dfrac{2\gamma}{3a} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.25}$$

$$
\boldsymbol{E}_N = \begin{pmatrix}
0 & \frac{KN}{\beta a} & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\
0 & \rho_0 g_0 \frac{N}{a} & 0 & \frac{N}{a} & 0 & \rho_0 & 0 & 0 \\
\frac{1}{a}\rho_0 g_0 & \frac{2N\gamma}{3a^2} & -\frac{\lambda}{a\beta} & 0 & \frac{\rho_0}{a} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{3.26}
$$

and let us recall, $\beta = \lambda + 2\mu, \gamma = \mu(3\lambda + 2\mu)/\beta = 3\mu K/\beta$ and $N = a^2(k^2 + l^2)$. For the case of material incompressibility, $K \to \infty$, we arrive at:

$$
1/\beta \to 0, \quad \lambda/\beta \to 1, \quad \gamma \to 3\mu \quad \text{and} \quad K/\beta \to 1. \tag{3.27}
$$

In the limit of the elastic mantle, $\eta(z) \to \infty$, i.e., $\xi(z) \to 0$, PDEs (3.24) should be consistent with the corresponding equations governing elastic free oscillations in the zero-frequency limit.

In the opposite limit of the inviscid mantle, $\eta(z) \to 0$, i.e., $\xi(z) \to \infty$, we obtain the static PDEs

$$
\boldsymbol{D}_N \boldsymbol{y}'(z) + \boldsymbol{E}_N \boldsymbol{y}(z) = 0, \tag{3.28}
$$

which can be shown to be equivalent with

$$
\boldsymbol{y}'(z) - \boldsymbol{A}_N \boldsymbol{y}(z) = 0, \tag{3.29}
$$

where we assumed $\mu \to 0$ in matrix $\boldsymbol{A}_N(z)$.

3.3 Boundary Conditions

Finally, we summarize the valid boundary conditions for the case of viscoelasticity. When referring to surface loads, we have to add a time dependency of the load. Again, we will follow formulation shown in *Farrell*, 1972, i.e. the Fourier representation of surface load γ_L takes the form

$$
\gamma_L(t) = \sum\nolimits_{kl} f(t) B_{kl}, \tag{3.30}
$$

where $f(t)$ describes the time evolution of the load. Hence the surface boundary conditions yield

$$
\begin{aligned}
T_z^{(-1)}(t, a) &= -g_0 f(t), \\
T_z^{(1)}(t, a) &= 0, \\
Q(t, a) &= -4\pi G f(t), \\
T_z^{(0)}(t, a) &= 0.
\end{aligned} \tag{3.31}
$$

Conditions at the bottom boundary, $z = b$, are remaining the same as in the elastic case, see expressions (2.58)–(2.59).

For the load applied at $t = 0$ and kept in effect continuously for any $t > 0$, i.e., for the Heaviside time dependence $f(t) = H(t)$, the Maxwell model responds elastically at $t = 0$. So that it is appropriate to require for $\boldsymbol{y}(t, z)$ initial condition in the form

$$\boldsymbol{y}(0, z) = \boldsymbol{y}^E(z). \tag{3.32}$$

The elastic solution $\boldsymbol{y}^E(z)$ can be obtained from the system (2.53) by integration from the point $z = b$, i.e., from the bottom side.

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Chapter 4

Numerical Methods and Techniques

In Chapter 3 we have derived the linear system of PDEs (3.24) with respect to time t and the depth z for the solution vector $\boldsymbol{y}(t, z)$ describing the response of the depth dependent, viscoelastic, Maxwell model of a flat layer. Now we are concerned with numerical methods applicable to this system. In our first step we will enforce discretization in space to get ODEs in time. Then we will realize modal decomposition to solve the eigenvalue problem. Applying the finite-difference technique, we undertake this in following paragraphs.

4.1 Discretization in Space: Ordinary Differential Equations in Time

It is well known that the viscoelastic responses of compressible models can be characterized by the exponential-like development in time. The spatial behaviour of the model is considerably different – when we refer to spherically symmetric models, the dependency can be expressed in terms of the spherical Bessel functions (*Wu & Peltier*, 1982).

For such system of PDEs, i.e. (3.24), a method based on discretization in the spatial domain, known as the method of lines (MOL), represents a powerful solution tool (*Hanyk*, 1999). From now on we consider only the spheroidal part of PDEs (3.24), so the solution vector $\boldsymbol{y}(t,z)$ consist of 6 spheroidal elements.

Let us now consider the staggered grids $\{z_i, i=1,\ldots,J\}$ and $\{x_j, j=1,\ldots,J\}$,

$$b = x_0 < z_1 < x_1 < z_2 < x_2 < \cdots < z_J < x_J = a, \tag{4.1}$$

spreading over the layer, $b \le z \le a$. In order to express the system of PDEs (3.24) on the grid $\{x_j\}$ by means of $\boldsymbol{y}(t,z)$ evaluated on the grid $\{z_i\}$, we employ expansions of $\boldsymbol{y}(t,z)$ and its first derivative evaluated at x_j by means of weighted sums of $\boldsymbol{y}_i(t) = \boldsymbol{y}(t,z_i)$,

$$\boldsymbol{y}(t,x_j) = \sum\nolimits_{i=0}^{J} \alpha_{ij}^{(0)} \boldsymbol{y}_i(t), \tag{4.2}$$

$$\boldsymbol{y}'(t,x_j) = \sum\nolimits_{i=0}^{J} \alpha_{ij}^{(1)} \boldsymbol{y}_i(t), \tag{4.3}$$

where $\alpha_{ij}^{(0)}$ and $\alpha_{ij}^{(1)}$ are the weights given by a choice of z_i and x_js. Using (4.2)–(4.3) we obtain $6J$ scalar ODEs in time for $6J+6$ uknown elements of $\boldsymbol{y}_i$ by expressing system of PDEs (3.24) on the grid $\{x_j\}$,

$$\sum_{i=0}^{J} \left[\alpha_{ij}^{(1)} - \boldsymbol{A}_j \alpha_{ij}^{(0)}\right] \dot{\boldsymbol{y}}_i(t) = \sum_{i=0}^{J} \left[\xi_j \left(\boldsymbol{D}_j \alpha_{ij}^{(1)} + \boldsymbol{E}_j \alpha_{ij}^{(0)}\right)\right] \boldsymbol{y}_i(t), \qquad i=1,\ldots,J \tag{4.4}$$

where $\boldsymbol{A}_j = \boldsymbol{A}_N(x_j), \boldsymbol{D}_j = \boldsymbol{D}_N(x_j), \boldsymbol{E}_j = \boldsymbol{E}_N(x_j)$ and $\xi_j = \xi(x_j)$. The last 6 necessary equations have origin in boundary conditions (2.57),

$$\boldsymbol{M}_J \boldsymbol{y}_J(t) = \begin{pmatrix} -g_0 \\ 0 \\ -4\pi G \end{pmatrix}, \qquad \boldsymbol{M}_J = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.5}$$

and the conditions (2.58), i.e. layer model $\mathcal{A}$,

$$\boldsymbol{M}_0 \boldsymbol{y}_0(t) = \boldsymbol{0}, \qquad \boldsymbol{M}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.6}$$

or the conditions (2.59), i.e. layer model $\mathcal{B}$,

$$\boldsymbol{M}_0 \boldsymbol{y}_0(t) = \boldsymbol{0}, \qquad \boldsymbol{M}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.7}$$

We define a discretized solution vector $\boldsymbol{y}(t)$ with $6J+6$ elements,

$$\boldsymbol{Y}(t) = \left(\boldsymbol{y}_0(t), \boldsymbol{y}_1(t), \ldots, \boldsymbol{y}_J(t)\right)^T, \tag{4.8}$$

for which ODEs and the boundary conditions (4.5) and (4.6) or (4.7) (differentiated in time) form the resulting system of $6J+6$ ODEs in time,

$$\boldsymbol{P}\dot{\boldsymbol{Y}}(t) = \boldsymbol{Q}\boldsymbol{Y}(t), \tag{4.9}$$

where

$$\boldsymbol{P} = \begin{pmatrix} \boxed{\boldsymbol{M}_0}_{\,3\times 6} & \boxed{0}_{\,3\times 6J} \\ \boxed{\boldsymbol{I}\alpha^{(1)}_{i1} - \boldsymbol{A}_1\alpha^{(0)}_{i1}}_{\,6\times(6J+6)} & \\ \vdots & \\ \boxed{\boldsymbol{I}\alpha^{(1)}_{J1} - \boldsymbol{A}_1\alpha^{(0)}_{J1}}_{\,6\times(6J+6)} & \\ \boxed{0}_{\,3\times 6J} & \boxed{\boldsymbol{M}_J}_{\,3\times 6} \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} \boxed{0}_{\,3\times(6J+6)} \\ \boxed{\xi_1\left(\boldsymbol{D}_1\alpha^{(1)}_{i1} + \boldsymbol{E}_1\alpha^{(0)}_{i1}\right)}_{\,6\times(6J+6)} \\ \vdots \\ \boxed{\xi_J\left(\boldsymbol{D}_J\alpha^{(1)}_{iJ} + \boldsymbol{E}_J\alpha^{(0)}_{iJ}\right)}_{\,6\times(6J+6)} \\ \boxed{0}_{\,3\times(6J+6)} \end{pmatrix}. \tag{4.10}$$

System (4.9) together with the initial condition (3.32) represents a purely initial-value formulation of the problem of the viscoelastic responses of a flat layer. For given grids $\{z_i\}$ and $\{x_j\}$, both matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are constant. The matrix $\boldsymbol{P}$ is regular for all models studied here, so we can write system (4.9) in equivalent form,

$$\dot{\boldsymbol{Y}}(t) = \boldsymbol{B}\boldsymbol{Y}(t), \qquad \boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{Q}. \tag{4.11}$$

This form is significant with the standard form a linear homogenous system of ODEs with constant coefficients (*Rektorys et al.*, 2000).

