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SEISMIC SURFACE WAVES

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Lecture notes for post-graduate studies

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Contents

Preface

Throwing a stone into water creates waves which propagate from the place of incidence. The amplitudes of these waves decrease rapidly with depth, so that their energy propagates practically only in the superficial layer. Consequently, these waves are referred to as *surface waves*. Note that water waves are frequently presented as the best-known type of waves, which we encounter in day-to-day use. However, the mathematical description of water waves is more complicated than the description of many other waves.

The electromagnetic waves propagating along the surface of conductors (skin-effect) represent another type of surface waves. Analogously, surface elastic waves can propagate along the surface of an elastic substance. Similar waves, which are generated by earthquakes, artificial explosions and analogous sources, and propagate along the Earth's surface, are referred to as *seismic surface waves*.

Despite some similarities which water waves and seismic surface waves display, there are substantial differences in the forces producing them. The main force forming water waves is gravitation (or rather gravity, i.e. the superposition of the gravitational force and the centrifugal force due to the Earth's rotation). For this reason, these waves are referred to as *gravitational waves* on water. At short periods, the effect of the surface tension is also significant, so that in this case we speak of capillary-gravitational waves. As opposed to this, seismic surface waves are produced predominantly by elastic forces; the effect of gravity is substantially smaller and, therefore, this effect is often neglected.

The most important information on the constitution of the Earth's interior has been obtained from studies of seismic body waves (longitudinal and transverse seismic waves propagating through the Earth). The division of the Earth into the Earth's crust, mantle and core, the later division into so-called Bullen zones, or the latest laterally inhomogeneous models of the Earth are all based predominantly on studies of seismic body waves. On the other hand, surface waves usually have the largest amplitudes on standard seismograms, and these waves also contribute considerably to the damaging effects of earthquakes. They, therefore, deserve the most attention. Nevertheless, some specific problems of an observational and theoretical nature caused that, initially, surface waves were considered in structural studies only exceptionally. Even now, in the present routine processing of seismograms at seismological observatories, surface waves are practically only used to determine the earthquake magnitude.

Interest in applying surface waves to structural studies began to increase in the middle of the 20th century. The following factors contributed to this progress:

- the construction of *long-period seismographs*, which made it possible to observe surface waves in broad frequency bands;
- advances in the surface-wave *theory*, e.g., the introduction of matrix methods, which have made it easier to consider complicated multilayered models of the medium;

- the application of *computers* which, e.g., have made it possible to solve transcendental dispersion equations for surface waves quickly.

Since then, surface waves have been used to treat many specific problems, such as: to study the existence and structure of the so-called low-velocity channel in the upper mantle; to distinguish between the continental and oceanic type of the Earth's crust; to determine the mean parameters of the Earth's crust in extended regions, including regions which are difficult to access (mountains, oceans, polar regions); to study lateral inhomogeneities in the Earth's crust, e.g., the position of faults. Another very promising application of surface waves seems to be the computation of complete synthetic seismograms by summing surface-wave modes. Surface waves also find technical applications in non-destructive testing of materials, electro-mechanical transducers and many others. Moreover, the mathematical methods used in the theory of surface waves are also applicable to some problems of propagation of elastic body waves, electromagnetic waves (e.g., waves in the ionosphere), temperature waves, in the physics of thin layers, etc.

Although surface waves are important from the scientific and practical points of view, less attention is usually paid to them in physics textbooks than to body waves. This is caused mainly by the more complicated physical character of surface waves. For example, it is difficult to imagine them in terms of rays propagating from the source. On the other hand, surface waves do not represent a principally new type of wave, but only an interference phenomenon of body waves. Therefore, the theory presented in these lecture notes can also be used in studies of other types of interference waves we encounter in seismology, physics and technical practice.

These lecture notes have been written for the purposes of post-graduate studies in geophysics, in particular for the corresponding part of the course of lectures on the attenution and dispersion of seismic waves, organised by the Universidade Federal da Bahia, Salvador, Brazil.

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Chapter 1

Main Types of Elastic Waves and Their Properties

In physics, waves are usually divided into *progressive* and *standing* waves. Seismic waves are also of these two types. Progressive seismic waves propagate away from seismic sources. In these lecture notes, we shall deal only with this type of waves. Standing seismic waves, known as the free oscillations of the Earth, represent vibrations of the Earth as a whole. These oscillations are generated by strong earthquakes.

From the point of view of the spatial concentration of energy, waves can be divided into *body waves* and *surface waves*. Body waves can propagate into the interior of the corresponding medium, whereas surface waves are concentrated along the surface of the medium. Acoustic waves in air, or electromagnetic waves in vacuum are examples of body waves. Examples of surface waves have already been mentioned in the Preface.

Note that, instead of surface waves, we should rather speak of a broader category of *guided waves*. Guided waves propagate along the surface of a medium (surface waves), along internal discontinuities, or other waveguides. Since seismic surface waves represent the most important type of seismic guided waves, we shall speak only of surface waves.

In these lecture notes we shall deal with various types of elastic waves. In order to obtain an initial idea of them, we shall give a brief review here; see Tab. 1.1. The corresponding derivations will be given in the following chapters.

Table 1.1. Principal types of progressive elastic waves.
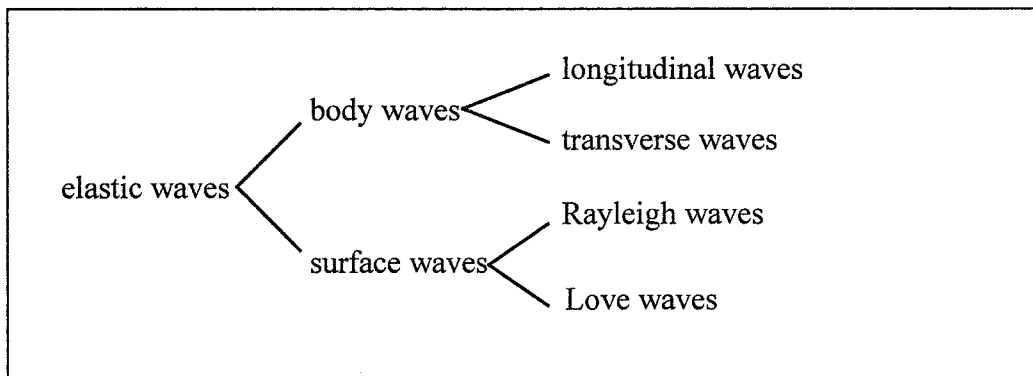
1.1 Body Waves

It follows from the theory of elasticity that there are two principal types of elastic body waves:

1) *Longitudinal waves*, also called compressional, dilatational or irrotational waves. In seismology, they are also called P waves (primary waves), because they represent the first waves appearing on seismograms. These

waves involve the compression and rarefaction of the material as the wave passes through it, but not rotation. Every particle of the medium, through which the longitudinal wave is passing, vibrates about its equilibrium position in the direction in which the wave is travelling (Fig. 1.1). Sound waves are examples of waves of this category.

2) *Transverse waves*, also called shear, rotational or equivoluminal waves. In seismology, they are also called S waves (secondary waves). These waves involve shearing and rotation of the material as the wave passes through it, but no volume change. The particle motion is perpendicular to the direction in which the wave is travelling (Fig. 1.1).
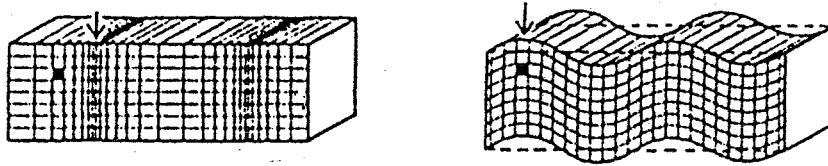
Fig. 1.1. Deformations of the medium when body waves propagate from left to right: longitudinal wave (on the left), and transverse wave (SV wave, i.e. a vertically polarised S wave, on the right). (After Fowler (1994)).

The velocities of longitudinal waves, α, and of transverse waves, β, in a homogeneous and isotropic medium are given by the formulae

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}, \qquad (1.1)$$

where λ and μ are the Lamé coefficients, and ρ is the density. Coefficient μ is the shear modulus, but coefficient λ has no immediate physical meaning.

Since Poison's relation, $\lambda = \mu$, holds in many elastic materials,

$$\alpha = \beta\sqrt{3}. \qquad (1.2)$$

This relation is frequently used in seismology.

Both longitudinal and transverse elastic waves can propagate in solid media. However, only longitudinal waves can propagate in liquids and gases; transverse waves cannot propagate in these media because $\mu = 0$ and, consequently, $\beta = 0$.

In a homogeneous anisotropic medium, three different waves can propagate, namely one quasi-longitudinal and two quasi-transverse waves (Pšencík, 1994).

Elastic body waves are reflected and transmitted at the discontinuities of elastic parameters. This increases the number of waves which are observed on seismograms.

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1.2 Surface Waves

Only longitudinal and transverse waves can propagate in a homogeneous, isotropic and unlimited medium. If the medium is bounded, another type of waves, surface waves, can be guided along the surface of the medium. These waves usually form the principal phase of seismograms. There are two types of surface elastic waves:

1) *Rayleigh waves*. These waves are elliptically polarised in the plane which is determined by the normal to the surface and by the direction of propagation (Fig. 1.2). Near the surface of a homogeneous half-space, the particle motion is a retrograde vertical ellipse (anticlockwise for a wave travelling to the right).

2) *Love waves*. The particle motion in these waves is transverse and parallel to the surface (Fig. 1.2). As opposed to Rayleigh waves, Love waves cannot propagate in a homogeneous half-space. Love waves can propagate only if the S-wave velocity generally increases with the distance from the surface of the medium.

(a) Rayleigh wave
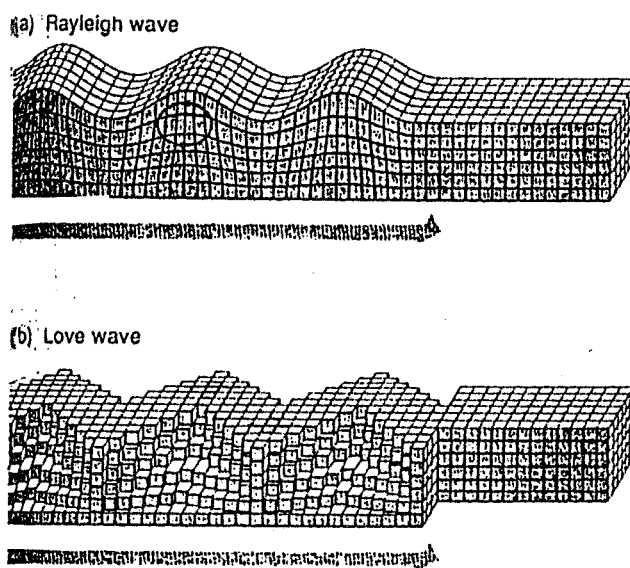
(b) Love wave

Fig. 1.2. The particle motion for surface waves: (a) Rayleigh waves and (b) Love waves. (After Fowler (1994)).

The simplest medium in which Rayleigh waves can propagate is a homogeneous isotropic half-space. The velocity of Rayleigh waves in this medium, c, is slightly less than the transverse wave velocity, $c \approx 0.9\beta$, and is independent of frequency. Thus, Rayleigh waves in this simple model of the medium are non-dispersive.

The simplest model in which Love waves can propagate consists of a homogeneous isotropic layer on a homogeneous isotropic half-space. Both the Rayleigh and Love waves in this model are already dispersive, i.e. their velocities are dependent on frequency.

As we have already mentioned, elastic surface waves do not represent principally new types of waves, but only *interference phenomena* of body

waves. Therefore, in principle, we could attempt to construct the wave field of surface waves (and of other guided waves) by summing body waves. However, this approach would be inconvenient if a large number of waves is to be taken into account (thin layers, large distances from the source). Therefore, a more appropriate mathematical description must be sought for surface waves, as well as for the other interference waves.

We shall emphasise the interference character of surface waves in many places of these lecture notes, in order to gain a deeper insight into the formation of these waves, to understand their specific properties, such as their dispersion and polarisation better, and to be able to apply the same mathematical approaches (e.g., matrix methods) both to surface-wave and body-wave problems.

1.3 Main Differences between Seismic Body Waves and Surface Waves

Let us summarise the main properties of seismic body waves and surface waves, as observed on seismograms of distant earthquakes:

1) Records of a seismic event begin with longitudinal waves, followed by transverse waves, and finally by surface waves.
2) Surface waves usually have larger amplitudes and longer periods.
3) Surface waves display a characteristic dispersion and polarisation.

An example of a seismogram in shown in Fig 1.3. Other examples will be given below.
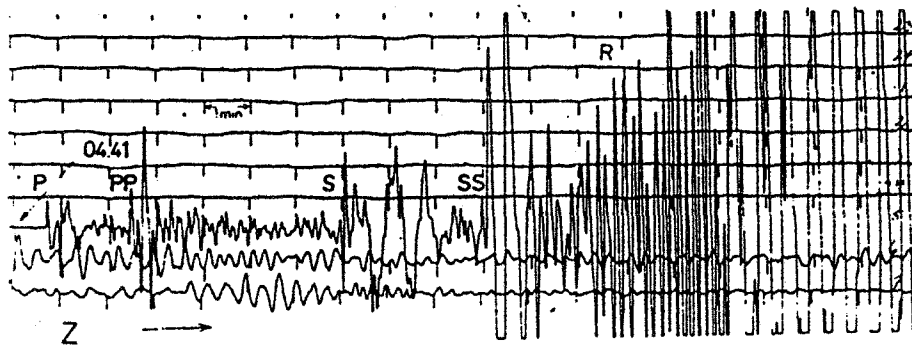
Fig 1.3. The China earthquake of November 13, 1965, recorded at Kiruna, Sweden. The higher-mode Rayleigh waves are exceptionally pronounced (the waves with higher frequencies at the beginning of the surface-wave group). (After Båth (1979)).

1.4 Dispersion of Waves

In these lecture notes we shall pay much attention to the dispersion of surface waves.

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We speak of the *dispersion of waves* if their velocity depends on frequency. A transient wave (a wave of a finite duration) changes its shape during propagation in a dispersive medium, because its individual spectral components propagate with different velocities. This distortion of waves causes some technical problems in the transmission of signals, and in the accurate measurements of their velocity. However, this phenomenon can be used to study the medium through which the waves have propagated. This has applications in seismology, but also in geomagnetism, physics of the magnetosphere and ionosphere, and in technical practice.

There are two types of wave dispersion:
- material dispersion;
- geometrical dispersion.

The *material dispersion* is due to the internal structure of substances. This type of dispersion is well known from optics, since the velocity of light in material media depends on frequency. This dispersion forms the basis of spectroscopy. The material dispersion of elastic waves is closely associated with their attenuation. This dispersion is usually relatively weak and, therefore, we shall not deal with it in these lecture notes.

The *geometrical dispersion* is due to the interference of waves. We encounter this dispersion when waves propagate in thin layers, various waveguides, or along the surface of a medium. We shall study this dispersion in detail in these lecture notes.

Chapter 2

Historical Development of the Theory of Elasticity and of the Theory of Seismic Surface Waves

> Science is the knowledge of many,
> orderly and methodically digested and arranged,
> so as to become attainable by one.
> (J.F.W. Herschel)

The theory of seismic waves is based on the theory of elasticity. In this chapter we shall deal with the historical development of these theories, especially with those aspects of the theory of elasticity which are closely related to the development of seismology. The theory of elasticity studies the behaviour of bodies subjected to forces, both as to their deformation as well as to their ultimate disruption under sufficiently large stresses.

In preparing this chapter we have drawn mainly on the books by Love (1927), Love (1911), Båth (1979), and the very comprehensive treatise by Todhunter and Pearson (1886).

2.1 Theory of Elasticity in the Seventeenth and Eighteenth Centuries

The modern theory of elasticity may be considered to have originated in 1821, when Navier first presented the equations for the equilibrium and motion of elastic solids. To understand the evolution of our modern conceptions, it is necessary to go back to the research of the seventeenth and eighteenth centuries, when experimental knowledge of the behaviour of strained bodies was gained and some special principles were formulated.

The first memoir requiring notice is the second dialogue of the *Discorsie Dimostrazioni matematiche* by Galileo Galilei (1638). This dialogue not only gave the impulse, but also determined the direction which was subsequently followed by many researchers. Galileo formulated conditions with regard to the fracture of solids (rods, beams and hollow cylinders). The noteworthy feature of his discussion is his assumption that the fibres of a strained beam are inextensible. He endeavoured to determine the resistance of a beam, one end of which is built into a wall, at the moment it tends to break under its own or an applied weight. He found that, with increasing load, the beam bends around an axis perpendicular to its length and situated in the plane of the wall. The problem to determine this axis is referred to as Galileo's problem. Although Galileo did not give any mathematical relations between load and deformation, his work was pioneering in the theory of elasticity.

Undoubtedly the next great landmark in the theory, initiated by Galileo's question, is the discovery of *Hooke's law*. This law provided the necessary experimental foundation for the theory. Hooke discovered this law in 1660, but

did not publish until 1676. In 1678 he formulated this law as follows: "Ut tensio sic vis", i.e. "The power of any spring is in the same proportion to the tension thereof". By "tension" Hooke understood, as he proceeded to explain, that which we now call "extension". Hooke did not probably apply this law to solving Galileo's problem. This application was made by Mariotte, who discovered the same law independently in 1680. Hooke in England and Mariotte in France then appropriated the experimental discovery of what we now term stress-strain relations.

In the interval between the discovery of Hooke's law and that of the general differential equations of elasticity by Navier, the attention of the researchers in the elasticity theory was chiefly directed to the solution and extension of Galileo's problem, and the related theories of the vibration of bars and plates, and the stability of columns. Many famous mathematicians and physicists, such as James Bernoulli, Daniel Bernoulli, Euler, Lagrange, Coulomb, Young, took part in these investigations. Although many special problems were solved during this period, these investigations did not lead to broad generalisations. The situation was complicated mainly by unresolved problems concerning the constitution of bodies. According to the Newtonian conception, material bodies are made up of small parts which act on one another by means of central forces. Newton regarded the "molecules" to have finite sizes and definite shapes. However, his successors gradually simplified the "molecules" into material points. The conception of material points was found to be very useful in many branches of mechanics. However, its application to the problems of elasticity often led to oversimplified results. In particular, the conception of material points, between which central forces act, leads to a smaller number of elastic constants than those which are actually necessary to describe real media.

Navier was the first to investigate the *general equations* of the theory of elasticity. He presented his memoir to the Paris Academy in 1821. He set out from the Newtonian conception of the constitution of bodies, and assumed that elastic reactions arise from the variations in the intermolecular forces which are due to changes in the molecular configuration. He regarded the molecules as material points, and assumed that the force between two molecules, whose distance is slightly increased, is proportional to the product of the increment of the distance and some function of the initial distance. His method consisted in forming an expression for the forces that act upon a displaced molecule, which then yielded the equations of motion of the molecule. The equations were thus obtained in terms of the displacements of the molecule. Navier assumed the material to be isotropic, and the equations of equilibrium and vibration to contain a single constant. We now know that an isotropic medium is characterised by two elastic constants. This demonstrates the simplifications arising from the conception of material points and central forces acting between them.

2.2 Propagation of Light and the Theory of Elasticity

In the same year, 1821, in which Navier's memoir was read to the Paris Academy, the study of elasticity received a powerful impulse from an

unexpected branch of physics – from *optics*. Fresnel announced that the observations of the interference of polarised light could be explained only by the hypothesis of transverse vibrations. Until then the undulatory theory of light conceived of light waves as longitudinal waves of condensation and rarefaction in a hypothetical light ether, i.e. as waves similar to sound waves. Although examples of transverse waves were already known earlier, e.g., waves on water, or transverse vibrations of strings, bars, membranes and plates, in no case were they examples of waves transmitted through a medium. Therefore, the principal question, which had to be answered first, was whether transverse waves can propagate inside an elastic medium. The theory of elasticity, and, in particular, the problem of the transmission of waves through an elastic medium, then attracted the attention of two other French mathematicians of high repute, namely Cauchy and Poisson. The former was a supporter, the latter a sceptical critic of Fresnel's ideas. The development of the theory of elasticity was largely due to the work of these two scientists. Their studies closely linked the development of elasticity theory with the problem of light propagation.

The present reader may be surprised that the problems of light propagation could influence the theory of elasticity. However, we should realise that the conception of the light ether was closely associated with the problems of elasticity. Electromagnetic waves were not yet known at that time, and it was, therefore, quite natural to consider light to be a specific type of mechanical waves. Only several decades later the conception of the ether was weakened by the theory of the electromagnetic field, and finally abandoned in the theory of relativity. Nevertheless, this unusual episode from the history of physical sciences is worth remembering.

2.3 Mathematical Theory of Elasticity

By the autumn of 1822, Cauchy had discovered most of the elements of the pure theory of elasticity. He had introduced the notion of stress at a point. He assumed that the stress state at a point is completely determined if the forces per unit area across all plane elements through the point were known. He had shown that, under simple assumptions, these forces could be expressed in terms of six components of stress. (Note that the same conception of stress is adopted in the present textbooks on continuum mechanics). Cauchy had also expressed the state of strain near a point in terms of six components of strain and determined the equations of motion. Assuming linear stress-strain relations, Cauchy derived the special form of these equations for isotropic solid bodies. The methods used in these investigations were quite different from those in Navier's memoir. In particular, no use was made of the hypothesis of material points and central forces. The resulting equations differ from Navier's in one important respect, namely Navier's equations contain a single elastic constant, while Cauchy's equations already contain two such constants.

Cauchy then extended his theory to the case of crystalline bodies. He made use of the hypothesis of material points between which there are forces of attraction and repulsion. In the general case of anisotropy (termed "aelotropy"

at the time), Cauchy found 15 true elastic constants; actually he found 21 independent constants, but 6 of these constants expressed the initial stress and vanished identically if the initial state was one of zero stress. Cauchy also applied his equations to the question of the propagation of light in crystalline as well as in isotropic media.

The first memoir by Poisson dealing with the problems of elasticity was read before the Paris Academy in 1828. Poisson obtained the equations of equilibrium and motion of isotropic elastic solids which were identical with those of Navier. The memoir is very remarkable for its numerous applications of the general theory to special problems.

Cauchy and Poisson, as well as other researchers, applied the theory of elasticity, the former two had developed on the basis of material points and central forces, to many problems of vibrations and of statical elasticity. It provided the means for testing its consequences experimentally, but adequate experiments were made much later.

Poisson (1831) used his theory to investigate the propagation of waves through an isotropic elastic solid of unlimited extent. He proved that two kinds of waves with different velocities could propagate in such a medium. He found that, at a large distance from the source of disturbance, the motion transmitted by the quicker wave was longitudinal, and the motion transmitted by the slower wave was transverse. This theory indicated that the ratio of these velocities was $\sqrt{3}{:}1$. Poisson also considered the vibration of a sphere.

Afterwards Stokes (1849) proved that the quicker wave was a wave of irrotational dilatation, and the slower wave was a wave of equivoluminal distorsion, characterized by differential rotation of the elements of the body. He also derived the well-known formulae for the velocities of the two waves, $\sqrt{(\lambda + 2\mu)/\rho}$ and $\sqrt{\mu/\rho}$, where ρ denotes the density, μ the rigidity, and $\lambda + (2/3)\mu$ the modulus of compression. These two velocities will be denoted here by α and β. This is the first time that we have come across waves P and S, now so well known in seismology. Stokes also proved that the two waves were separated completely at a sufficiently large distance from the initially disturbed region. At shorter distances they are superposed for part of the time. Note that the "dilatational wave" is now also called the "longitudinal wave" or "compressional wave". Analogously, the "distortional wave" is also termed the "transverse wave" or "shear wave".

Green (1839) was dissatisfied with the hypothesis of material points and central forces on which the theory was based, and he sought a new foundation. Starting from what is now called the principle of the conservation of energy he propounded a new method of obtaining the equations of elasticity. He derived the potential energy of the strained elastic body, expressed in terms of the components of strain, and then applied the methods which are used in analytical mechanics. Green stated that this approach "appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other". He deduced the equations of elasticity, in the general case containing 21 constants. In the case of isotropy there are two constants, and the equations are

the same as those of Cauchy's first memoir. The revolution which Green effected in the elements of the theory is comparable in importance with that produced by Navier's discovery of the general equations. Kelvin supported the existence of Green's strain-energy function on the basis of the first and second laws of thermodynamics.

The methods of Navier, of Poisson, and of Cauchy's later memoirs lead to equations of motion containing fewer constants than occur in the equations obtained by the methods of Green, and of Cauchy's first memoir. The questions in dispute are as follows: Is elastic anisotropy to be characterised by 21 constants or by 15, and is elastic isotropy to be characterised by two constants or by one? The two theories were called the "multi-constant" theory and the "rari-constant" theory, respectively. The importance of the discrepancy was first emphasised by Stokes in 1845. He made the observation that resistance to compression and resistance to shearing are the two fundamental kinds of elastic resistance, and he definitely introduced the two principal moduli of elasticity. The two parameters are now called the modulus of compressibility and the modulus of rigidity.

Much attention was also paid to the ratio of lateral contraction to longitudinal extension of a bar under tractive load. This ratio is often called "Poisson's ratio". From his theory Poisson deduced that this ratio must be 1/4. However, experiments on some materials did not support this result. The experimental evidence led Lamé to adopt also the multi-constant equations, and after the publication of his book in 1852 they were generally adopted.

We have already mentioned Poisson's discovery of longitudinal and transverse waves which can propagate through the interior of a solid elastic body. This theory takes no account of the existence of a boundary. When the waves from a source reach the boundary, they are reflected, but in general the longitudinal wave, on reflection, gives rise to both kinds of waves, and the same is true of the transverse wave. Any subsequent state of the body can be represented as the result of superposing waves of both kinds reflected one or more times at the boundary. Without mathematical analysis it is not easy to see what the properties of the resulting wave will be. In 1887, Lord Rayleigh discovered that a specific wave can be formed near the free surface of a homogeneous body. The wave has the following main properties:

1) it propagates along the surface at a certain velocity, less than both α and β;

2) it does not penetrate far beneath the surface because its amplitude decreases exponentially with distance from the surface;

3) it is elliptically polarised in the plane determined by the normal to the surface and by the direction of propagation.

In Lord Rayleigh's work the surface was regarded as an unlimited plane, and the waves could be of any length. Gravity was neglected, and it was found that the wave velocity was independent of the wavelength. Such waves have since been called *Rayleigh waves*, after the person who had discovered them theoretically. (Note that Love (1911) called them "Rayleigh-waves", but the hyphen was later omitted). These waves belong to the category of so-called

surface waves, since their propagation is practically restricted to a certain zone close to the surface of the medium.

A noteworthy feature of the surface wave discovered by Rayleigh is that the vertical component at the surface is larger than the horizontal component. The ratio of the two is nearly 3:2, if Poisson's ratio is taken to be 1:4. This ratio appeared to be important in later seismological applications of Rayleigh's theory; see below. In the paper cited Rayleigh remarked: "It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."

The German scientist A. Schmidt published a paper in 1888 in which he discussed the propagation of waves through the Earth's interior. He emphasised that in general the wave velocity must increase with depth in the Earth and as a consequence of this, the wave paths must be curved, and bent downwards. At about the same time, Knott in England investigated the energy of reflected and refracted waves.

We have seen that the main types of waves, now regularly found on seismograms, had been discovered by mathematicians long before any seismic records were obtained.

2.4 Beginnings of Seismology

Seismology became an independent science around the turn of the nineteenth and twenties centuries. Its theoretical foundations, especially the theory of elasticity, had already been developed in the first half of the nineteenth century, as we have mentioned in the preceding section. However, the theoretical foundations and the observations of earthquakes were completely separated from each other until the end of the nineteenth century. Thanks to the construction of seismographs, it was then possible to combine the two disciplines.