4.2 Modal Decomposition: The Eigenvalue Problem

A solution of ODEs (4.11) as a fundamental system can be written as a linear combination of the constituents (see e.g. *Rektorys et al.*, 2000)

$$e^{s_p t} \quad \text{or} \quad R_q(t)e^{s_q t}, \tag{4.12}$$

where s_p is any nondegenerate eigenvalue of $\boldsymbol{B}$, s_q any Q-degenerate eigenvalue of $\boldsymbol{B}$ and R_q a polynomial of the matrix degree $Q-1$. Thus, eigenanalysis of matrix $\boldsymbol{B}$ can reveal a substantial information about the behavior of the solution of ODEs (4.11). We can easily see the correspondence between a generalized eigenvalue problem

$$s\boldsymbol{P}\boldsymbol{Y} = \boldsymbol{Q}\boldsymbol{Y}, \tag{4.13}$$

and ODEs (4.9), while a standard eigenvalue problem,

$$s\boldsymbol{Y} = \boldsymbol{B}\boldsymbol{Y} \tag{4.14}$$

matches ODEs 4.11. In the case of regular matrix $\boldsymbol{P}$ both eigenproblems are formally equivalent.

For a given nondegenerate eigenvalue $s = s_p$ the corresponding eigenvector $\boldsymbol{Y}_p$ with $6J+6$ elements gathers the discretized $(J+1)$ eigenvectors U_p, V_p, etc. The response of viscoelastic models to the time impulse, i.e., for $f(t) = \delta(t)$ is traditionally expressed by the surface load Love numbers (*Peltier*, 1974),

$$h(t) = h^E \delta(t) + \sum\nolimits_p r_p^{(h)} \exp(s_p t) , \tag{4.15}$$

$$l(t) = l^E\, \delta(t) + \sum\nolimits_p r_p^{(l)} \exp(s_p t) , \tag{4.16}$$

$$k(t) = k^E \delta(t) + \sum\nolimits_p r_p^{(k)} \exp(s_p t) , \tag{4.17}$$

where h^E, l^E and k^E are the elastic load Love numbers and the sums on the right-hand sides describe the non-elastic response. The surface Love numbers are related to the 1st, 2nd and 5th elements of solution vector $\boldsymbol{y}(t)$ by the definition (*Farrell*, 1972)

$$\begin{pmatrix} y_1(z) \\ y_2(z) \\ y_5(z) \end{pmatrix} = \Phi_N \begin{pmatrix} h_N(z)/g_0 \\ l_N(z)/g_0 \\ -k_N(z) \end{pmatrix} , \tag{4.18}$$

with Φ_N the coefficient of the expansion of the surface potential of the point mass load. In order to evaluate the non-elastic part of of the Love numbers, we employ formulas for the partial modal amplitudes developed by *Tromp & Mitrovica*, II., 1999

$$r_p^{(h)} = \frac{\tau}{2s_p} U_p(g_0 U_p + F_p) , \tag{4.19}$$

$$r_p^{(l)} = \frac{\tau}{2s_p} V_p(g_0 U_p + F_p) , \quad \tau = \frac{M}{a}\frac{2n+1}{4\pi} , \tag{4.20}$$

$$r_p^{(k)} = -\frac{\tau}{2g_0 s_p} F_p(g_0 U_p + F_p) , \tag{4.21}$$

where M is the mass of the Earth.

4.3 Numerical Techniques

The numerical implementation of the viscoelastic response of a flat layer can be divided in three together connected parts. Let us now name and describe each one of them.

4.3.1 "YBY" Phase

The "YBY" phase and its Fortran code provides numerical implementation of the **impulse** response of a selected model. Elastic response is evaluated through both numerical and analytical solution routines. The viscoelastic response is integrated in time with adaptive step-size routines from *Numerical Recipes* - ODEINT. We employ the concept of stiffness[1] and simultaneously the routines `stiffpq` and `stiffb`, both developed by dr. L. Hanyk[2]. These stiff time-integrators can substantially reduce the number of time steps. However, the reduction of the number of time steps in our geometrical approach was not so big as expected. We will discuss more than this in the following chapter.

Output is provided in different ways, depending on model and further uses of the data. The main interest lies on the behaviour of the vertical and horizontal displacements (temrs U and V

[1] Stiffness occurs in a problem where there are two or more very different scales of the independent variable on which the dependent variables are changing (see *Hanyk*, 1999).

[2] We are very grateful to dr. Hanyk for extending us the code and discussing concerning problems.

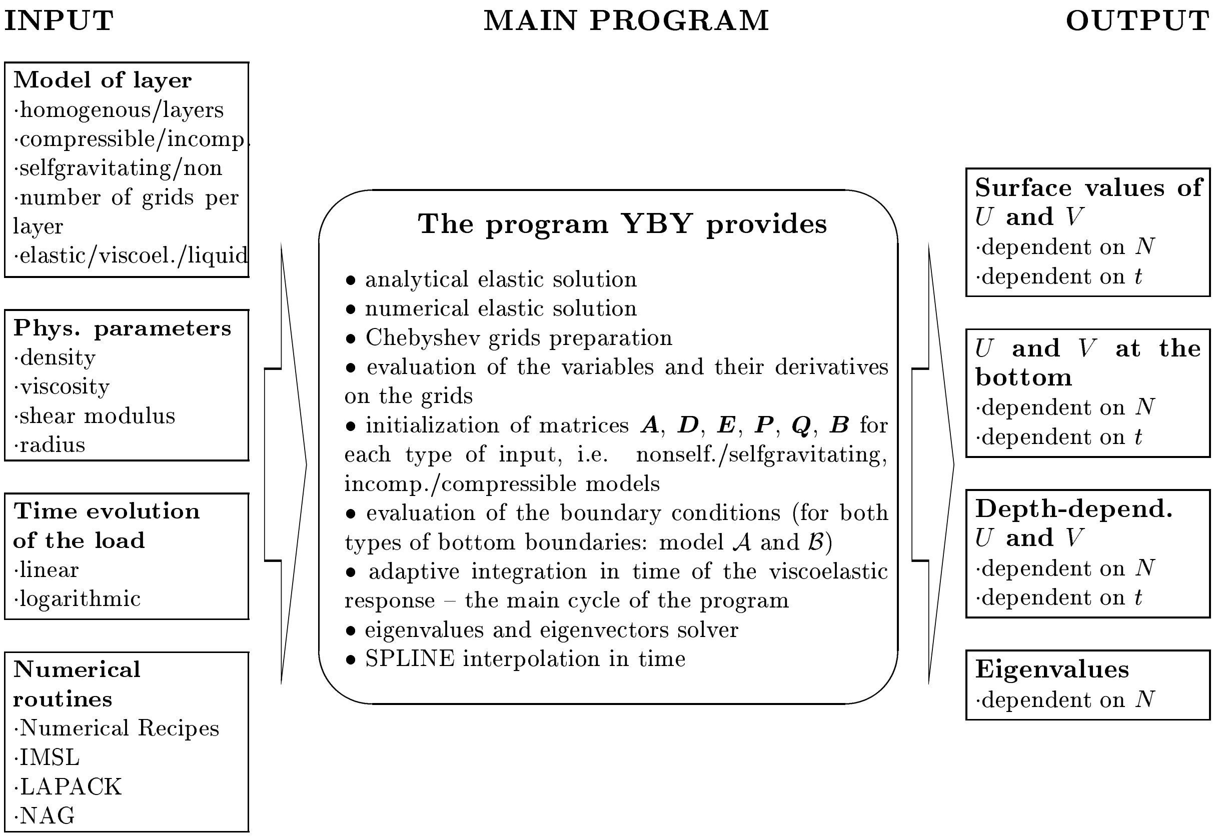

Fig. 4.1. Schematic graph of the "YBY" phase when evaluating the impulse response of a given model of flat layer. The input paragraph is actually represented by several files containing external configuration, routines and data files. The output takes very similar form. We obtain separate files for surface, bottom boundary, depth-dependence and eigenvalues.

of the solution vector $\boldsymbol{y}$) at the surface, at the bottom boundary and through the layers (depth dependence). These temrs are evaluated for different orders of decomposition N and different times t. Hence we obtain the Fourier modes which stand as an input data for the second phase. More detailed schema of the "YBY" phase can be seen in Fig. 4.1.