Observations of earthquakes and their effects have been made in populated areas as far as history goes. Reports on earthquakes exist at least as far back as 1800 B.C. The first instruments for earthquake observations were the seismoscopes used in China about one century A. D. Information from ancient times, however, does not generally satisfy modern scientific requirements on observations. In order to express earthquake effects (so-called macroseismic observations) quantitatively, intensity scales were introduced. The first more commonly used intensity scale was proposed by De Rosi in Italy between 1874 and 1878. Such a quantification of an earthquake by a single number was still too far from the requirements of the mathematical theory of elasticity.

The most important breakthrough in the study of earthquakes and the Earth's interior was undoubtedly the installation of seismographs. In 1880 seismographs were constructed in Japan by the Englishmen Gray, Milne and Ewing. They were mainly intended for recording Japanese earthquakes. The first record of a distant earthquake was obtained in 1889. This earthquake occurred in Japan and the record was made in Potsdam.

Very soon it was noticed that the records of distant earthquakes displayed two very distinct stages, the first characterised by a very feeble motion, the second by a much larger motion. These stages were called the "preliminary tremor" and the "main shock" (the "main shock" was also often described as the "large waves" or sometimes as the "principal portion"). The idea that these two waves might be dilatational and distortional waves, emerging at the surface, took firm root among seismologists for a time. In the light of increasing knowledge this idea had to be abandoned.

As mentioned above, Rayleigh suggested that the surface waves he had investigated might play an important part in earthquakes. This suggestion was not, at first, well received by seismologists, mainly because the records did not show a preponderance of vertical motion in the main shock. It was first systematically applied to the interpretation of seismic records by Oldham (1900). He recognised two distinct phases in the preliminary tremors, and showed that their travel times to distant stations correspond to the propagation through the body of the Earth of waves travelling with practically constant velocities. On the other hand, the main shock is recorded at times which correspond to the propagation over the surface of the Earth of waves travelling with a different nearly constant velocity. Oldham, therefore, proposed to identify the first and second phases of the preliminary tremors respectively with dilatational and distorsional waves, travelling along nearly straight paths through the body of the Earth, and he proposed to regard the main shock as Rayleigh waves. The suggestion that the first and second phases of the preliminary tremors should be regarded as dilatational and distorsional waves, transmitted through the body of the Earth, was generally accepted. However, the proposed identification of the main shock with Rayleigh waves was less favourably received for two reasons: partly on account of the difficulty already mentioned with regard to the ratio of the horizontal and vertical displacements; partly because observation showed that a large part of the motion transmitted in the main shock was a horizontal motion at right angles to the direction of propagation (these waves are now called Love waves).

Lamb (1904) considered in detail the waves produced by impulsive pressure suddenly applied at a point of the surface. The motion recorded at a distant point begins suddenly at a time corresponding to the arrival of the longitudinal wave. The surface rises rather sharply, and then subsides very gradually without oscillation. At the time corresponding to the arrival of the transverse wave a slight motion occurs. This is followed, at the time corresponding to the arrival of the Rayleigh wave, by a much larger motion, after which the motion gradually subsides without oscillation. The subsidence is indefinitely prolonged.

Lamb's theory easily accounted for some of the most prominent features of seismic records, namely the first and second phases of the preliminary tremors and the larger disturbance of the main shock. However, it did not account for the existence of horizontal motion at right angles to the direction of propagation. Such motions are observed both in the second phase of the preliminary tremors and in the main shock. The existence of such motions in the second phase of the preliminary tremors could be accounted for easily by

assuming a different kind of initial disturbance, for example by a sudden horizontal shearing motion, or by a couple applied locally. But no assumption as to the nature of the disturbance at the source was able to account for the relatively large horizontal displacements in the main shock which were transverse to the direction of propagation. Moreover, the theory did not account for the approximately periodic oscillations which were a prominent feature in all seismic records. Lamb suggested that these might be due to a succession of primitive shocks. Nevertheless, such an explanation seemed to be rather artificial.

All the controversies between theory and observations were resolved in an excellent way by Love (1911). Instead of a homogeneous half-space, which was considered by Rayleigh and Lamb, Love considered an elastic medium consisting of a layer on a half-space. The main properties of surface waves in this medium already agreed with observations. In particular, he found that a new type of surface waves can propagate in a layer on a half-space. These waves are polarised in the horizontal plane perpendicularly to the direction of propagation, so that give a good explanation of the transverse motion in the main shock. Such waves have since been called *Love waves*.

The propagation of Rayleigh waves in a layer on a half-space has been studied in many papers, starting with those by Bromwich (1898) and Love (1911). Love found that the ratio of the horizontal and vertical components of these waves was already close to the observed values. For a review we refer the reader to Ewing et al. (1957); see also the papers by Bolt and Butcher (1960), and by Money and Bolt (1966).

Rayleigh and Love waves in a layer on a half-space, and in all more complicated models of the medium, are dispersive. The dispersion equation for Love waves in one layer on a half-space was derived by Love (1911), for two layers on a half-space by Stoneley and Tillotson (1928), and for three layers on a half-space by Stoneley (1937).

We shall deal with Rayleigh and Love waves in the simplest models of the medium in Chapters 5 to 7, after explaining the necessary principles of continuum mechanics in Chapter 3 and of the theory of elastic waves in Chapter 4.

2.5 Studies of Other Types of Surface Waves

We have seen that the main shock (now called the "main phase" of a seismogram) was originally interpreted as a body wave, but later it was found to be formed by surface waves. In particular, this seismic phase is formed by the fundamental modes (fundamental tones) of Rayleigh and Love waves. Similarly, also other waves on seismograms were at first interpreted as body waves, but later identified with surface waves.

2.5.1 Channel waves and higher modes

Press and Ewing (1952) found two short-period large-amplitude waves on the records of surface waves crossing North America. The existence of these waves

was then also confirmed in other regions, but only if the path between the epicentre and the station was continental. Consequently, these waves have been used by some authors to determine whether the Earth's crust beneath a given area is continental or oceanic. They have sometimes been referred to as "continental waves".

Press and Ewing (1952) originally interpreted these waves as multiply reflected waves in the granitic layer of the Earth's crust. The transverse wave with a velocity of about $3.5\,\mathrm{km\,s}^{-1}$ and periods ranging from 0.5 to 6 s was thus termed the Lg wave, and the wave with the polarisation of Rayleigh waves, with a velocity of $3.0\,\mathrm{km\,s}^{-1}$ and periods of 8 to 12 s, was termed the Rg wave. A record of these waves is reproduced in Fig. 2.1.
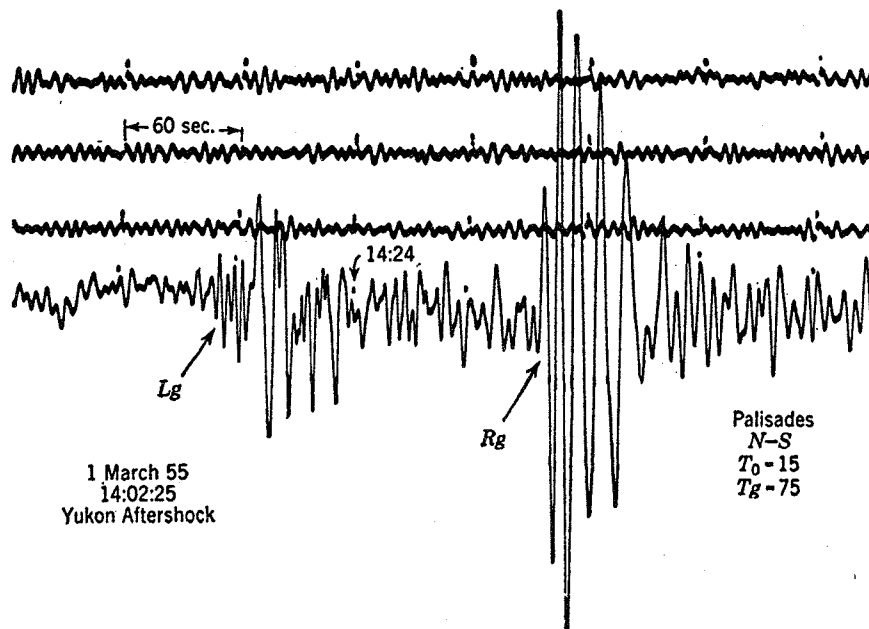
Fig. 2.1. Lg and Rg waves from the Yukon earthquake of March 1, 1955, recorded by a horizontal seismograph at Palisades. (After Ewing et al. (1957)).

Båth (1954) distinguished two phases in the Lg train, and termed them $Lg1$ and $Lg2$. His interpretation was influenced by Gutenberg's proposal of the existence of a low-velocity channel in the Earth's crust (Gutenberg, 1951). Båth explained the $Lg1$ as a wave reflected at the Earth's surface and refracted by the velocity gradient above the channel, whereas the $Lg2$ propagated along the axis of the channel. A modified explanation of the $Lg1$ and $Lg2$, as waves propagating in two crustal low-velocity channels, was also proposed.

Caloi (1953) identified a prominent phase, called Sa, with an arrival velocity of $4.4\,\mathrm{km\,s}^{-1}$ and periods ranging from 10 to 30 s. This wave was explained by some investigators as a wave guided by the low-velocity channel in the asthenosphere (thus, Sa denotes an S wave in the asthenosphere). Press and Ewing (1955) suggested an explanation involving "whispering gallery" propagation in the mantle by multiple grazing reflections from the Mohorovicic

discontinuity. The analogous wave with a vertical component was designated *Pa*.

The interpretation of the above-mentioned waves as channel waves provides a good explanation of their velocities, but not of their amplitudes. Namely, a body wave confined to a low-velocity channel inside the crust or upper mantle should have small surface amplitudes, which contradicts observations. Therefore, a new interpretation was required.

Oliver and Ewing (1957, 1958) were the first to suggest that channel waves were associated with higher modes of surface waves. In particular, the velocities and periods of the observed waves were associated with the extreme values of the group velocity for higher modes. Since then, this interpretation of "channel waves" has generally been accepted. For example, Anderson and Toksöz (1963) stated that "the continental *Sa* wave with periods of 14 to 20 s is unquestionably associated with the long-period maximum of the first higher Love mode ... . The vertical component, sometimes reported, is probably associated with Rayleigh motion and the different designation is desirable".

Note that the explanation of *Lg*, *Rg*, *Sa* and *Pa* waves as higher modes does not require the postulation of low-velocity zones. The computations of synthetic seismograms have demonstrated that these waves may exist in structures without any low-velocity channel.

Hence, also other prominent seismic waves, namely *Lg*, *Rg*, *Sa* and *Pa*, which were originally interpreted as body waves, have finally been identified with surface waves.

For more detailed references see, e.g., Kovach (1965), and Pec and Novotny (1977).

2.5.2 *PL* waves and leaking modes

When broad-band seismograms became available, it was found that the *P*-wave group sometimes contained a long-period component, which was designated *PL*. At first, this long-period motion was attributed to the focal mechanism. However, *PL* waves were later identified with the high-velocity surface waves which correspond to complex roots of dispersion equations. These surface-wave modes with complex velocities are called *leaking modes*, since the imaginary part of their velocity is associated with the leakage of their energy into deeper parts of the Earth. Consequently, these waves can be observed only at epicentral distances shorter than about 2 000 km.

This interpretation leads to a surprising conclusion that even the initial part of a seismogram, which is traditionally explained in terms of body waves, may also contain a surface-wave component.

A surface wave of a similar physical nature has also been observed in seismic prospecting in shallow water. The wave has the following characteristics:

    1) Large amplitudes and long duration.

    2) Numerous repetitions of the wave pattern, and even the character of almost pure sine waves in some cases. This indicates that the wave contains one or several discrete frequencies.

3) Occurrence usually when a hard stratum exists at or near the sea floor.

Burg et al. (1951) explained these waves as leaking modes. They stated that the waves propagated by multiple reflections at angles of incidence between the normal and the critical angle for total reflection, under the condition of constructive interference. A slight leakage of energy, which occurs with each reflection from the bottom, is compensated by automatic gain control. This causes the recorded amplitudes to remain approximately constant for many seconds.

For more details we refer the reader, e.g., to Ewing et al. (1957).

2.5.3 Microseisms

Seismic noise, called microseisms, is permanently present on seismograms. This noise has numerous natural causes (sea waves, wind, variations of the atmospheric pressure), and civilisation causes (traffic, vibrations of heavy machines, swinging of high buildings). The most intensive microseisms usually have periods close to 6 s.

The physical nature of microseisms is not quite clear, but in most cases they are composed predominantly of surface waves, including their higher modes.

For details and references see, e.g., Ewing et al. (1957) and Båth (1979).

Chapter 3

Principles of Continuum Mechanics

In this chapter we shall derive the basic equations of the theory of elasticity which are required in the theory of seismic wave propagation. To be able to use these equations with confidence, one must know their origin and derivations. Therefore, the discussion of the basic ideas here will be rather comprehensive and detailed.

In preparing this chapter we have drawn mainly on the textbooks by Brdicka (1959), Fung (1965, 1969), Sedov (1973), and the lecture notes by Novotny (1976).

3.1 Mathematical Models in Physics

In order to simplify the mathematical and physical description of studied phenomena, various simplifications and models are used, for example, simplified models of the medium, models of physical processes, various principles, etc. The usual idealisations of material objects in mechanics are the mass point (particle), rigid body, and continuum. The model of a continuum is used in mechanics when the deformations of a body cannot be neglected.

The concept of a continuum is derived from mathematics. For example, the system of real numbers is a continuum since between any two particular real numbers there is another particular real number. Therefore, there are infinitely many real numbers between any two particular real numbers.

The *continuum* in mechanics is a medium with a *continuous distribution of matter*. The molecular and atomic structures of matter are ignored in this model of the medium. In other words, a material continuum is a material for which the densities of mass, momentum, and energy exist in the mathematical sense. The mechanics of such a material is continuum mechanics.

When the fine structure of matter attracts our attention, continuum mechanics cannot be used. In these cases we should use particle physics and statistical physics. The duality of continuum and particles resembles the situation in modern optics, in which light is treated sometimes as waves and sometimes as particles.

Continuum mechanics is usually divided into:
- *the theory of elasticity* (we shall deal with this theory in this chapter),
- hydromechanics (the mechanics of fluids, i.e. the mechanics of liquids and gases),
- the theory of plasticity.

The main *advantage* of the concept of a continuum consists in the possibility of applying the mathematical theory of *continuous functions*, and *differential* and *integral* calculi.

The same body (e.g., the Earth) may be regarded as a mass point, rigid body or continuum in different physical problems. For example, in studying the motions of the Earth in the Galaxy, we shall probably consider the Earth to be a

particle. In studies of its rotation, precession or polar wobble, we shall usually consider the Earth as a rigid body, or even as a continuum in detailed studies of these phenomena. In studying the deformations of the Earth due to the gravitational effects of the Moon and Sun, or in the theory of seismic waves, we consider the Earth to be a continuum; the Earth as a particle or a rigid body is not adequate to these purposes.

3.2 Displacement Vector

Real bodies are deformed by the action of forces. The description of the deformation is based on a comparison of the instantaneous state (volume and shape) of the body with some previous state, which will be regarded as an original state. In this section we shall study the corresponding displacements, and in the next section we shall seek some quantities which can be used to describe the deformations.

Note that we shall speak of two types of points, which should be distinguished: mass points (particles) of a continuum, and points of an Euclidean space. At different times, a certain particle is located generally at different points of space.
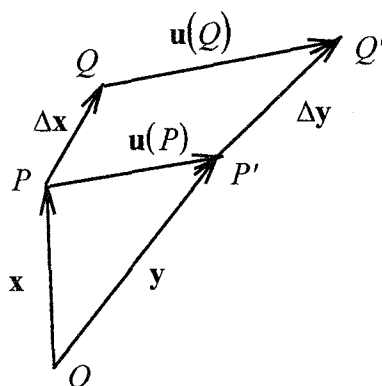
Fig. 3.1. Displacements of two neighbouring points, P and Q.

Therefore, we shall compare a continuum in two states, namely in the *original* (unstrained) state, and in the *deformed* (new, strained) state. Introduce a Cartesian coordinate system, its origin being denoted by O. (The description in curvilinear systems leads to certain problems, but in this chapter we shall not use curvilinear coordinates). We consider the reference frames to be right-handed, but this specification will only be needed later on, in particular in Eq. (3.65).

Consider a particle at point P in the original state, which is moved to point P' in the deformed state (Fig. 3.1). Denote the radius vector of point P by $\mathbf{x} = (x_1, x_2, x_3)$, and of point P' by $\mathbf{y} = (y_1, y_2, y_3)$. The new position, given by vector $\mathbf{y}$, depends on the initial position $\mathbf{x}$, on the acting forces, physical properties of the continuum and the time between the original and new states. In this section, we shall study only the first dependence, i.e. we shall study the general relations which, under certain assumptions, must be valid between the

coordinates of the new and original states. Thus, we shall study the vector function

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) \ , \qquad (3.1a)$$

or in terms of components,

$$y_1 = y_1(x_1,x_2,x_3), \quad y_2 = y_2(x_1,x_2,x_3), \quad y_3 = y_3(x_1,x_2,x_3) \ . \qquad (3.1b)$$

In this chapter we shall always assume that there is a one-to-one correspondence between the original and deformed configurations, i.e. that the inverse function exists:

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) \ . \qquad (3.2)$$

The displacement of a particle from an original to a deformed position can be described by the corresponding *displacement vector* $\mathbf{u} = (u_1,u_2,u_3)$,

$$\mathbf{u} = \mathbf{y} - \mathbf{x} \ . \qquad (3.3)$$

We shall usually consider the displacement vector as a function of the coordinates of the original state:

$$\mathbf{u} = \mathbf{y}(\mathbf{x}) - \mathbf{x}, \quad \text{i.e.} \quad \mathbf{u} = \mathbf{u}(\mathbf{x}) \ . \qquad (3.4)$$

In this case we speak of the *Lagrangian description* of motion.

However, we can also express the displacement vector as a function of the coordinates of the deformed state:

$$\mathbf{u} = \mathbf{y} - \mathbf{x} = \mathbf{y} - \mathbf{x}(\mathbf{y}), \quad \text{i.e.} \quad \mathbf{u} = \mathbf{u}(\mathbf{y}) \ .$$

In this case we speak of the *Eulerian description*. This description is frequently used in hydrodynamics. Here we shall use the Lagrangian description, with exceptions in Section 3.4.

We shall *assume* that the *displacement vector* and its *first derivatives are continuous* functions of coordinates. These assumptions will simplify many mathematical considerations.

In a neighbourhood of point P, let us consider another point, Q, which will be displaced to point Q' in the deformed state (Fig. 3.1). The radius vectors of points Q and Q' are $\mathbf{x} + \Delta\mathbf{x}$ and $\mathbf{y} + \Delta\mathbf{y}$, respectively. Using the Taylor expansion, we get

$$u_j(Q) = u_j(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) =$$

$$\qquad (3.5)$$

$$= u_j(x_1,x_2,x_3) + \sum_{k=1}^{3}\left(\frac{\partial u_j}{\partial x_k}\right)_{(x_1,x_2,x_3)} \Delta x_k + \ldots = u_j(P) + \sum_{k=1}^{3}\left(\frac{\partial u_j}{\partial x_k}\right)_{P} \Delta x_k + \ldots \ ,$$

where $j = 1, 2, 3$.

To simplify the formulae which follow, let us introduce Einstein's summation convention: If any suffix occurs twice in a single term, it is to be put equal to 1, 2 and 3 in turn and the results added. For example:

a) $$\underbrace{\sum_{k=1}^{3} \frac{\partial u_j}{\partial x_k} \Delta x_k}_{standard\ notation} = \underbrace{\frac{\partial u_j}{\partial x_k} \Delta x_k}_{summation\ convention} \quad ;$$

b) $$|\Delta \mathbf{x}|^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = \sum_{k=1}^{3} \left(\Delta x_k\right)^2 = \sum_{k=1}^{3} \Delta x_k \Delta x_k = \underbrace{\Delta x_k \Delta x_k}_{sum.\ convention} \quad .$$

An index that is summed over is called *a dummy index*. Since a dummy index only indicates summation, it is immaterial which symbol is used. Thus, $\Delta x_k \Delta x_k$ in the last example may be replaced by $\Delta x_i \Delta x_i$, etc.

Using this summation convention and neglecting the higher-order terms in (3.5), we get approximately

$$\boxed{u_j(Q) = u_j(P) + \left(\frac{\partial u_j}{\partial x_k}\right)_P \Delta x_k} \ . \tag{3.6}$$

Let us briefly discuss the consequences of the mathematical assumptions adopted above. The continuity of displacement **u** guarantees that an originally continuous body will also remain continuous during the deformation. The continuity of $\partial u_j / \partial x_k$ guarantees the existence of the total differential of the displacement. Consequently, formula (3.6) can then be made as accurate as required by choosing point Q sufficiently close to P. This formula will play an important role in the theory which follows.

On the other hand, we should also mention some places where the assumptions of continuity are not satisfied, in particular:

- cracks, faults, cavities, etc. (discontinuity of **u**),
- contact of solid and liquid media (discontinuity of the tangential components of **u**),
- discontinuities of material constants (discontinuity of $\partial u_j / \partial x_k$); specific phenomena, namely the reflection and transmission of elastic waves, occur at these discontinuities.

3.3 Strain Tensor

3.3.1 Tensor of finite strain

If the displacement is known for every particle in a body, we can construct the deformed body from the original. Hence, a deformation can be described by the

displacement field. However, the displacement vector describes the translation, rotation and pure deformation (strain) of the medium. But we are not interested in translation and rotation; these motions are studied in detail in the mechanics of rigid bodies. We are only interested in those quantities which characterise the strain. There are two approaches to obtaining these characteristics:

1) subtracting the translation and rotation from the displacement;
2) considering changes in distances.

The first approach is convenient and simple if only small strains are considered (Bullen, 1965; Ewing et al., 1957). However, in the case of large deformations of a continuum, the separation of translation, rotation and pure strain in the displacement vector is much more complicated (Novozhilov, 1958). Although we shall not consider large strains in the following chapters, we shall use the second approach because this approach is more general.

It is evident that the change in the size and shape of a body will be determined in full if the changes in the distances of two arbitrary points are known. However, it will be more convenient to consider the squares of these distances instead of the distances themselves; see the discussion in Subsection 3.3.2.

Denote the distance between points P and Q in the original state by $\overline{PQ}$ (Fig. 3.1). The square of this distance can be expressed as (if the summation convention is used)

$$\overline{PQ}^2 = \Delta\mathbf{x} \cdot \Delta\mathbf{x} = \Delta x_i \Delta x_i \ . \tag{3.7}$$

It follows from the quadrangle $PP'Q'Q$ and Eq. (3.6) that

$$\mathbf{u}(P) + \Delta\mathbf{y} = \Delta\mathbf{x} + \mathbf{u}(Q) = \Delta\mathbf{x} + \mathbf{u}(P) + \left(\frac{\partial\mathbf{u}}{\partial x_i}\right)_P \Delta x_i \ .$$

By comparing the beginning and end of this equation, we see that

$$\Delta\mathbf{y} = \Delta\mathbf{x} + \left(\frac{\partial\mathbf{u}}{\partial x_i}\right)_P \Delta x_i \ . \tag{3.8}$$

We shall omit suffix P hereafter.

Introduce the Kronecker symbol (Kronecker delta)

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j \ . \end{cases} \tag{3.9}$$

Then, for example, $\Delta x_k = \delta_{ik}\Delta x_i$.

Formula (3.8) can then be expressed in components as

$$\Delta y_k = \left(\delta_{ik} + \frac{\partial u_k}{\partial x_i}\right)\Delta x_i \ . \tag{3.10}$$

Consequently,

$$\overline{P'Q'}^2 = \Delta \mathbf{y} \cdot \Delta \mathbf{y} = \Delta y_k \Delta y_k = \left(\delta_{ik} + \frac{\partial u_k}{\partial x_i} \right) \Delta x_i \left(\delta_{jk} + \frac{\partial u_k}{\partial x_j} \right) \Delta x_j \ . \quad (3.11)$$

Note that we have used different dummy indices, i and j, in the latter formula.

Let us introduce nine quantities ε_{ij}, referred to as the *components of the tensor of finite strain*, by the relation

$$\boxed{\overline{P'Q'}^2 - \overline{PQ}^2 = 2\varepsilon_{ij}\Delta x_i \Delta x_j} \ . \quad (3.12)$$

Since $\overline{PQ}^2 = \Delta x_i \Delta x_i = \delta_{ij}\Delta x_i \Delta x_j$, and $\overline{P'Q'}^2$ is given by (3.11), we get

$$2\varepsilon_{ij} = \left(\delta_{ik} + \frac{\partial u_k}{\partial x_i} \right) \left(\delta_{jk} + \frac{\partial u_k}{\partial x_j} \right) - \delta_{ij} \ . \quad (3.13)$$

Taking into account that

$$\delta_{ik}\delta_{jk} = \delta_{ij} \ , \quad \delta_{ik}\frac{\partial u_k}{\partial x_j} = \frac{\partial u_i}{\partial x_j} \ ,$$

we arrive at the following formula for the components of the tensor of finite strain:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} \right)} \ . \quad (3.14)$$

The set of nine elements ε_{ij} constitutes the *tensor of finite strain*. This tensor is also referred to as *Green's strain tensor* or the Lagrangian strain tensor. This tensor is obviously symmetric, i.e. $\varepsilon_{ji} = \varepsilon_{ij}$. Consequently, only six of its components are independent. Let us write out in full two components of this tensor:

$$\varepsilon_{11} = \frac{1}{2}\left[2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2 \right] \ ,$$

$$(3.15)$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2} \right] \ , \text{etc.}$$

Since the derivatives of the displacement vector have been calculated at point P, see (3.8), we shall also regard components ε_{ij} as defined at point P, and speak of the tensor of finite strain at point P.

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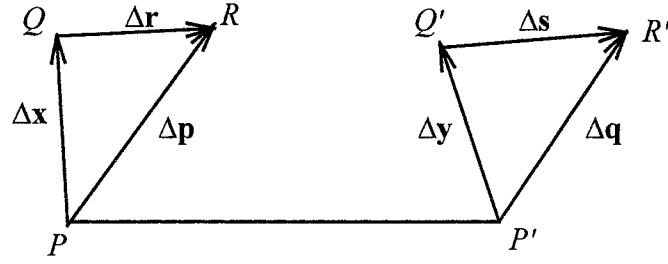
Fig. 3.2. Deformation of the neighbourhood of point P.

We are seeking quantities which describe *all* strains in a small vicinity of point P, i.e. the changes in distances of any two points from this vicinity. We have so far only considered the distances from point P. Therefore, let us now consider two points, Q and R, in a vicinity of point P, but different from P (Fig. 3.2). Let the particles which are at points P, Q and R in the original state be displaced to P', Q' and R', respectively. Let the position of point R relative to P be determined by vector $\Delta\mathbf{p}$, and the position of R' relative to P' by $\Delta\mathbf{q}$. According to (3.8),

$$\Delta\mathbf{q} = \Delta\mathbf{p} + \frac{\partial\mathbf{u}}{\partial x_i}\Delta p_i \ . \tag{3.16}$$

The vectors between points Q, R and Q', R' are (Fig. 3.2)

$$\Delta\mathbf{r} = \Delta\mathbf{p} - \Delta\mathbf{x} , \quad \Delta\mathbf{s} = \Delta\mathbf{q} - \Delta\mathbf{y} , \tag{3.17}$$

respectively. By inserting (3.16) and (3.8) into $\Delta\mathbf{s}$, we get

$$\Delta\mathbf{s} = \Delta\mathbf{r} + \frac{\partial\mathbf{u}}{\partial x_i}\Delta r_i \ . \tag{3.18}$$

We have arrived at the formula for $\Delta\mathbf{s}$ which is quite analogous to (3.8) for $\Delta\mathbf{y}$. By performing an analogous derivation as above between (3.8) and (3.10), we would obtain

$$\overline{Q'R'}^2 - \overline{QR}^2 = 2\varepsilon_{ij}\Delta r_i \Delta r_j \ . \tag{3.19}$$

This means that the change in the distance (actually in its square) of two points, both different from P, is also described by quantities ε_{ij} defined at point P. Hence, we have proved that the tensor of finite strain at a given point describes the strain of the small vicinity of this point in full.