4.3.2 "ZBZ" Phase

The second part, called "ZBZ", provides many important steps. First, the program prepares model of parabolic and cylindrical load and its Fourier coefficients, reads the output data from the "YBY" phase and interpolates in time and in the order N. Next step is multiplication of the Fourier coefficients and the horizontal and vertical displacements. The inverse Fourier transform of these data is the final step in numerical part of the code.

4.3.3 Visualization Phase

The last step in the whole process is visualization of the time evolution of the viscoelastic response. The final part of program "ZBZ" prepares headers and data sets as an input for visualization program Amira. The files are created for each time step and contain the coordinate grid and values of vertical and horizontal displacements on the grid.

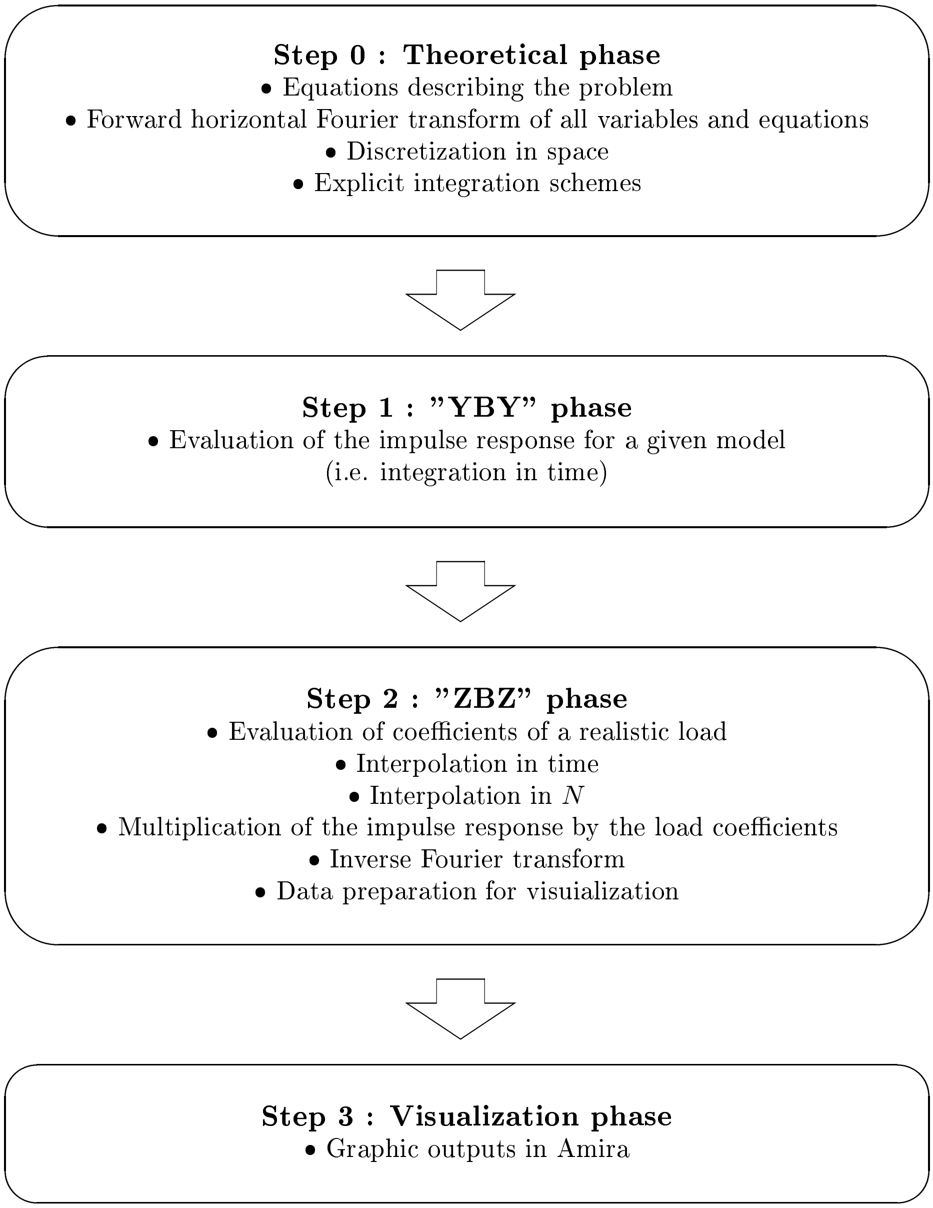

Fig. 4.2. Summary of the work presented in Chapters 2–4. Steps 1 and 2 are represented by Fortran code as was described in previous sections, the visualization part represents data manipulation with commercial program Amira.

A small schematic table summarizing the main outcomes of Chapters 2–4, i.e., a journey from the theory to the results can be seen in Fig. 4.2.

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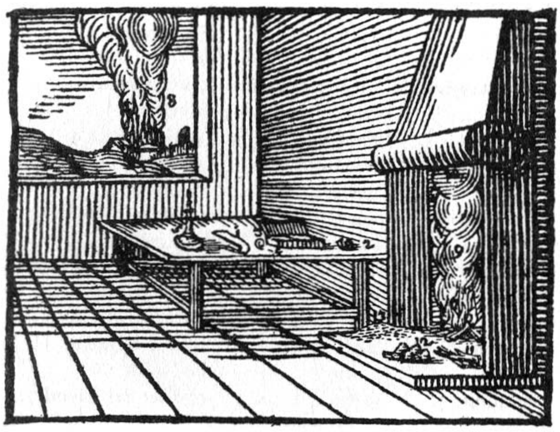

Chapter 5

Results

This chapter should summarize the results developed in Chapters 2–4 and show the main outcomes in the way of graphs and pictures. First we will focus upon correspondence between the Cartesian geometry approach which had to be developed in the previous chapters and the spherical geometry approach dealt by e.g. Peltier, 1974, Hanyk, 1999, etc. Next we will move our attention on results coming from the numerical modelling. We will deal with models of layer (ie. homogenous/layered, compressible/incompressible, selfgravitating/non-selfgravitating etc.), possible discretizations and also we will discuss the concerning problems.

5.1 Cartesian Geometry versus Spherical geometry

In this section we will compare the main results of both formalisms, the Cartesian and the spherical, respectively. First, we will focus upon the elastic case and secondly we will move on the viscoelastic case.

5.1.1 Elasticity

At the end of Chapter 2 we have arrived at the set of ODEs for the elastic solution vector $\boldsymbol{y}^E$, cf. (2.53),

$$\boldsymbol{y}^{E\prime} = \boldsymbol{A}\,\boldsymbol{y}^E, \tag{5.1}$$

where the matrix $\boldsymbol{A}$ holds, cf. (2.54),

$$\boldsymbol{A} = \begin{pmatrix} 0 & \frac{\lambda N}{\beta a} & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{a} & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\ 0 & \rho_0 g_0 \frac{N}{a} & 0 & \frac{N}{a} & 0 & \rho_0 & 0 & 0 \\ \frac{1}{a}\rho_0 g_0 & \frac{N}{a^2}(\gamma+\mu) & -\frac{\lambda}{a\beta} & 0 & \frac{\rho_0}{a} & 0 & 0 & 0 \\ -4\pi G\rho_0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4\pi G\rho_0\frac{N}{a} & 0 & 0 & \frac{N}{a^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu\frac{N}{a^2} & 0 \end{pmatrix}, \tag{5.2}$$

with $N = a^2(k^2+l^2)$, a the scaling length and k and l both the wave numbers. To see the spherical representation, we can use e.g. monograph by *Hanyk*, 1999, expression (2.50),

$$\boldsymbol{A} = \begin{pmatrix} -\frac{2\lambda}{r\beta} & \frac{N\lambda}{r\beta} & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\ \frac{4\gamma}{r^2} - \frac{4\rho_0 g_0}{r} & -\frac{2N\gamma}{r^2} + \frac{N\rho_0 g_0}{r} & -\frac{4\mu}{r\beta} & \frac{N}{r} & -\frac{(n+1)\rho_0}{r} & \rho_0 & 0 & 0 \\ -\frac{2\gamma}{r^2} + \frac{1}{r}\rho_0 g_0 & \frac{N\gamma + (N-2)\mu}{r^2} & -\frac{\lambda}{r\beta} & -\frac{3}{r} & \frac{\rho_0}{r} & 0 & 0 & 0 \\ -4\pi G\rho_0 & 0 & 0 & 0 & -\frac{n+1}{r} & 1 & 0 & 0 \\ -\frac{4\pi G\rho_0(n+1)}{r} & 4\pi G\frac{N\rho_0}{r} & 0 & 0 & 0 & \frac{n-1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r} & \frac{1}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu\frac{N-2}{r^2} & -\frac{3}{r} \end{pmatrix}, \tag{5.3}$$

where $N = n(n+1)$. The symbol n denotes a spherical degree (or angular order) of the spherical harmonics representation and the symbol r denotes the radius.

To start the discussion, let us first mention what connects both matrices together. The system consists of two independent systems, one with 6×6 matrix $(a_{1..6,1..6})$ containing the spheroidal coefficients of $\boldsymbol{u}$ and $\boldsymbol{\tau}^E$, and the second with 2×2 matrix $(a_{7..8,7..8})$, connecting the toroidal coefficients (cf. *Peltier*, 1974 or *Hanyk*, 1999).

The main difference between those two approaches can be immediately seen on the diagonal of the matrix $\boldsymbol{A}$. The Cartesian matrix (5.2) has all the diagonal terms $a_{ii}, i = 1 \dots 8$ equal to zero. The explanation ensues from comparison of the expressions for the expansion terms (2.28)–(2.32) in monograph by *Hanyk*, 1999 and (2.31)–(2.35) in Chapter 2, respectively. We can see that in expressions in spherical geometry there exist among others both differentiated (with respect the depth) and non-differentiated terms in sequence: U, V, W, F. The differentiated terms appear on the left-hand side of the system (2.53) and the non-differentiated terms are to be appeared on the right-hand side. But those non-differentiated terms cannot be found in expressions described in Cartesian geometry (2.31)–(2.35). The explanation for such behaviour can be found in general expression for the divergence differential operator acting on vector entities, cf. expression (B.37) in Appendix B in monograph by *Hanyk*, 1999 and expressions (B.20)–(B.21) in Appendix B, Section B.2.

For the non-diagonal terms we can conclude in similar direct and generalized way following from the geometrical distincts. The geometrical intuition (i.e. the Cartesian approach as a limit case of the spherical) could one allow to use similar limit expressions like

$$r \to a \text{ (near infinity)} \quad \text{and} \quad n \to \infty. \tag{5.4}$$

Comparing both matrices (5.2) and (5.3), we can state that the following expressions justify such behaviour, so that the geometrical idea succeeds

$$\begin{aligned} n/r &\to 0, \\ 1/r &\to 1/a, \\ N/r &\to N/a, \\ 1/r^2 &\to 0, \\ N/r^2 &\to 0, \end{aligned} \tag{5.5}$$

and, let us recall, that the symbol $N = n(n+1)$ on the left-hand side comes from the spherical representation.

But we can find one exception, which does not justify these rules. Term a_{31} in Cartesian matrix (5.2) differs in its behaviour against the rule $1/r \to 1/a$ (cf. term a_{31} in spherical matrix (5.3)). The origin of such fact can not be found on the journey through the formalism leading to the matrix $\boldsymbol{A}$, but in the Poisson equation and its initial form (2.7).

5.1.2 Viscoelasticity

In the viscoelastic case we will compare both matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ coming from the system (3.24) for the solution vector $\boldsymbol{y}$

$$\dot{\boldsymbol{y}}' - \boldsymbol{A}\,\dot{\boldsymbol{y}} = \xi \left[\boldsymbol{D}\,\boldsymbol{y}' + \boldsymbol{E}\,\boldsymbol{y} \right], \tag{5.6}$$

where the Cartesian matrices hold, cf. (3.25) and (3.26),

$$\boldsymbol{D} = \begin{pmatrix} -\frac{K}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2\gamma}{3a} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.7}$$

$$\boldsymbol{E} = \begin{pmatrix} 0 & \frac{KN}{\beta a} & \frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 & 0 & 0 \\ 0 & \rho_0 g_0 \frac{N}{a} & 0 & \frac{N}{a} & 0 & \rho_0 & 0 & 0 \\ \frac{1}{a}\rho_0 g_0 & \frac{2N\gamma}{3a^2} & -\frac{\lambda}{a\beta} & 0 & \frac{\rho_0}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

The spherical representation of was again taken from monograph by *Hanyk*, 1999, expressions (3.24) and (3.25):

$$\boldsymbol{D} = \begin{pmatrix} -\frac{K}{\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4\gamma}{3r} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2\gamma}{3a} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.9}$$

$$
\boldsymbol{E} = \begin{pmatrix}
-\dfrac{2K}{r\beta} & \dfrac{KN}{r\beta} & \dfrac{1}{\beta} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{1}{\mu} & 0 & 0 & 0 & 0 \\
\dfrac{8\gamma}{3r^2} - \dfrac{4\rho_0 g_0}{r} & \dfrac{4N\gamma}{3r^2} + \dfrac{N\rho_0 g_0}{r} & -\dfrac{4\mu}{r\beta} & \dfrac{N}{r} & -\dfrac{(n+1)\rho_0}{r} & \rho_0 & 0 & 0 \\
-\dfrac{4\gamma}{3r^2} + \dfrac{1}{r}\rho_0 g_0 & \dfrac{2N\gamma}{3r^2} & -\dfrac{\lambda}{r\beta} & -\dfrac{3}{r} & \dfrac{\rho_0}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\mu} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{3}{r}
\end{pmatrix}. \tag{5.10}
$$

When referring to differences between the Cartesian and the spherical matrix $\boldsymbol{E}$, we can use the same arguments as in the (previous) elastic case. Hence we obtain the same relations

$$
\begin{aligned}
n/r &\to 0, \\
1/r &\to 1/a, \\
N/r &\to N/a, \\
1/r^2 &\to 0, \\
N/r^2 &\to 0,
\end{aligned} \tag{5.11}
$$

we can make the same conclusions and again find one exception, which does not correspond to these rules. The term e_{42} does not confirm the rule $N/r^2 \to 0$ and the origin of such behaviour can be found in the expression (3.15) for divergence of the time derivative of the elastic part of Cauchy stress tensor $\boldsymbol{\tau}^E$ together with expressions (3.10) for elements of vector $\boldsymbol{y}$.

Comparing matrices $\boldsymbol{D}$, we can see that it is needed to find an explanation for behaviour of the term d_{31}. The zero value of this term in Cartesian matrix comes from zero value of the term b_{33}, what can be seen in expression (2.44).

Tab. 5.1. Physical Parameters of the Homogeneous Layer Model and Other Constants

layer thickness a	6371 km
density ρ_0	5517 $\mathrm{kg\,m^{-3}}$
Lamé elastic parameter λ	3.5288×10^{11} Pa
shear modulus μ	1.4519×10^{11} Pa
bulk modulus K	4.4967×10^{11} Pa
viscosity η	10^{21} Pa s
β	$\lambda + 2\mu$
γ	$3\mu K/\beta$
ξ	μ/η
Newton gravitational constant G	6.6732×10^{-11} $\mathrm{m^3 s^{-2} kg^{-1}}$

5.2 Numerical Implementation

We illustrate the results from the reformulation of IV/MOL approach by some output of the numerical modelling. First, we will focus upon comparison of the analytical and numerical elastic solution for the homogenous and incompressible model. Next, we will move on the depth-dependent, Maxwell viscoelastic Earth models, both incompressible and compressible. We are concerned of homogenous or layered models (such as core-mantle model or simple PREM). Physical properties of the homogenous model can be seen in Table 5.1.

5.2.1 Analytical and Numerical Solution for Homogenous and Incompressible Model

At the end of Chapter 2 we have developed analytical solution for the homogenous and incompressible model of layer.