As mentioned above, we could also describe the strain in Eulerian coordinates. The inverse relation (3.2), i.e. $\mathbf{x} = \mathbf{x}(\mathbf{y})$, yields

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$$\Delta x_i = \frac{\partial x_i}{\partial y_j} \Delta y_j ,$$

where the higher-order terms have been neglected. By substituting $x_i = y_i - u_i$, we get

$$\Delta x_i = \left(\delta_{ij} - \frac{\partial u_i}{\partial y_j} \right) \Delta y_j .$$

Then

$$\overline{P'Q'}^2 - \overline{PQ}^2 = \Delta y_k \Delta y_k - \Delta x_k \Delta x_k =$$

$$= \delta_{ij} \Delta y_i \Delta y_j - \left(\delta_{ki} - \frac{\partial u_k}{\partial y_i} \right) \Delta y_i \left(\delta_{kj} - \frac{\partial u_k}{\partial y_j} \right) \Delta y_j .$$

Introduce another tensor of finite strain, η_{ij}, by the formula

$$\overline{P'Q'}^2 - \overline{PQ}^2 = 2\eta_{ij} \Delta y_i \Delta y_j . \qquad (3.20)$$

We then arrive at

$$\eta_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i} \frac{\partial u_k}{\partial y_j} \right) . \qquad (3.21)$$

This tensor is very similar to ε_{ij}, but the sign with the last term is opposite. Tensor η_{ij} is called *Almansi's strain tensor* or the Eulerian strain tensor. We shall not use this tensor here.

3.3.2 Other strain measures

We should not assume that Green's and Almansi's strain tensors, defined above, are the only ones suitable for describing deformation. They are, of course, the most natural ones because the squares of distances can simply be expressed by means of Pythagoras' theorem. However, there are also other possibilities.

We can use the set of nine first derivatives of the displacement field, arranged into the "deformation gradient matrix" with elements $a_{ij} = \partial u_i / \partial x_j$. The symmetric part of this matrix is the matrix of infinitesimal strain, which will be introduced below.

Other strain measures are Cauchy's strain tensor,

$$C_{ij} = \frac{\partial y_k}{\partial x_i} \frac{\partial y_k}{\partial x_j} ,$$

Finger's strain tensor

$$B_{ij} = \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k},$$

and their analogues in Eulerian coordinates. These tensors may be convenient for some special purposes.

We have mentioned these other strain measures only as examples, but we shall not use them here. For further details we refer the reader to Fung (1969). Now we shall return to the traditional approaches.

3.3.3 Physical meaning of the components of the tensor of finite strain

a) *Interpretation of* ε_{11}, ε_{22} *and* ε_{33}.

Consider an elementary abscissa, PQ, which is parallel to the x_1-axis in the original state, i.e. $\Delta \mathbf{x} = (\Delta x_1, 0, 0)$; see Fig. 3.3. As $\Delta x_2 = \Delta x_3 = 0$, Eq. (3.12) takes the simple form

$$|\Delta \mathbf{y}|^2 - |\Delta \mathbf{x}|^2 = 2\varepsilon_{11}(\Delta x_1)^2 . \tag{3.22}$$

Consequently,

$$|\Delta \mathbf{y}| = \sqrt{1 + 2\varepsilon_{11}} \, \Delta x_1 . \tag{3.23}$$
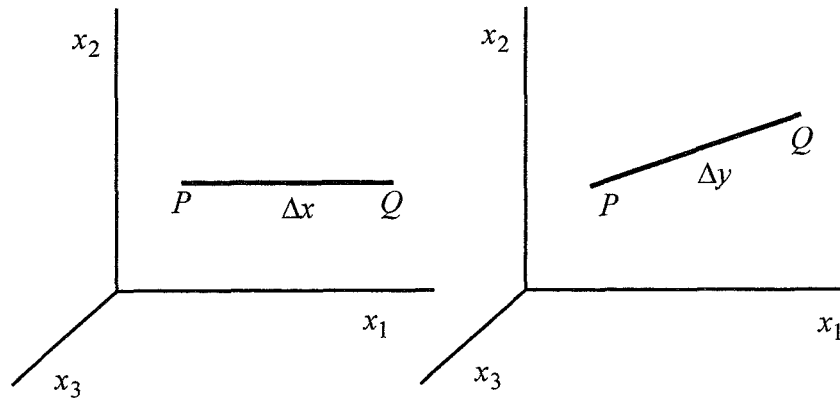
Fig. 3.3. Physical meaning of ε_{11}.

The relative extension of the abscissa PQ is defined by

$$E_1 = \frac{|\Delta \mathbf{y}| - |\Delta \mathbf{x}|}{|\Delta \mathbf{x}|} . \tag{3.24}$$

Using (3.23), this extension can be expressed as

$$E_1 = \sqrt{1 + 2\varepsilon_{11}} - 1 . \tag{3.25}$$

Hence, component ε_{11} characterises the relative extension of an element which was originally parallel to the x_1-axis. Analogously, components ε_{22} and ε_{33} characterise the extensions along the second and third axes, respectively.

b) *Interpretation of ε_{12}, ε_{13} and ε_{23}.*

Now let us consider two perpendicular vectors in the original state, $\Delta\mathbf{x}^{(1)} = (\Delta x_1, 0, 0)$ and $\Delta\mathbf{x}^{(2)} = (0, \Delta x_2, 0)$; see Fig. 3.4. The corresponding vectors $\Delta\mathbf{y}^{(1)}$ and $\Delta\mathbf{y}^{(2)}$ in the deformed state have, according to Eq. (3.10), the following components:

$$\Delta y_i^{(1)} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) \Delta x_j \text{, but only } \Delta x_1 \neq 0 \text{ ;}$$

$$(3.26)$$

$$\Delta y_i^{(2)} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} \right) \Delta x_j \text{, but only } \Delta x_2 \neq 0 \text{ .}$$

The scalar product of these vectors is

$$\Delta\mathbf{y}^{(1)} \cdot \Delta\mathbf{y}^{(2)} = \Delta y_i^{(1)} \Delta y_i^{(2)} = \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_i}{\partial x_1} \frac{\partial u_i}{\partial x_2} \right) \Delta x_1 \Delta x_2 = 2\varepsilon_{12} \Delta x_1 \Delta x_2 \text{ .}$$

$$(3.27)$$
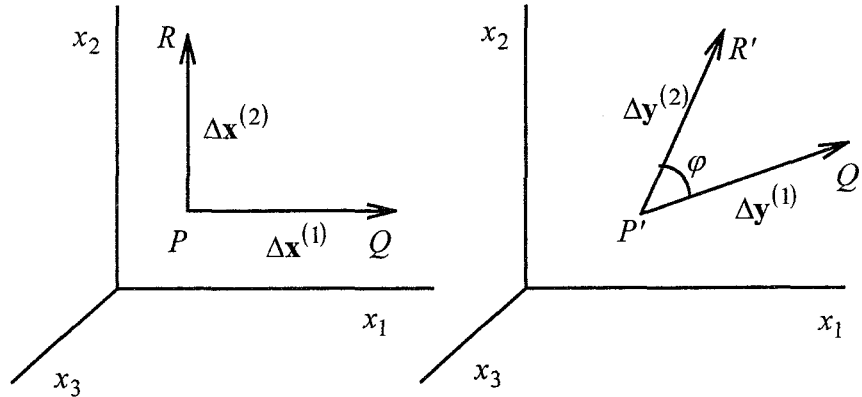
Fig. 3.4. Physical meaning of ε_{12}.

Denote by φ the angle between vectors $\Delta\mathbf{y}^{(1)}$ and $\Delta\mathbf{y}^{(2)}$. The angle $\alpha_{12} = 90° - \varphi$ represents the change of the right angle (decrease of the right angle) due to deformation. The scalar product can also be expressed as

$$\Delta\mathbf{y}^{(1)} \cdot \Delta\mathbf{y}^{(2)} = \left| \Delta\mathbf{y}^{(1)} \right| \left| \Delta\mathbf{y}^{(2)} \right| \cos\varphi \text{ .} \qquad (3.28)$$

Using (3.23) to express $\left| \Delta\mathbf{y}^{(1)} \right|$ and $\left| \Delta\mathbf{y}^{(2)} \right|$, we arrive at

$$\sin\alpha_{12} = \cos\varphi = \frac{2\varepsilon_{12}}{\sqrt{1+2\varepsilon_{11}}\sqrt{1+2\varepsilon_{22}}} \ . \qquad (3.29)$$

Hence, component ε_{12} characterises the change of the right angle between two line elements, one of which was parallel in the original state to the x_1-axis, and the second was parallel to the x_2-axis. The physical meaning of the remaining components ε_{13} and ε_{23} is analogous.

3.3.4 Principal axes of strain

Let us study the geometric changes of an infinitesimal vicinity of a point due to deformation. It follows from definition (3.12) of the strain tensor that

$$|\Delta\mathbf{y}|^2 = |\Delta\mathbf{x}|^2 + 2\varepsilon_{ij}\Delta x_i \Delta x_j = \Delta x_i \Delta x_i + 2\varepsilon_{ij}\Delta x_i \Delta x_j = (\delta_{ij} + 2\varepsilon_{ij})\Delta x_i \Delta x_j \ . \qquad (3.30)$$

Let us assume that, in the deformed state, this vicinity takes the shape of a sphere of radius C, i.e. the points on the surface of the vicinity satisfy the condition $|\Delta\mathbf{y}| = C$. Equation (3.30) then takes the form

$$C^2 = \left(\delta_{ij} + 2\varepsilon_{ij}\right)\Delta x_i \Delta x_j = A_{ij}\Delta x_i \Delta x_j \ , \qquad (3.31)$$

where $A_{ij} = \delta_{ij} + 2\varepsilon_{ij}$. Equation (3.31) is the equation of a quadric in variables $\Delta x_1, \Delta x_2$ and Δx_3. If follows from the physical character of the problem that this quadric is an ellipsoid (generally a tri-axial ellipsoid). Thus, a sphere in the deformed state is obtained from an ellipsoid in the original state.

The opposite statement also holds true, which could be proved by applying Almansi's tensor (3.20). Thus, an infinitesimal sphere in the original state changes due to the deformation into a tri-axial ellipsoid. The axes of the corresponding ellipsoid are called the principal axes of strain. These axes, being perpendicular in the original state, remain perpendicular also in the deformed state.

3.3.5 Tensor of infinitesimal strain

The tensor of finite strain, ε_{ij}, contains products of the derivatives of the displacement vector, $\partial u_i / \partial x_j$. These products represent non-linear terms, which complicate the solution of many problems. However, in many applications, these quadratic terms may be neglected.

We shall assume hereafter that the derivatives of the displacement are small, i.e.

$$\left|\frac{\partial u_i}{\partial x_j}\right| << 1 \ , \qquad (3.32)$$

so that their mutual products are small quantities of the second order, which may be neglected in comparison with the derivatives themselves. In this case, the tensor of finite strain ε_{ij} simplifies to yield the tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

(3.33)

which is called the *tensor of infinitesimal strain* or Cauchy's infinitesimal strain tensor. In speaking of the strain tensor only, we shall have in mind the tensor of infinitesimal strain (3.33).

The components of strain tensor e_{ij} have a simple physical meaning. If ε_{11} is small and the higher-order terms are neglected, Eq. (3.25) simplifies to read

$$E_1 = \sqrt{1 + 2\varepsilon_{11}} - 1 \approx 1 + \varepsilon_{11} - 1 = \varepsilon_{11} \approx e_{11} .$$

(3.34)

Thus, in the case of small deformations, components e_{11}, e_{22} and e_{33} are equal to the relative extensions of the line elements which, in the original state, were parallel to the coordinate axes.

Furthermore, for small deformations it follows from (3.29) that

$$\sin\alpha_{12} \approx 2\varepsilon_{12} \approx 2e_{12} .$$

Consequently, $\sin\alpha_{12}$ is small and may be approximated by α_{12}, so that

$$\alpha_{12} \approx 2e_{12} .$$

(3.35)

Thus, component e_{12} is equal to half the change of the corresponding right angle.

It can also be proved that, on condition (3.32), the difference between Green's and Almansi's tensors disappears, so that we can put

$$\eta_{ij} = \varepsilon_{ij} = e_{ij} .$$

(3.36)

Let us return to Eq. (3.6), which describes the displacements in the vicinity of point P. The first term on the right-hand side, $u_j(P)$, can be interpreted as a component of the translation of the whole vicinity, and term $\dfrac{\partial u_j}{\partial x_k}\Delta x_k$ describes the rotation and deformation of the vicinity. Therefore, if derivatives $\partial u_j / \partial x_k$ are small (as we assume here), not only the deformations of the vicinity, but also its rotations are small. In this case, tensor ε_{ij} may be replaced by e_{ij}. In other words, we may replace tensor ε_{ij} by e_{ij} if both the deformations and also the rotations are small; small deformations alone are not sufficient for this simplification.

small force

small deformation, but large rotation of this part of the bar
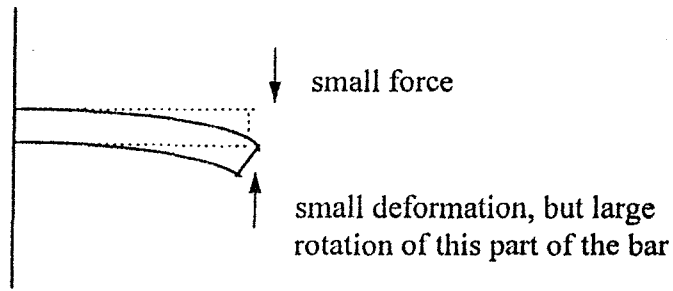
Fig. 3.5. Deformation of a bar or plate.