5.2.2

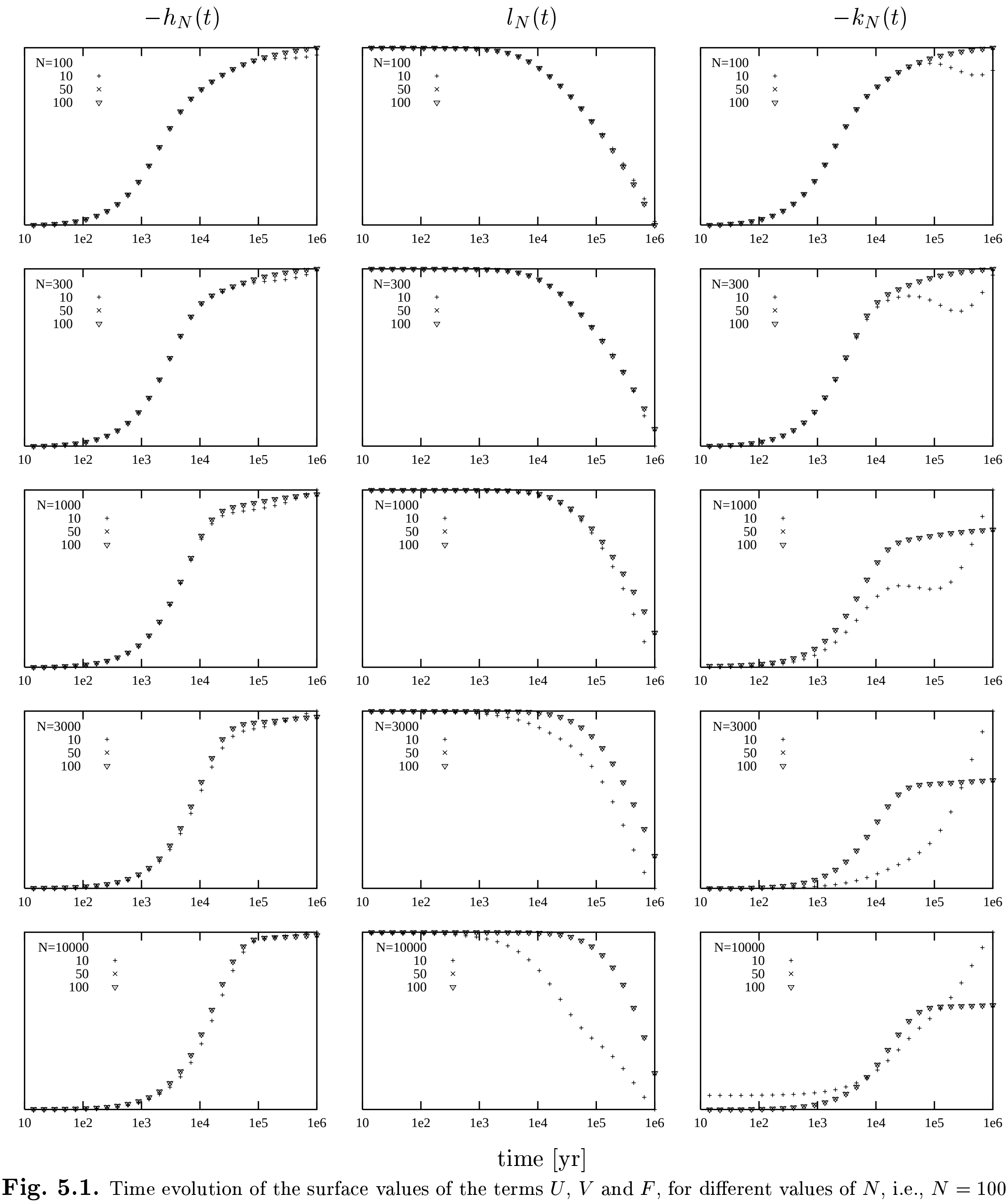

Fig. 5.1. Time evolution of the surface values of the terms U, V and F, for different values of N, i.e., $N = 100$, 300, 1000, 3000 and 10000, of the homogeneous incompressible model evaluated by the IV/MOL approach. The responses have been calculated with various density of the spatial discretization: symbols + denote values obtained with 11 equidistant spatial grid points, × pertain to 51 grid points and △ to 101 grid points ($J = 10$, 50 and 100, respectively).

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Chapter 6

Conclusions

We have finalized the geometrical reformulation of the initial value/method of lines approach to viscoelastic response from the spherical geometry to the Cartesian geometry. The field partial differential equations (2.16)–(2.18) in the material-local form have been subjected to the Fourier horizontal decomposition. With the elastic constitutive relation (2.13) being considered first, the procedure of derivation of the boundary-value problem for the ordinary differential equations (2.53) with respect to the depth has been exposed. We have obtained analytical elastic solution for the homogenous and incompressible model of layer following the analysis presented by *Wu & Peltier*, 1982. The similar procedure to the elastic case and generalized for the Maxwell viscoelastic constitutive relation (2.18), has been introduced to derive the partial differential equations (3.24) with respect to both time and the depth. In accordance with the idea of the method of lines, the spatial discretization has been enforced. This has led to the set of ordinary differential equations with respect to time in the form appropriate for the applications of stiff integrators (*Hanyk*, 1999).

The results we have achieved, led us to the following conclusions:

- The geometrical reformulation developed in this thesis can be applied to regional modelling of the viscoelastic response

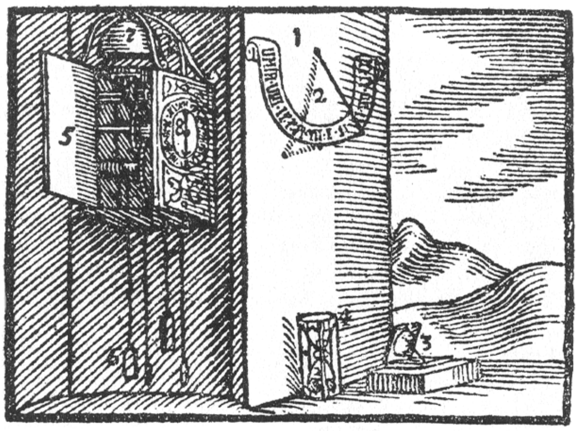
Problems, simplifications we have used,

Further improvements, further use, postseismicity as an internal source contra external source...

We have used models of glaciers (parabolic and cylindrical) as an external force. Further possible applications can lead to internal forces such as postseismic deformation.

Appendices

Hodiny. Horologia. Uhrwerke.



Appendix A

Fourier Transform

The mathematical approach, which is used for quantitative description of some geophysical models, is to be developed in this appendix. First we mention theoretical aspects of the Fourier transform. Secondly, we show some useful identities which may be helpful on journey leading to the final results.

A.1 Defining Fourier Transform

Let $\mathbb{R}_n$ be a n-dimensional space of real numbers and L_1 be a space of finitely integrable functions. For complex function $f \in L_1(\mathbb{R}_n)$ we define its Fourier transform $\mathcal{F}^{(A,B,k)}f$ with the formula

$$(\mathcal{F}^{(A,B,k)}f)(\xi) \equiv A^n \int_{\mathbb{R}_n} e^{-ik(x,\xi)} f(x)dx, \quad \xi \in \mathbb{R}_n, \tag{A.1}$$

and the inverse Fourier transform $\mathcal{F}_{-1}^{(A,B,k)}f$ with the formula

$$(\mathcal{F}_{-1}^{(A,B,k)}f)(\xi) \equiv B^n \int_{\mathbb{R}_n} e^{ik(x,\xi)} f(x)dx, \quad \xi \in \mathbb{R}_n, \tag{A.2}$$

where parameters $A, B, k \in \mathbb{R}\backslash\{0\}$ are related through

$$AB = \frac{|k|}{2\pi}, \tag{A.3}$$

and where we also used a formula for the scalar product of $x \in \mathbb{R}_n, y \in \mathbb{R}_n$ in space $\mathbb{R}_n$,

$$(x, y) \equiv \sum_{j=1}^{n} x_j y_j.$$

There are different ways to define the Fourier transform[1], lots of them differs only by values of the parameters A, B and k. The most frequent values of parameters A, B, k are $A = 1$, $B = 1/(2\pi)$, $k = 1$, hence we obtain well-known formulae for the Fourier transform and the inverse Fourier transform

$$\mathcal{F}f = \int_{\mathbb{R}_n} e^{-i(x,\xi)} f(x)dx, \tag{A.4}$$

$$\mathcal{F}_{-1}f = (2\pi)^{-n} \int_{\mathbb{R}_n} e^{i(x,\xi)} f(x)dx. \tag{A.5}$$

The relation (A.3) ensures relevance of the theorem of inversion:

$$\mathcal{F}_{-1}^{(A,B,k)}\mathcal{F}^{(A,B,k)}f = \mathcal{F}^{(A,B,k)}\mathcal{F}_{-1}^{(A,B,k)}f = f. \tag{A.6}$$

A.2 Properties and Identities

Let us now denote the Fourier transform $\mathcal{F}^{(A,B,k)}f$ and the inverse Fourier transform $\mathcal{F}_{-1}^{(A,B,k)}f$ of a general function $f \in L_1(\mathbb{R}_n)$ by the symbols $\hat{f}$ and $\check{f}$, respectively. Following relations ensue directly from the definitions (A.1)–(A.2)

$$\check{f} = (B/A)^n \hat{f}(-\xi), \qquad \overline{\check{f}(\xi)} = (B/A)^n \hat{\overline{f}}(\xi), \qquad \check{\overline{f}}(\xi) = (B/A)^n \overline{\hat{f}(\xi)}, \tag{A.7}$$

where the overline symbol stands for complex conjugation.

[1] Our choice was similar to *Kopáček*, 2001

Connection between shifting and multiplying by an independent parameter is frequently used property of the Fourier transform

$$\widehat{f(x-z)}(\xi) = e^{-ik(\xi,z)}\hat{f}(\xi), \; z \in \mathbb{R}_n, \tag{A.8}$$

$$\hat{f}(\xi-\zeta) = \widehat{e^{ik(x,\zeta)}f}(x)(\xi), \; \zeta \in \mathbb{R}_n, \tag{A.9}$$

$$\widehat{f(\epsilon x)}(\xi) = |\epsilon|^{-n}\hat{f}(\xi/\epsilon), \; \epsilon \in \mathbb{R}\backslash\{0\}. \tag{A.10}$$

We can see, that shifting in the "time" ("frequency") domain corresponds to multiplication by the exponential in the "frequency" ("time") domain. Following identities are direct consequences of the definitions (A.1)–(A.2):

$$\int_{\mathbb{R}_n} \hat{f} g dx = \int_{\mathbb{R}_n} f \hat{g} dx, \qquad \int_{\mathbb{R}_n} \check{f} g dx = \int_{\mathbb{R}_n} f \check{g} dx, \tag{A.11}$$

for functions $f, g \in L_1(\mathbb{R}_n)$.