For example, small deformations of bars or plates (due to small stresses) cannot often be described by e_{ij} if the elements of the body are rotated through angles which are not small; see Fig. 3.5. This situation often occurs with "one-dimensional" bodies (thin bars) or "two-dimensional" bodies (plates). In these cases we must use the tensor of finite strain ε_{ij}, not the tensor of infinitesimal strain e_{ij}. In "three-dimensional" bodies, e.g., within the Earth, small deformations may be described by e_{ij}, since they are usually associated with small rotations.

3.3.6 Volume dilatation

Consider a small parallelepiped in the original state, the edges of which coincide with the principal axes of strain. Denote the lengths of the edges by d_1, d_2, d_3, respectively. The volume of the parallelepiped is $V = d_1 d_2 d_3$. In the deformed state, these edges will again be perpendicular, and their lengths will be (neglecting higher-order terms)

$$d_1 + e_{11}d_1, \quad d_2 + e_{22}d_2, \quad d_3 + e_{33}d_3, \tag{3.37}$$

respectively. Therefore, the new volume will be

$$V' = d_1 d_2 d_3 (1 + e_{11})(1 + e_{22})(1 + e_{33}) = V(1 + e_{11} + e_{22} + e_{33}) . \tag{3.38}$$

The *volume dilatation* (cubical dilatation), defined by

$$\vartheta = \frac{V' - V}{V} , \tag{3.39}$$

then reads

$$\vartheta = e_{11} + e_{22} + e_{33} . \tag{3.40}$$

The theory of quadratic surfaces indicates that the sum $e_{11} + e_{22} + e_{33}$ is an invariant, i.e. a quantity which is independent of the choice of the coordinate system (the frame remaining orthogonal). Consequently, quantity ϑ describes

the relative change of an arbitrary infinitesimal volume which surrounds the considered point.

Dilatation ϑ allows us to divide the deformation into the voluminal and shape parts. In the obvious identity

$$e_{ij} = \frac{1}{3}\vartheta\delta_{ij} + \left(e_{ij} - \frac{1}{3}\vartheta\delta_{ij}\right) \ , \tag{3.41}$$

denote the individual terms on the right-hand side by

$$f_{ij} = \frac{1}{3}\vartheta\delta_{ij} \ , \quad g_{ij} = e_{ij} - \frac{1}{3}\vartheta\delta_{ij} \ . \tag{3.42}$$

These expressions yield

$$f_{ii} = f_{11} + f_{22} + f_{33} = \frac{1}{3}\vartheta\delta_{ii} = \vartheta \ , \tag{3.43}$$

and

$$g_{ii} = e_{ii} - \frac{1}{3}\vartheta\delta_{ii} = e_{ii} - \vartheta = 0 \tag{3.44}$$

in view of (3.40). Thus, the voluminal changes are described by tensor f_{ij}. Tensor g_{ij} describes the changes when the volume does not change, i.e. this tensor describes the shape changes. Tensor g_{ij} is called the deviatoric (or distortional) strain tensor.

3.4 Stress Vector and Related Problems

3.4.1 Body forces and surface forces

In particle mechanics, we study two types of interactions between particles: by action at a distance and by collision. An analogous division of forces is convenient also in continuum mechanics. Therefore, we shall divide the forces acting in a continuum into two groups according to their "action radius":

1) *Body forces*, also called *voluminal forces*, which have a large action radius. Examples of body forces are gravitational forces, electromagnetic forces, inertial force (in dynamic problems), and also fictitious forces in non-inertial reference frames (Coriolis and centrifugal forces).

2) *Surface forces*, which have a small action radius. Examples of such forces are hydrostatic pressure, aerostatic pressure, and forces due to the mechanical contact of two bodies.

This separation of forces facilitates the formulation and solution of many problems because:

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1) the effect of forces with a small action radius may be approximated by a surface integral (surface forces) instead of a more complicated volume integral;
2) body forces vanish in some limits, e.g., in Eq. (3.55) given below. These forces may also be neglected in some problems, e.g., in many problems of elastic wave propagation.

3.4.2 Stress vector

A deformed continuum at rest resembles a rigid body. Therefore, we shall *assume* that some notions and equations from rigid-body mechanics can also be applied in continuum mechanics. However, these analogies will be no more than basic assumptions. This approach will only facilitate the formulation of the basic equations of continuum mechanics, but cannot be regarded as a derivation of these equations. Namely, the general equations of continuum mechanics cannot, in principle, be derived from more special equations for a rigid body or a mass point. The validity of the general equations can be verified only by comparing their solutions with experiments.

Let us start with the description of the stress state in a continuum. Consider a point, P, and an element of a surface, ΔS, drawn through this point (Fig. 3.6). Denote the normal to ΔS at point P by $\vec{v}$. Vector $\vec{v}$ enables us to define the positive and negative sides of the element ΔS (upper and lower sides in Fig. 3.6, respectively).
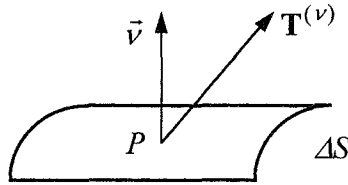
Fig. 3.6. Stress vector.

In analogy to the static equilibrium of a rigid body, we shall *assume* that, in a deformed continuum at rest, the effect of all surface forces exerted across the small element ΔS is statically equivalent to a single force $\Delta \mathbf{H}$, acting at point P in a definite direction, together with couple $\Delta \mathbf{G}$, acting also at P about a definite axis.

Let us indefinitely diminish surface element ΔS by any continuous process, always keeping point P within the element. From physical considerations it seems reasonable to *assume* that, in ordinary materials, vector $\Delta \mathbf{H}/\Delta S$ tends to a non-zero limit, $\mathbf{T}^{(v)}$, whereas vector $\Delta \mathbf{G}/\Delta S$ tends to the zero vector. The vector

$$\mathbf{T}^{(v)} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{H}}{\Delta S} = \frac{d \mathbf{H}}{d S} \qquad (3.45)$$

is called *the stress vector* or traction at point P; see Fig. 3.6. Note that the direction of stress $\mathbf{T}^{(\nu)}$ need not coincide with the direction of normal $\vec{\nu}$. Vector $\mathbf{T}^{(\nu)}$ is the vector acting on the unit infinitesimal surface, the normal of which is $\vec{\nu}$. The stress at P varies, in general, with the direction of normal $\vec{\nu}$.

Vector $\mathbf{T}^{(\nu)}$ can be decomposed into a normal component (in the direction of $\vec{\nu}$) and a tangential (shear) component, which is perpendicular to $\vec{\nu}$. We then speak of normal and tangential (shear) stresses, respectively.

Analogously, body forces acting in a vicinity of point P are assumed to be statically equivalent to force $\Delta\mathbf{K}$ and couple $\Delta\mathbf{L}$, acting at point P. In diminishing the volume ΔV of the vicinity to zero, we shall assume that force $\Delta\mathbf{K}/\Delta V$ tends to a non-zero limit, $\mathbf{F}$, whereas couple $\Delta\mathbf{L}/\Delta V$ tends to the zero vector. Force $\mathbf{F}$ is the body force acting on the unit infinitesimal volume.

Note that, in some problems, also the couples of body and surface forces are assumed to be non-zero. In these cases we speak of moment media. There more complicated models are sometimes used to describe media with a characteristic microstructure, such as some composite materials, fibreglass, and others. These models have also been used in some studies of mechanical processes in the vicinity of an earthquake focus. If these couples are non-zero, further terms must be added in the conditions of equilibrium and the equations of motion, given below. However, so far no fundamental application has been found for the couple-stress theory, hence we shall not discuss it further in this chapter.

3.4.3 Conditions of equilibrium in integral form

It is well-known from rigid-body mechanics that a rigid body is in static equilibrium if the total applied force and total applied torque are zero.

We shall *assume* that an arbitrary part of a continuum, in the deformed state at rest, is in equilibrium under the same conditions as if this part were a rigid body. This means that we shall express these conditions of equilibrium in the following form:

$$\iint_S \mathbf{T}^{(\nu)}\, dS + \iiint_V \mathbf{F}\, dV = 0 \ , \tag{3.46}$$

$$\iint_S \left(\mathbf{y} \times \mathbf{T}^{(\nu)} \right) dS + \iiint_V (\mathbf{y} \times \mathbf{F}) dV = 0 \ , \tag{3.47}$$

where V is the volume of the part of the continuum, S is its surface, $\mathbf{T}^{(\nu)}$ is the surface force acting from the side of the outward normal $\vec{\nu}$, $\mathbf{F}$ is the body force, and $\mathbf{y}$ is the radius vector of the point under consideration (in the deformed state, i.e. the Eulerian radius vector). The first of these equations requires the resultant force to be equal to zero, and the second equation requires the resultant torque to be equal to zero. The validity of these equations will be discussed in the next subsection, together with the equations of motion.

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3.4.4 Equations of motion in integral form

Using D'Alembert's principle, the equations of motion can easily be obtained from the conditions of equilibrium by adding the inertial forces.

Consider any portion of a material body. Let the volume of this portion at any time t be denoted by $V = V(t)$. Let $\mathbf{y}$ be the radius-vector of a particle, $\mathbf{v}$ be its velocity, and ρ be the density of the material at the corresponding point. Integral

$$\mathbf{P} = \iiint\limits_V \rho\mathbf{v}\, \mathrm{d}V$$

is the linear momentum, and

$$\mathbf{L} = \iiint\limits_V (\mathbf{y} \times \rho\mathbf{v})\, \mathrm{d}V$$

is the angular momentum of this part of the body. Derivative $\mathrm{d}\mathbf{P}/\mathrm{d}t$ is the corresponding inertial force.

Hence, by adding the inertial terms on the right-hand sides of Eqs. (3.46) and (3.47), we arrive at the *equations of motion of a continuum* in the form

$$\iint\limits_S \mathbf{T}^{(\nu)}\, \mathrm{d}S + \iiint\limits_V \mathbf{F}\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \rho\mathbf{v}\, \mathrm{d}V \quad , \qquad (3.48)$$

$$\iint\limits_S \left(\mathbf{y} \times \mathbf{T}^{(\nu)}\right) \mathrm{d}S + \iiint\limits_V (\mathbf{y} \times \mathbf{F})\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V (\mathbf{y} \times \rho\mathbf{v})\, \mathrm{d}V . \qquad (3.49)$$

It should be noted that no demand was made on domain $V(t)$ other than that it must consist of the same material particles at all times. Equations (3.48) and (3.49) are applicable to any material body which may be considered as a continuum. Boundary surface S may coincide with the external boundary of the body, but it may also include only a small portion thereof..

Equations (3.48) and (3.49) represent the linear momentum theorem and the angular momentum theorem, respectively, applied to an arbitrary part of a continuum in the deformed configuration. These equations are also referred to as the *laws of motion of a continuum*, since they are considered to be valid generally.

In other words, we *postulate* that the general equations of motion of a continuum have the forms (3.48) and (3.49). (We ignore the possible couples of body and surface forces, if any). As mentioned in Subsection 3.4.2, these equations cannot be derived from the simpler equations of motion for a mass point or a rigid body, since the continuum is a more general medium. We have only used some analogies from rigid-body mechanics in seeking a probable form of the general equations. The range of validity of Eqs. (3.48) and (3.49) can be estimated only by comparing the results of the theory, based on these equations, with experiments. On the other hand, the equations of motion of a

rigid body and of a mass point follow from the general equations (3.48) and (3.49) as their special cases.

Continuum mechanics is founded on Newton's laws of motion, and the generalisation of these laws for a continuum was given by Euler already in the 18th century. However, the forms of the equations of motion which are usually used in practice (see below) were derived much later.

3.4.5 One property of the stress vector

Consider a surface element, ΔS, in the interior of a body, and denote the normal to it by $\vec{v}$ (Figs. 3.6 and 3.7). Let $\mathbf{T}^{(+)} = \mathbf{T}^{(v)}$ be the stress vector representing the action of material from the positive side of element ΔS on the material on the negative side. Similarly, $\mathbf{T}^{(-)} = \mathbf{T}^{(-v)}$ denotes the action of material on the negative side of ΔS on that on the positive side.

In view of Newton's third law, vectors $\mathbf{T}^{(+)}$ and $\mathbf{T}^{(-)}$ are equal in magnitude and opposite in direction:

$$\mathbf{T}^{(-)} = -\mathbf{T}^{(+)} \ . \tag{3.50}$$

Another way of stating this result is that the stress vector is a function of the vector normal to a surface. When the orientation of the normal vector reverses, the stress vector reverses too.
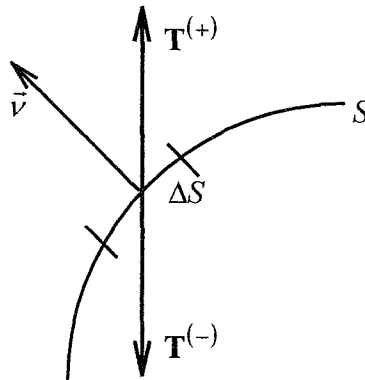
Fig. 3.7. Stress vectors acting on opposite sides of a surface.

Moreover, formula (3.50) also follows from the equation of motion (3.48). To prove this, let us shift surface element ΔS by a small distance δ in the positive and negative directions of normal $\vec{v}$, and consider the "pill box" between these two parallel surfaces (Fung, 1969). If δ shrinks to zero, while ΔS remains finite, the volume integrals vanish, as well as the contribution of surface forces on the sides of the pill box. Equation (3.48) then implies, for small ΔS, that

$$\mathbf{T}^{(+)} \Delta S + \mathbf{T}^{(-)} \Delta S = 0 \ ,$$

which yields (3.50).

3.5 Stress Tensor

3.5.1 Components of the stress tensor

In the previous Section 3.4 we introduced the basic assumption that the action of forces with a small "action radius" (surface forces) across any infinitesimal surface element can be described by a stress vector. Thus, to describe the stress state at a point, it is necessary to know the *stresses* acting *on all infinitesimal surfaces* drawn through this point. This means that surfaces of any shape should be considered.