Finally, we show relationship between the Fourier transform and derivatives, i.e. multiplication by an independent variable:

- Let $f \in \mathcal{C}^s(\mathbb{R}_n)$, i.e. let f be continuous to the order of s and let derivation $D^\alpha f \in L_1(\mathbb{R}_n)$, where α denotes the order of the derivation and must satisfy $\alpha \leq s$. Then we obtain

$$\widehat{D^\alpha f}(\xi) = (ik\xi)^\alpha \hat{f}(\xi) \tag{A.12}$$

- Let $f \in L_1(\mathbb{R}_n)$ and $x^\alpha f(x) \in L_1(\mathbb{R}_n)$. Then for $\alpha \leq s$ and $\hat{f} \in \mathcal{C}^s(\mathbb{R}_n)$, yields

$$D^\alpha \hat{f}(\xi) = \mathcal{F}((-ikx)^\alpha f(x))(\xi) \tag{A.13}$$

Thus, we may say in short that differentiation in the "time" domain corresponds to multiplication by independent variable in the "frequency" domain and conversely.

There are different ways to define the Fourier transform[2], lots of them differs only by values of the paramaters A, B and k.

[2]Our choice was similar to *Kopáček*, 2001

Mudrctví. Philosophia. Die Weltweißheit.

Appendix B

Discovering the Expansions

In this Appendix we show the exact steps which lead to the resulting expansions shown in Chapters 2 and 3. We define three differential operators, which act on scalar, vector and second-order tensor fields. Collected expressions from the apparatus allow us to converse the field partial differential equations (PDEs) into the system of ordinary differential equations (ODEs). At the end of this Appendix we arrive at expansions for the Poisson equation and for the forcing term in equation of motion.

B.1 Differential Operators in Cartesian Coordinates

Let $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ be the unit basis vectors of a Cartesian system x, y and z, denoted as $(x, y, z) \equiv (\xi_1, \xi_2, \xi_3)$. The differential operators grad, div and rot acting on the scalar and vector functions

$$\begin{aligned} f &= f(x,y,z), && \text{(B.1)} \\ \boldsymbol{u} &= \boldsymbol{u}(z,y,z) = u_x(x,y,z)\boldsymbol{e}_x + u_y(x,y,z)\boldsymbol{e}_y + u_z(x,y,z)\boldsymbol{e}_z && \text{(B.2)} \end{aligned}$$

can be written as follows:

$$\begin{aligned} \operatorname{grad} f \equiv \nabla f &= \sum_{i=1}^{3} \frac{\partial f}{\partial \xi_i} \boldsymbol{e}_i, && \text{(B.3)} \\ \operatorname{div} \boldsymbol{u} \equiv \nabla \cdot \boldsymbol{u} &= \sum_{i=1}^{3} \frac{\partial u_i}{\partial \xi_i}, && \text{(B.4)} \\ \operatorname{rot} \boldsymbol{u} \equiv \nabla \times \boldsymbol{u} &= \sum_{ijk} \varepsilon_{ijk} \frac{\partial u_k}{\partial \xi_j} \boldsymbol{e}_i, && \text{(B.5)} \\ \operatorname{grad} \boldsymbol{u} \equiv \nabla \boldsymbol{u} &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial \xi_i} \right) \boldsymbol{e}_i \boldsymbol{e}_j, && \text{(B.6)} \end{aligned}$$

where ε_{ijk} denotes the Levi-Civita symbol ($\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{312} = -\varepsilon_{213} = 1$, otherwise 0). Expression for the Laplace operator $\Delta \equiv \operatorname{div} \operatorname{grad} \equiv \nabla \cdot \nabla \equiv \nabla^2$ can be obtained by substituting (B.3) into (B.4),

$$\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial \xi_i^2}. \tag{B.7}$$

Let us also mention here two useful identities, which follow from the expressions for the differential second-order operators,

$$\begin{aligned} \Delta \boldsymbol{u} &= \nabla \nabla \cdot \boldsymbol{u} - \nabla \times \nabla \times \boldsymbol{u}, && \text{(B.8)} \\ \nabla \cdot (\nabla \boldsymbol{u})^T &= \nabla \nabla \cdot \boldsymbol{u}. && \text{(B.9)} \end{aligned}$$

B.2 Differential Operators Acting on Fourier Expansions

Let $B_{kl}(x,y)$ be the scalar basis function defined by (2.19) and $\boldsymbol{G}_{kl}^{(-1)}, \boldsymbol{G}_{kl}^{(1)}$ and $\boldsymbol{G}_{kl}^{(0)}$ the vector basis functions defined by (2.21)–(2.23). Denoting the horizontal derivatives of the scalar basis function (2.19) we obtain expressions which allow us to rewrite our further steps more clearly:

$$B_{kl}^x(x,y) = \frac{\partial B_{kl}}{\partial x} = ikB_{kl}, \quad B_{kl}^y(x,y) = \frac{\partial B_{kl}}{\partial y} = ilB_{kl}. \tag{B.10}$$

Hence the vector basis functions (2.21)–(2.23) now hold,

$$\begin{aligned} \boldsymbol{G}_{kl}^{(-1)}(x,y) &= B_{kl}(x,y)\boldsymbol{e}_z, && \text{(B.11)} \\ \boldsymbol{G}_{kl}^{(1)}(x,y) &= aB_{kl}^x(x,y)\boldsymbol{e}_x + aB_{kl}^y(x,y)\boldsymbol{e}_y, && \text{(B.12)} \\ \boldsymbol{G}_{kl}^{(0)}(x,y) &= -aB_{kl}^y(x,y)\boldsymbol{e}_x + aB_{kl}^x(x,y)\boldsymbol{e}_y. && \text{(B.13)} \end{aligned}$$

The expansions of scalar and vector functions can be expressed as

$$f(x,y,z) = \sum_{kl} F_{kl}(z) B_{kl}(x,y), \tag{B.14}$$

$$\boldsymbol{u}(x,y,z) = \sum_{kl} \left[U_{kl}(z) \boldsymbol{G}_{kl}^{(-1)}(x,y) + V_{kl}(z) \boldsymbol{G}_{kl}^{(1)}(x,y) + W_{kl}(z) \boldsymbol{G}_{kl}^{(0)}(x,y) \right]. \tag{B.15}$$

The scalar components of $\boldsymbol{u} = (u_x, u_y, u_z)$ yield from (B.11)–(B.13) and (B.15),

$$u_x(x,y,z) = \boldsymbol{e}_x \cdot \boldsymbol{u} = \sum_{kl} \left[aV_{kl} B_{kl}^x - aW_{kl} B_{kl}^y \right], \tag{B.16}$$

$$u_y(x,y,z) = \boldsymbol{e}_y \cdot \boldsymbol{u} = \sum_{kl} \left[aV_{kl} B_{kl}^y + aW_{kl} B_{kl}^x \right], \tag{B.17}$$

$$u_z(x,y,z) = \boldsymbol{e}_z \cdot \boldsymbol{u} = \sum_{kl} U_{kl} B_{kl}. \tag{B.18}$$

From now on we suppress both subscripts k and l of the coefficients $F_{kl}, U_{kl}, V_{kl}, W_{kl}, X_{kl}$, and of the horizontal derivatives of the basis function, B_{kl}^x and B_{kl}^y. Expressions (B.3)–(B.4) for the first-order differential operators acting on expansions (B.14)–(B.15) using (B.16)–(B.18) yields:

$$\begin{aligned} \nabla f &= \sum_{kl} \left[F' B_{kl} \boldsymbol{e}_z + F \left(B^x \boldsymbol{e}_x + B^y \boldsymbol{e}_y \right) \right] \\ &= \sum_{kl} \left[F' \boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{a} F \boldsymbol{G}_{kl}^{(1)} \right], \end{aligned} \tag{B.19}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= \sum_{kl} \left[a(-k^2 V + lkW - l^2 V - klW) + U' \right] B_{kl} \\ &= \sum_{kl} X B_{kl}, \end{aligned} \tag{B.20}$$

$$\text{with} \quad X = U' - \tfrac{N}{a} V, \tag{B.21}$$

$$\nabla \times \boldsymbol{u} = \sum_{kl} \left[-\tfrac{N}{a} W \boldsymbol{G}_{kl}^{(-1)} - W' \boldsymbol{G}_{kl}^{(1)} + (-\tfrac{1}{a} U + V') \boldsymbol{G}_{kl}^{(0)} \right], \tag{B.22}$$

where $N = a^2(k^2 + l^2)$ and the prime $'$ stands for the derivative with respect to z. The second-order differential operators take the form as follows:

$$\nabla \cdot \nabla f = \sum_{kl} \left[F'' - \tfrac{N}{a^2} F \right] B_{kl}, \tag{B.23}$$

$$\nabla \nabla \cdot \boldsymbol{u} = \sum_{kl} \left[\left(U'' - \tfrac{N}{a} V' \right) \boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{a} \left(U' - \tfrac{N}{a} V \right) \boldsymbol{G}_{kl}^{(1)} \right], \tag{B.24}$$

$$\begin{aligned} \nabla \times \nabla \boldsymbol{u} = \sum_{kl} \Big[& \tfrac{N}{a} \left(\tfrac{1}{a} U - V' \right) \boldsymbol{G}_{kl}^{(-1)} + \left(\tfrac{1}{a} U' - V'' \right) \boldsymbol{G}_{kl}^{(1)} \\ & + \left(\tfrac{N}{a^2} W - W'' \right) \boldsymbol{G}_{kl}^{(0)} \Big], \end{aligned} \tag{B.25}$$

$$\begin{aligned} \nabla \cdot \nabla \boldsymbol{u} = \sum_{kl} \Big[& \left(U'' - \tfrac{N}{a^2} U \right) \boldsymbol{G}_{kl}^{(-1)} + \left(V'' - \tfrac{N}{a^2} V \right) \boldsymbol{G}_{kl}^{(1)} \\ & + \left(W'' - \tfrac{N}{a^2} W \right) \boldsymbol{G}_{kl}^{(0)} \Big], \end{aligned} \tag{B.26}$$

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{u} = \sum_{kl} \Big[& \tfrac{N}{a} \left(-V' + \tfrac{1}{a} U \right) \boldsymbol{G}_{kl}^{(-1)} + \left(-V'' + \tfrac{1}{a} U' \right) \boldsymbol{G}_{kl}^{(1)} \\ & + \left(-W'' + \tfrac{N}{a^2} W \right) \boldsymbol{G}_{kl}^{(0)} \Big]. \end{aligned} \tag{B.27}$$

B.3 Strain Tensor

The scalar components of symmetric strain tensor $\boldsymbol{e}$, defined by

$$\boldsymbol{e} = \frac{1}{2} \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right], \tag{B.28}$$

can be expressed in the Cartesian coordinates using the components of $\nabla \boldsymbol{u}$ by (B.6), and the expansion of these components can be simply obtained by substitution from (B.16)–(B.18) as follows:

$$\begin{pmatrix} e_{xx} & 2e_{xy} & 2e_{xz} \\ " & e_{yy} & 2e_{yz} \\ " & " & e_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \\ " & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \\ " & " & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{B.29}$$

$$= \sum_{kl} \begin{pmatrix} a(-k^2V + klW)B_{kl} & a(l^2W - k^2W - 2klV)B_{kl} & a(V'B^x - W'B^y) + UB^x \\ " & a(-l^2V - klW)B_{kl} & a(V'B^y + W'B^x) + UB^y \\ " & " & U'B_{kl} \end{pmatrix} \tag{B.30}$$

Symmetry of the tensors is indicated by the double quotes. From the form of (B.30) we can easily obtain the first invariant of $\boldsymbol{e}$,

$$\overline{\boldsymbol{e}} \equiv e_{xx} + e_{yy} + e_{zz} = \sum\nolimits_{kl} (U' - \tfrac{N}{a}V)B_{kl} = \nabla \cdot \boldsymbol{u}. \tag{B.31}$$

We express the Fourier expansion of the vector $\boldsymbol{e}_z \cdot \boldsymbol{e}$,

$$\begin{aligned} \boldsymbol{e}_z \cdot \boldsymbol{e} &= \boldsymbol{e}_x e_{zx} + \boldsymbol{e}_y e_{zy} + \boldsymbol{e}_z e_{zz} \\ &= \sum\nolimits_{kl} \left[U'\boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{2}(\tfrac{1}{a}U + V')\boldsymbol{G}_{kl}^{(1)} + \tfrac{1}{2}W'\boldsymbol{G}_{kl}^{(0)} \right], \end{aligned} \tag{B.32}$$

and the expansion of the $\nabla \cdot \boldsymbol{e}$,

$$(\nabla \cdot \boldsymbol{e})_j = \sum_i \frac{\partial e_{ij}}{\partial \xi_i} = \frac{1}{2} \sum_i \left(\frac{\partial^2 u_j}{\partial \xi_i^2} + \frac{\partial^2 u_i}{\partial \xi_i \partial \xi_j} \right), \tag{B.33}$$

where the individual components hold,

$$\begin{aligned} 2(\nabla \cdot \boldsymbol{e})_x &= (-2ak^2VB^x + 2ak^2WB^y - aklVB^y - aklWB^x - al^2VB^x \\ &\quad + al^2WB^y + U'B^x + aV''B^x - aW''B^y), \end{aligned} \tag{B.34}$$

$$\begin{aligned} 2(\nabla \cdot \boldsymbol{e})_y &= (-aklVB^x + aklWB^y - ak^2VB^y - ak^2WB^x - 2al^2VB^y \\ &\quad -2al^2WB^x + U'B^y + aV''B^y + aW''B^x), \end{aligned} \tag{B.35}$$

$$2(\nabla \cdot \boldsymbol{e})_z = (-ak^2V' + aklW' - k^2U - al^2V' - aklW' - l^2U + 2U'')B_{kl}. \tag{B.36}$$

Hence we arrive at:

$$\begin{aligned} \nabla \cdot \boldsymbol{e} &= \sum\nolimits_{kl} \left[\frac{1}{2} \left(2U'' - \tfrac{N}{a}V' - \tfrac{N}{a^2}U\right) \boldsymbol{G}_{kl}^{(-1)} + \frac{1}{2} \left(V'' + \tfrac{1}{a}U' - 2\tfrac{N}{a^2}V\right) \boldsymbol{G}_{kl}^{(1)} \right. \\ &\quad \left. + \frac{1}{2} \left(W'' - \tfrac{N}{a^2}W\right) \boldsymbol{G}_{kl}^{(0)} \right], \end{aligned} \tag{B.37}$$

where we used identities

$$klB^x = k^2B^y, \tag{B.38}$$

$$klB^y = l^2B^x. \tag{B.39}$$

B.4 Elastic Stress Tensor and Elastic Stress Vector of Traction

The elastic stress tensor $\boldsymbol{\tau}^E$ is defined by the elastic constitutive relation, cf. (2.13), with 1-D depth-dependent distribution of Lamé parameters, $\lambda(z)$ and $\mu(z)$,

$$\boldsymbol{\tau}^E = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + 2\mu\boldsymbol{e} = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right), \tag{B.40}$$

where $\lambda = (K - \frac{2}{3}\mu)$ and $\boldsymbol{e}$ is the strain tensor defined by (B.28). Tensor $\boldsymbol{\tau}^E$ is symmetric. To obtain the components of $\boldsymbol{\tau}^E$ we use expressions for the strain tensor $\boldsymbol{e}$ (B.29)–(B.30) and expansion (B.20),

$$\begin{pmatrix} \tau^E_{xx} & \tau^E_{xy} & \tau^E_{xz} \\ " & \tau^E_{yy} & \tau^E_{yz} \\ " & " & \tau^E_{zz} \end{pmatrix} = \begin{pmatrix} \lambda\nabla\cdot\boldsymbol{u} + 2\mu\frac{\partial u_x}{\partial x} & \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \\ " & \lambda\nabla\cdot\boldsymbol{u} + 2\mu\frac{\partial u_y}{\partial y} & \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \\ " & " & \lambda\nabla\cdot\boldsymbol{u} + 2\mu\frac{\partial u_z}{\partial z} \end{pmatrix}$$

$$= \sum_{kl} \left(\begin{matrix} \left[\lambda X + 2\mu a\left(-k^2V + klW\right)\right]B_{kl} & \mu a(l^2W - k^2W - 2klV)B_{kl} & \dots \\ " & \left[\lambda X + 2\mu a(-l^2V - klW)\right]B_{kl} & \dots \\ " & " & \dots \end{matrix} \right.$$

$$\left. \begin{matrix} \dots & \mu(UB^x + aV'B^x - aW'B^y) \\ \dots & \mu(UB^y + aV'B^y + aW'B^x) \\ \dots & (\lambda X + 2\mu U')B_{kl} \end{matrix} \right). \tag{B.41}$$

The expression for the scalar product of $\boldsymbol{e}_z\cdot\boldsymbol{\tau}^E$, i.e. the stress vector of traction $\boldsymbol{T}^E_z$, can be found as follows, using (B.20) and (B.32),

$$\begin{aligned} \boldsymbol{T}^E_z \equiv \boldsymbol{e}_z\cdot\boldsymbol{\tau}^E &= \lambda\boldsymbol{e}_z\nabla\cdot\boldsymbol{u} + 2\mu\boldsymbol{e}_z\cdot\boldsymbol{e} \\ &= \sum\nolimits_{kl}\left[(\lambda X + 2\mu U')B_{kl}\boldsymbol{e}_z\right. \\ &\quad + \mu(UB^y + aV'B^y + aW'B^x)\boldsymbol{e}_y \\ &\quad \left. + \mu(UB^x + aV'B^x - aW'B^y)\boldsymbol{e}_x\right] \\ &= \sum\nolimits_{kl}\left[T^{E(-1)}_z\boldsymbol{G}^{(-1)}_{kl} + T^{E(1)}_z\boldsymbol{G}^{(1)}_{kl} + T^{E(0)}_z\boldsymbol{G}^{(0)}_{kl}\right], \end{aligned} \tag{B.42}$$

where

$$T^{E(-1)}_z = 2\mu U' + \lambda X = \beta U' - \lambda\frac{N}{a}V, \tag{B.43}$$

$$T^{E(1)}_z = \mu(V' + \frac{1}{a}U), \tag{B.44}$$

$$T^{E(0)}_z = \mu W', \tag{B.45}$$

with $\beta = \lambda + 2\mu$ and again with suppressed both subscripts k and l of the coefficients $T^E_{z-1,kl}$, $T^E_{z1,kl}$ and $T^E_{z0,kl}$. In the next step we will express the expansion of $\nabla\cdot\boldsymbol{\tau}^E$, exactly

$$\nabla\cdot\boldsymbol{\tau}^E = \lambda\nabla\nabla\cdot\boldsymbol{u} + \lambda'\boldsymbol{e}_z\nabla\cdot\boldsymbol{u} + 2\mu\nabla\cdot\boldsymbol{e} + 2\mu'\boldsymbol{e}_z\cdot\boldsymbol{e}. \tag{B.46}$$