To simplify the problem, we shall further assume that we can restrict ourselves to plane surfaces only. Thus, we adopt another assumption that the stress state at a point will be described if the *stresses* acting on *all plane infinitesimal surfaces* drawn through this point are known. We shall show in Subsection 3.5.2 that, on certain continuity assumptions, it will be even sufficient to know these stresses only on three perpendicular plane elements. For this purpose, we shall introduce suitable notations here.

Consider plane element ΔS which is perpendicular to the i-th coordinate axis, so that its normal $\vec{v}$ is parallel to the i-th axis, and has the same orientation as this axis. Let $\mathbf{T}^{(i)} = (T_1^{(i)}, T_2^{(i)}, T_3^{(i)})$ be the stress vector acting on this plane element. Introduce a new notation,

$$\boxed{\tau_{ij} = T_j^{(i)}}, \tag{3.51}$$

where $i, j = 1, 2, 3$; see Fig 3.8. The array of nine quantities τ_{ij} will be called the *stress tensor*, and the individual quantities τ_{ij} will be called the *components of the stress tensor*.
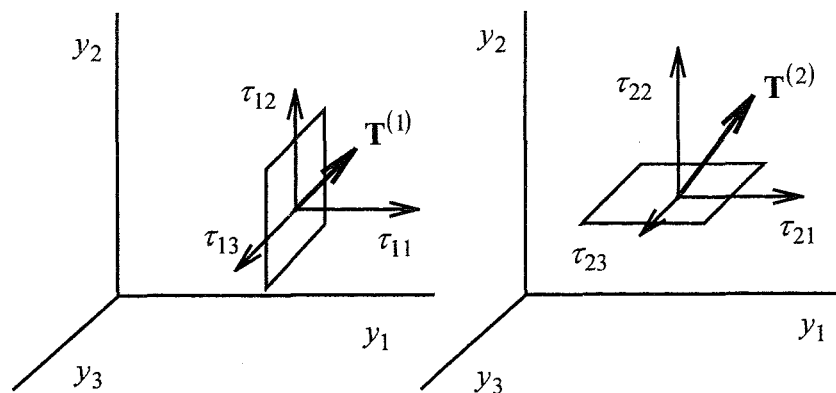
Fig. 3.8. Introduction of the components of the stress tensor.

Let us repeat the meaning of the individual subscripts in component τ_{ij}. Subscript i indicates that the corresponding plane element is perpendicular to the i-th axis, i.e. its normal $\vec{v}$ is parallel to the i-th axis. Subscript j denotes the j-th component of the corresponding force. For example, τ_{11}, τ_{12} and τ_{13} are

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the components of the force acting on a surface element which is perpendicular to the first axis. Note that, according to (3.50), the force acting on the "negative" side of this element has the opposite components, i.e. $-\tau_{11}$, $-\tau_{12}$, $-\tau_{13}$.

3.5.2 Cauchy's formula

Now, we shall show that the nine components of stress tensor τ_{ij} are sufficient to describe the stress state at a particular point. Let us consider point P and an arbitrary plane element drawn through this point. Denote the unit vector which is normal to the element by $\vec{v} = (v_1, v_2, v_3)$. Construct a small tetrahedron with one vertex at point P, three faces parallel to the coordinate planes, and the fourth face perpendicular to $\vec{v}$; see Fig. 3.9. Introduce the following notations: σ is the area of the face with normal $\vec{v}$, i.e. the area of triangle ABC, h is the distance of triangle ABC from vertex P, $V = \frac{1}{3}\sigma h$ is the volume of the tetrahedron, σ_1, σ_2 and σ_3 are the areas of the triangles which are perpendicular to the individual coordinate axes, i.e. the areas of triangles PBC, PCA and PAB, respectively. Assuming the normal $\vec{v}$ to be a unit vector, $|\vec{v}| = 1$, we have $\sigma_i = \sigma v_i$, $i = 1, 2, 3$.
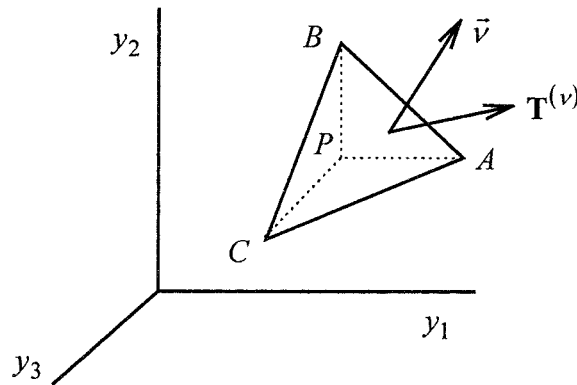
Fig. 3.9. Tetrahedron used to decompose the stress vector.

The condition of equilibrium (3.46) for the tetrahedron can be expressed as

$$\iint_{\sigma} \mathbf{T}^{(v)} \, dS - \iint_{\sigma_1} \mathbf{T}^{(1)} \, dS - \iint_{\sigma_2} \mathbf{T}^{(2)} \, dS - \iint_{\sigma_3} \mathbf{T}^{(3)} \, dS + \iiint_{V} \mathbf{F} \, dV = 0 , \quad (3.52)$$

where by the integration over σ we understand the integration over triangle ABC, etc. In the integrals over σ_1, σ_2 and σ_3 we have used the negative signs because the outward normals of the individual triangles are opposite to the orientation of the corresponding coordinate axes; see the definitions of vectors $\mathbf{T}^{(1)}$, $\mathbf{T}^{(2)}$ and $\mathbf{T}^{(3)}$ in Subsection 3.5.1. The i-th component of (3.52) is

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$$\iint_{\sigma} T_i^{(v)} \, dS - \iint_{\sigma_1} \tau_{1i} \, dS - \iint_{\sigma_2} \tau_{2i} \, dS - \iint_{\sigma_3} \tau_{3i} \, dS + \iiint_{V} F_i \, dV = 0 \ . \qquad (3.53)$$

Assume that the *body forces* and *surface forces* are *continuous* functions of coordinates. Then, according to the mean value theorem, Eq. (3.53) can be expressed as

$$T_i^{(v)}(P_\sigma)\sigma - \tau_{1i}(P_1)\sigma_1 - \tau_{2i}(P_2)\sigma_2 - \tau_{3i}(P_3)\sigma_3 + F_i(P^*)V = 0 \ ,$$

where P_σ, P_1, P_2, P_3 are points on surfaces $\sigma, \sigma_1, \sigma_2, \sigma_3$, respectively, and P^* is an interior point of the tetrahedron. Using $\sigma_i = \sigma v_i$ and dividing through by σ, we have

$$T_i^{(v)}(P_\sigma) - \tau_{1i}(P_1)v_1 - \tau_{2i}(P_2)v_2 - \tau_{3i}(P_3)v_3 + F_i(P^*)\frac{1}{3}h = 0 \ . \qquad (3.54)$$

Let the tetrahedron shrink towards point P so that point P is kept as a vertex, and the direction of normal $\vec{v}$ is kept fixed during this process. In the limit $h \to 0$, since the forces are continuous, all the points in Eq. (3.54) tend to point P and the body force vanishes. Hence

$$T_i^{(v)}(P) = \tau_{ji}(P)v_j \ . \qquad (3.55)$$

Omitting the letter P, we finally get

$$T_i^{(v)} = \tau_{ji}v_j \ . \qquad (3.56)$$

Thus, the stress vector $\mathbf{T}^{(v)}$ acting on a surface element with unit normal $\vec{v}$ is completely determined by the components of stress tensor τ_{ji}. Consequently, the stress state at a point is described in full by the nine components τ_{ji}.

We shall derive below that the stress tensor is symmetric (if the body and surface forces couples are zero), i.e. $\tau_{ij} = \tau_{ji}$. Consequently, Eq. (3.56) will usually be expressed as

$$\boxed{T_i^{(v)} = \tau_{ij}v_j} \ . \qquad (3.57)$$

This formula is referred to as Cauchy's formula.

Finally, note that the body forces have vanished in Eq. (3.55), since they decrease proportionally to the volume, i.e. as h^3, whereas the surface forces decrease as h^2. Since the inertial force is a body force, Eq. (3.57) also holds in dynamic problems.

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3.5.3 Conditions of equilibrium in differential form

The conditions of equilibrium in differential form can be derived from the integral conditions of equilibrium in several ways. In the elementary derivation, the equilibrium of an infinitesimal parallelepiped is usually considered (Fung, 1969). Here we shall give a shorter derivation which is based on the application of Gauss' theorem.

Gauss' theorem can be expressed as

$$\iiint_V \text{div}\, \mathbf{A}\, dV = \iint_S A_n\, dS = \iint_S \mathbf{A}\vec{v}\, dS = \iint_S A_j v_j\, dS \ , \tag{3.58}$$

where $\mathbf{A}$ is a continuous vector with continuous derivatives, and $\vec{v}$ is the unit outward normal. Denoting the radius vector by $\mathbf{y} = (y_1, y_2, y_3)$ and using $\text{div}\, \mathbf{A} = \partial A_j / \partial y_j$, we arrive at another form of Gauss' theorem:

$$\iiint_V \frac{\partial A_j}{\partial y_j}\, dV = \iint_S A_j v_j\, dS \ . \tag{3.59}$$

The i-th component of the integral condition of equilibrium (3.46) is

$$\iint_S T_i^{(v)}\, dS + \iiint_V F_i\, dV = 0 \tag{3.60}$$

or, using (3.56),

$$\iint_S \tau_{ji} v_j\, dS + \iiint_V F_i\, dV = 0 \ . \tag{3.61}$$

Putting $A_j = \tau_{ji}$ and using Gauss' theorem (3.59), the surface integral in (3.61) may be expressed as a volume integral:

$$\iiint_V \left(\frac{\partial \tau_{ji}}{\partial y_j} + F_i \right) dV = 0 \ . \tag{3.62}$$

Since the integrand in (3.62) is assumed to be continuous, and volume V is arbitrary, integral (3.62) will be equal to zero only if the integrand is also equal to zero (see the explanation below). This yields the condition of equilibrium in the form

$$F_i + \frac{\partial \tau_{ji}}{\partial y_j} = 0 \ . \tag{3.63}$$

Let us explain how we proceed from Eq. (3.62) to Eq. (3.63) in greater detail. Denote the integrand in Eq. (3.62) by

$$f = \frac{\partial \tau_{ji}}{\partial y_j} + F_i \; ,$$

and assume that this function is continuous. Assume further that there is a point $\overline{P}$ where function f is non-zero, say, positive:

$$f(\overline{P}) > 0 \; . \qquad (3.64)$$

Since f is continuous and positive at $\overline{P}$, this function must also be positive in a certain vicinity of this point, $\overline{V}$. However, any integral of a positive function is positive, so that

$$\iiint\limits_{\overline{V}} f \, \mathrm{d} V > 0 \; .$$

This contradicts Eq.(3.62), which must be satisfied for any volume. This means that our assumption (3.64) is wrong, and we must put $f = 0$ everywhere, which yields Eq. (3.63).

Now, let us consider the second integral condition of equilibrium, i.e. Eq. (3.47). For example, for the first component we have

$$\iint\limits_{S} \left(y_2 T_3^{(v)} - y_3 T_2^{(v)} \right) \mathrm{d} S + \iiint\limits_{V} \left(y_2 F_3 - y_3 F_2 \right) \mathrm{d} V = 0 \; . \qquad (3.65)$$

Rearrange the surface integral in this equation by means of Cauchy's formula (3.57), Gauss' theorem (3.59) and the condition of equilibrium (3.63):

$$\iint\limits_{S} \left(y_2 T_3^{(v)} - y_3 T_2^{(v)} \right) \mathrm{d} S = \iint\limits_{S} \left(y_2 \tau_{j3} v_j - y_3 \tau_{j2} v_j \right) \mathrm{d} S =$$

$$= \iiint\limits_{V} \left[\frac{\partial \left(y_2 \tau_{j3} \right)}{\partial y_j} - \frac{\partial \left(y_3 \tau_{j2} \right)}{\partial y_j} \right] \mathrm{d} V = \iiint\limits_{V} \left[\tau_{23} + y_2 \left(-F_3 \right) - \tau_{32} - y_3 \left(-F_2 \right) \right] \mathrm{d} V \; .$$

After inserting this expression into Eq. (3.65), since several terms vanish, we get

$$\iiint\limits_{V} \left(\tau_{23} - \tau_{32} \right) \mathrm{d} V = 0 \; . \qquad (3.66)$$

As the stress tensor is assumed to be continuous, we arrive at

$$\tau_{23} = \tau_{32} \; .$$

From the second and third components of Eq. (3.47), we would obtain $\tau_{13} = \tau_{31}$ and $\tau_{12} = \tau_{21}$, respectively. Consequently, we arrive at the condition of symmetry of the stress tensor,

$$\boxed{\tau_{ij} = \tau_{ji}} \, .$$
(3.67)

Note that the stress tensor is not symmetric in moment media (media with a microstructure) because additional terms are present in the integral condition of equilibrium.

We have seen that the integral condition of equilibrium (3.47) does not yield a new differential equation, but only the condition of symmetry of the stress tensor. Using this symmetry, the condition of equilibrium (3.63) can be expressed as

$$F_i + \frac{\partial \tau_{ij}}{\partial y_j} = 0 \, .$$
(3.68)

When the difference between the Lagrangian and Eulerian coordinates may be neglected (see the discussion in the next subsection), we can express the *conditions of equilibrium* also as

$$\boxed{F_i + \frac{\partial \tau_{ij}}{\partial x_j} = 0} \, .$$
(3.69)

The conditions of equilibrium are frequently used in this form.

3.5.4 Equations of motion in differential form

In deriving the differential conditions of equilibrium, we first modified integral conditions (3.46) and (3.47) to Eqs. (3.62) and (3.66), respectively. To derive the equations of motion in differential form, it would, therefore, be sufficient to modify the inertial terms on the right-hand sides of Eqs. (3.48) and (3.49) to similar volume integrals. In other words, we would need to interchange the order of the differentiation with respect to time t and the integration over volume V. However, special care should be paid to this step because volume V also varies with time t; see Fung (1969). To avoid this problem, we shall derive the equations of motion in differential form directly from the differential conditions of equilibrium by applying D'Alembert's principle to them.