First, application of the grad operator (B.6) onto (B.20) gives,

$$\nabla(\nabla\cdot\boldsymbol{u}) = \sum\nolimits_{kl}\left[X'\boldsymbol{G}^{(-1)}_{kl} + X\boldsymbol{G}^{(1)}_{kl}\right] \tag{B.47}$$

Second, applying $\boldsymbol{e}_z$ on (B.20) yields,

$$\boldsymbol{e}_z(\nabla\cdot\boldsymbol{u}) = \sum\nolimits_{kl} X\boldsymbol{G}_{kl}^{(-1)}. \tag{B.48}$$

Third, the expression for $\nabla\cdot\boldsymbol{e}$ is taken from (B.37), fourth, the term $\boldsymbol{e}_z\cdot\boldsymbol{e}$ is taken from (B.32) and finally, we may conclude that the expansion of $\nabla\cdot\boldsymbol{\tau}^E$ equals to

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau}^E &= \sum\nolimits_{kl}\Big\{\left[\lambda X' + \lambda' X + \mu(2U'' - \tfrac{N}{a}V' - \tfrac{N}{a^2}U) + 2\mu' U'\right]\boldsymbol{G}_{kl}^{(-1)} \\ &\quad + \left[\tfrac{\lambda}{a}X + \mu(V'' + U' - 2\tfrac{N}{a^2}V) + \mu'(\tfrac{1}{a}U + V')\right]\boldsymbol{G}_{kl}^{(1)} \\ &\quad + \left[\mu(W'' - \tfrac{N}{a^2}W) + \mu' W'\right]\boldsymbol{G}_{kl}^{(0)}\Big\}. \end{aligned} \tag{B.49}$$

Differentiation of (B.43)–(B.45) with respect to z gives

$$T_z^{E(-1)'} = \lambda' X + \lambda X' + 2\mu' U' + 2\mu U'', \tag{B.50}$$

$$T_z^{E(1)'} = \mu'(\tfrac{1}{a}U + V') + \mu(\tfrac{1}{a}U' + V''), \tag{B.51}$$

$$T_z^{E(0)'} = \mu' W' + \mu W''. \tag{B.52}$$

If we substitute these expressions into (B.49) and evaluate the term X from (B.21) and (B.43), we obtain the final form,

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau}^E &= \sum\nolimits_{kl}\Big[(T_z^{E(-1)'} - \tfrac{N}{a}T_z^{E(1)})\boldsymbol{G}_{kl}^{(-1)} \\ &\quad + (T_z^{E(1)'} + \tfrac{\lambda}{a\beta}T_z^{E(-1)} - \tfrac{N}{a^2}(\gamma+\mu)V)\boldsymbol{G}_{kl}^{(1)} \\ &\quad + (T_z^{E(0)'} - \mu\tfrac{N}{a^2}W)\boldsymbol{G}_{kl}^{(0)}\Big], \end{aligned} \tag{B.53}$$

where we defined,

$$\gamma = \frac{\mu(3\lambda + 2\mu)}{\beta}, \tag{B.54}$$

and used identity following from the last expression,

$$\gamma + \mu = \frac{4\mu(\lambda+\mu)}{\beta}. \tag{B.55}$$

B.5 Forcing Term Expansion and Poisson Equation

Now let us express the expansion for the forcing term $\boldsymbol{f}$ in (2.16). As we mentioned in Chapter 1, we assume 1-D depth-dependent distribution of density $\rho_0 = \rho_0(z)$. Introducing the Fourier expansion of φ_1, cf. (2.26),

$$\varphi_1 = \sum\nolimits_{kl} F_{kl}B_{kl}, \tag{B.56}$$

the expansion of $\boldsymbol{f}$ can be found (both subscripts k and l are suppressed for the coefficient F_{kl}, U_{kl} and V_{kl}),

$$\begin{aligned}\boldsymbol{f} &= -\rho_0\nabla\varphi_1 + \nabla\cdot(\rho_0\boldsymbol{u})g_0\boldsymbol{e}_z - \nabla(\rho_0 g_0\boldsymbol{e}_z\cdot\boldsymbol{u}) \\ &= -\rho_0\nabla\varphi_1 + \rho_0\nabla\cdot\boldsymbol{u}g_0\boldsymbol{e}_z - \rho_0\nabla(g_0\boldsymbol{e}_z\cdot\boldsymbol{u}) \\ &= \sum\nolimits_{kl}\Big[-\rho_0\left(F'\boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{a}F\boldsymbol{G}_{kl}^{(1)}\right) + \rho_0 g_0\left(U' - \tfrac{N}{a}V\right)\boldsymbol{G}_{kl}^{(-1)} \\ &\quad - \rho_0\left((g_0U)'\boldsymbol{G}_{kl}^{(-1)} + \tfrac{g_0}{a}U\boldsymbol{G}_{kl}^{(1)}\right)\Big] \\ &= \sum\nolimits_{kl}\Big[\left(-\rho_0 g_0\tfrac{N}{a}V_{kl} - \rho_0 Q_{kl}\right)\boldsymbol{G}_{kl}^{(-1)} + \tfrac{1}{a}\left(-\rho_0 F_{kl} - \rho_0 g_0 U_{kl}\right)\boldsymbol{G}_{kl}^{(1)}\Big], \end{aligned} \tag{B.57}$$

where the auxiliary coefficient $Q_{kl}(z)$ holds,

$$Q_{kl} = F'_{kl} + 4\pi G\rho_0 U_{kl}, \tag{B.58}$$

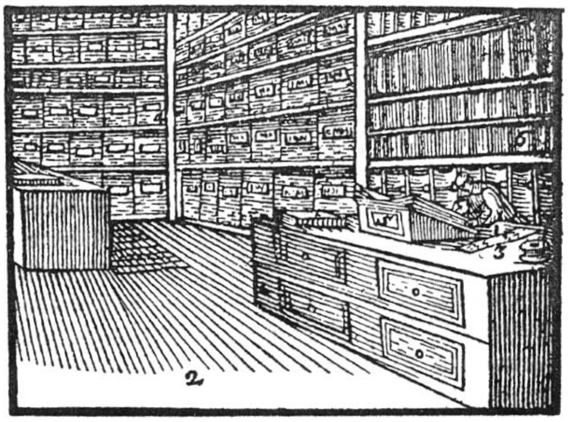
and where the Poisson equation (2.7) for the initial field φ_0 has been used in the case, that the value of initial gravitational acceleration g_0 is assumed to be positive,

$$g'_0 - 4\pi G\rho_0 = 0. \tag{B.59}$$

The expansion of the left-hand side of the Poisson equation (2.17) can be derived in two steps:

$$\begin{aligned}
\nabla\varphi_1 + 4\pi G\rho_0 \boldsymbol{u} &= \sum\nolimits_{kl} \Big[\big(F' + 4\pi G\rho_0 U\big) \boldsymbol{G}^{(-1)}_{kl} + \big(\tfrac{1}{a}F + 4\pi G\rho_0 V\big) \boldsymbol{G}^{(1)}_{kl} \\
&\quad + \; 4\pi G\rho_0 W \boldsymbol{G}^{(0)}_{kl} \Big] \\
&= \sum\nolimits_{kl} \Big[Q\boldsymbol{G}^{(-1)}_{kl} + \big(\tfrac{1}{a}F + 4\pi G\rho_0 V\big) \boldsymbol{G}^{(1)}_{kl} \\
&\quad + \; 4\pi G\rho_0 W \boldsymbol{G}^{(0)}_{kl} \Big], \qquad \text{(B.60)} \\
\nabla\cdot(\nabla\varphi_1 + 4\pi G\rho_0 \boldsymbol{u}) &= \sum\nolimits_{kl} \left[Q'_{kl} - \tfrac{N}{a^2}F_{kl} - 4\pi G\rho_0 \tfrac{N}{a} V_{kl} \right] B_{kl}. \qquad \text{(B.61)}
\end{aligned}$$

Knihárna (knihkupectví). Bibliopolium. Der Buchladen.



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