The inertial force per unit volume is

$$\mathbf{F}_{iner} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho \mathbf{v}\right) \, ,$$
(3.70)

ρ being the density, $\mathbf{v}$ the velocity, and t time. Assume that the time variations of density ρ may be neglected. Velocity $\mathbf{v}$, in Lagrangian coordinates, is a function of the form $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3, t)$, where coordinates x_1, x_2, x_3 describe

the original position, i.e. they are independent of time t. Consequently, the total derivative with respect to time is equal to the corresponding partial derivative:

$$\mathbf{F}_{iner} = -\rho \frac{d\mathbf{v}}{dt} = -\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} , \qquad (3.71)$$

where $\mathbf{u}$ is the displacement vector.

According to d'Alembert's principle, the equation of motion can be obtained from the condition of equilibrium, in our case from Eq. (3.68), by adding the inertial force. Consequently,

$$F_i + \frac{\partial \tau_{ij}}{\partial y_j} = \rho \frac{\partial^2 u_i}{\partial t^2} . \qquad (3.72)$$

Since $y_k = x_k + u_k$, the following relation holds between the derivatives of the stress tensor in Lagrangian and Eulerian coordinates:

$$\frac{\partial \tau_{mn}}{\partial x_j} = \frac{\partial \tau_{mn}}{\partial y_k} \frac{\partial y_k}{\partial x_j} = \frac{\partial \tau_{mn}}{\partial y_k} \left(\delta_{kj} + \frac{\partial u_k}{\partial x_j} \right) . \qquad (3.73)$$

Assuming that the products of the derivatives are small and may be neglected, relation (3.73) simplifies to read

$$\frac{\partial \tau_{mn}}{\partial x_j} = \frac{\partial \tau_{mn}}{\partial y_j} . \qquad (3.74)$$

This means that the difference between the Lagrangian and Eulerian descriptions vanishes in this case.

The *equations of motion* of a continuum can then be expressed in the following final form:

$$\boxed{F_i + \frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}} . \qquad (3.75)$$

This is one of the most important equations in continuum mechanics, and the basic equation in the theory of elastic waves.

To complete the description, let us also derive the equations of motion in Eulerian coordinates. The velocity of an element of a continuum in these coordinates is a function of the type

$$\mathbf{v} = \mathbf{v}\left(y_1(t), y_2(t), y_3(t), t\right) .$$

The derivative of its i-th component with respect to time t is then

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{\partial v_i}{\partial y_j}\dot{y}_j + \frac{\partial v_i}{\partial t} = \frac{\partial v_i}{\partial y_j}v_j + \frac{\partial v_i}{\partial t} \ .$$

Consequently, the equation of motion in Eulerian coordinates can be expressed as

$$F_i + \frac{\partial \tau_{ij}}{\partial y_j} = \rho \left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial y_j}v_j \right) \ . \tag{3.76}$$

In the cases when the last term on the right-hand side may be neglected, this equation takes a form similar to Eq. (3.75). However, this last term cannot be neglected in many problems of hydrodynamics, which causes the corresponding equations to be non-linear and, consequently, difficult to solve.

3.6 Stress-Strain Relations

3.6.1 Rheological classification of substances

We have not yet considered the relations between strain and stress, but such relations will be needed in solving the equations of motion. The relation between strain and stress depends on the type of substance and on many other factors. This is different for gases, liquids and solids, but there are great differences even between substances of the same phase. The study of these relations is the subject of *rheology*. The relations between strain and stress in real substances may be very complicated, so that various simplified models are introduced in rheology.

Let us briefly describe the main properties of elastic, viscous and plastic substances.

A substance is said to be *elastic* if the strain completely vanishes on removal of load. A special type of elastic substance is a linear elastic substance, in which the strain and stress are directly proportional.

A substance is said to be *viscous* if any force, however small, produces strains in the substance which increase indefinitely with time. Hence, time enters into the relation between the strain and stress. Liquids are examples of such materials. A special type of viscous substance is the so-called Newtonian substance (Newtonian liquid), in which a linear relation exists between the stress and the rate of strain.

A *plastic* substance starts to flow only when a certain stress limit is exceeded. Plastic behaviour is exhibited, e.g., by metals under very large pressure and by many macro-molecular materials.

Many materials, such as asphalt, pitch, glass and others, exhibit intermediate properties between the properties of solids and liquids. A simple model for these materials is a *visco-elastic* substance, which combines the properties of a Newtonian viscous liquid and of a linear elastic material. We mention this model here, since some properties of rocks may also be described as properties of visco-elastic substances, e.g., the attenuation of seismic waves in rocks.

However, hereafter we shall restrict ourselves only to a linear elastic continuum.

3.6.2 Generalised Hooke's law

The classical Hooke's law describes deformation only in the direction of the acting force. However, we have seen that strain and stress are complicated quantities of a tensor character. Therefore, we shall generalise the classical Hooke's law by assuming that a general linear relation exists between the stress and strain tensors:

$$\boxed{\tau_{ij} = C_{ijkl}e_{kl}} \ . \tag{3.77}$$

This relation is referred to as the generalised Hooke's law, and quantities C_{ijkl} are called elastic coefficients. Relation (3.77) describes well the behaviour of many substances, such as crystals and many other anisotropic materials. As a special case, it also describes the properties of many isotropic substances.

The total number of coefficients C_{ijkl} is $3^4 = 81$. However, as a consequence of the symmetry of the stress and strain tensors, the number of independent elastic coefficients reduces to $6 \times 6 = 36$. Moreover, the elastic coefficients are also symmetric with respect to interchanging of the first and second pairs of the subscripts, i.e. $C_{ijkl} = C_{klij}$, which follows from energetic considerations. In this way, the number of independent elastic coefficients reduces to 21. This number of elastic coefficients appears in the triclinic crystallographic structure. For crystals of a higher symmetry, the number of independent coefficients reduces further, so that the monoclinic structure is characterised by 13 independent elastic coefficients, rhombic by 9, and cubic by 3 independent elastic coefficients.

An isotropic medium, which has the same properties in all directions, is characterised by 2 elastic coefficients. The Lamé coefficients, λ and μ, are usually used in theoretical papers as these two coefficients. The generalised Hooke's law for an isotropic medium then takes the form

$$\boxed{\tau_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij}}, \tag{3.78}$$

where $\vartheta = \operatorname{div} \mathbf{u} = e_{11} + e_{22} + e_{33}$ is the volume dilatation. Coefficient μ can be identified as the *shear modulus* (rigidity), but coefficient λ has no immediate physical interpretation.

The independent elastic coefficients are usually sought by analysing the changes of these coefficients under various rotations of the coordinate frame (Brdicka, 1959; Fung, 1965; Pšencík, 1994). We shall not perform these tedious calculations here, but shall only briefly derive Hooke's law for an isotropic medium in the form of (3.78). We shall start by assuming that the deformation of an isotropic body consists of two independent parts, namely of a dilatation part and a shearing part. This idea was adopted in the middle of the 19th century on the basis of extensive experiments.

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We have proved that the tensor of infinitesimal deformations e_{ij} can be divided into dilatation and shearing parts,

$$e_{ij} = f_{ij} + g_{ij} = \frac{1}{3}\vartheta\delta_{ij} + \left(e_{ij} - \frac{1}{3}\vartheta\delta_{ij}\right) \; ; \tag{3.79}$$

see Subsection 3.3.6. An analogous identity can also be applied to the stress tensor,

$$\tau_{ij} = p_{ij} + q_{ij} \; , \tag{3.80}$$

where

$$p_{ij} = \frac{1}{3}\kappa\delta_{ij}, \quad q_{ij} = \tau_{ij} - \frac{1}{3}\kappa\delta_{ij}, \quad \kappa = \tau_{11} + \tau_{22} + \tau_{33} \; . \tag{3.81}$$

According to these analogies we may expect stresses p_{ij} to produce changes of volume, and stresses q_{ij} to produce changes of shape. Therefore, we shall assume that two coefficients exist, k_1 and k_2, where k_1 expresses the proportionality between the dilatation parts of the stress and strain tensors, and k_2 expresses the proportionality between the shearing parts:

$$p_{ij} = k_1 f_{ij}, \quad q_{ij} = k_2 g_{ij} \; . \tag{3.82}$$

By inserting these expressions into Eq. (3.80) and using the definitions of f_{ij} and g_{ij} in Eq. (3.79), we obtain

$$\tau_{ij} = k_1 f_{ij} + k_2 g_{ij} = \frac{1}{3}(k_1 - k_2)\vartheta\delta_{ij} + k_2 e_{ij} \; . \tag{3.83}$$

Introducing a new notation for the elastic coefficients,

$$\lambda = \frac{1}{3}(k_1 - k_2), \quad 2\mu = k_2 \; , \tag{3.84}$$

we immediately arrive at formula (3.78).

3.7 Equations of Motion

The general equation of motion (3.75) cannot be used in practice unless the relation between stress and strain is specified, e.g., in the form of the generalised Hooke's law (3.77). Here we shall specify the equations of motion for a homogeneous isotropic medium.

3.7.1 Equations of motion for a homogeneous isotropic medium

Insert Hooke's law for an isotropic medium, i.e. Eq. (3.78), into the equation of motion (3.75):

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$$F_i + \frac{\partial}{\partial x_j}\left[\lambda \vartheta \delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \rho\frac{\partial^2 u_i}{\partial t^2} \ . \tag{3.85}$$

This equation is sometimes called the Navier-Green equation.

For a homogeneous isotropic medium, i.e. assuming elastic coefficients λ and μ to be constant, we get

$$F_i + \lambda\frac{\partial \vartheta}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}\right) = \rho\frac{\partial^2 u_i}{\partial t^2} \ . \tag{3.86}$$

Remember the following notations:

$$\vartheta = \operatorname{div}\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_j}{\partial x_j} \ ,$$

$$\frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} = \nabla^2 u_i \ , \tag{3.87}$$

$$\operatorname{grad}\vartheta = \left(\frac{\partial \vartheta}{\partial x_1}, \frac{\partial \vartheta}{\partial x_2}, \frac{\partial \vartheta}{\partial x_3}\right) \ ,$$

where ∇^2 is Laplace's operator. Equation (3.86) can now be expressed in terms of displacements as

$$F_i + (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j^2} = \rho\frac{\partial^2 u_i}{\partial t^2} \ . \tag{3.88}$$

This equation represents the i-th component $(i = 1,2,3)$ of the following vector equation,

$$\boxed{\mathbf{F} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}} \ . \tag{3.89}$$

By the Laplacian of a vector we understand the application of the Laplacian to the individual components, i.e.

$$\nabla^2\mathbf{u} = \left(\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3\right) \ . \tag{3.90}$$

However, it should be noted that such a simple definition of $\nabla^2\mathbf{u}$ may be introduced only in Cartesian coordinates. For example, in spherical or cylindrical coordinates it has a more complicated form.

Equations (3.88) and (3.89) are the required equations of motion for a homogeneous isotropic medium. Further, if **F** is replaced by a body force, **g**, which is related to the unit mass, i.e.

$$\mathbf{F} = \rho\mathbf{g} \ , \tag{3.91}$$

equation (3.89) takes the form

$$\rho\mathbf{g} + (\lambda + \mu)\,\text{grad}\,\text{div}\,\mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} \ . \tag{3.92}$$

This form of the equation of motion is frequently used in the theory of seismic waves.

3.7.2 Wave equations

Let us derive two special forms of the equation of motion for a homogeneous isotropic medium, known as the wave equations. Neglect the body force **F** in Eq. (3.89), which is acceptable in many problems of wave propagation. Apply the divergence operator to this equation and change the order of the derivatives in the second and third terms:

$$(\lambda + \mu)\,\text{div}\,\text{grad}\,\text{div}\,\mathbf{u} + \mu\nabla^2\,\text{div}\,\mathbf{u} = \rho\frac{\partial^2(\text{div}\,\mathbf{u})}{\partial t^2} \ .$$

Since the Laplacian $\nabla^2 = \text{div}\,\text{grad}$, we arrive at a *scalar wave equation* for volume dilatation $\vartheta = \text{div}\,\mathbf{u}$,

$$\boxed{\nabla^2\vartheta = \frac{1}{\alpha^2}\frac{\partial^2\vartheta}{\partial t^2}} \ , \tag{3.93}$$

where the velocity of propagation of dilatation changes (longitudinal waves, compressional waves) is

$$\boxed{\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}} \ . \tag{3.94}$$

Similarly, again put $\mathbf{F} = 0$, denote $\Omega = \text{curl}\,\mathbf{u}$ and apply the operator curl to Eq. (3.89). We shall arrive at a *vector wave equation*,

$$\boxed{\nabla^2\Omega = \frac{1}{\beta^2}\frac{\partial^2\Omega}{\partial t^2}} \ , \tag{3.95}$$

where the velocity of the propagation of distortion changes (transverse waves, shear waves) is

$$\boxed{\beta = \sqrt{\frac{\mu}{\rho}}} \quad . \qquad\qquad (3.96)$$

It follows from these equations that two types of elastic waves can propagate in a homogeneous isotropic medium, namely longitudinal and transverse waves. We shall use wave equations (3.93) and (3.95) many times in the following chapters.

3.8 A Review of the Most Important Formulae

From the seismological point of view, let us summarise the most important formulae which have been derived in this chapter:

- the expression for the tensor of infinitesimal strain e_{ij} in terms of displacement vector **u**,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \; ; \qquad\qquad (3.33)$$

- the equation of motion of a continuum,

$$F_i + \frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \; ; \qquad\qquad (3.75)$$

- the generalised Hooke's law,

$$\tau_{ij} = C_{ijkl} e_{kl} \; ; \qquad\qquad (3.77)$$

- the generalised Hooke's law for an isotropic medium,

$$\tau_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij} \; ; \qquad\qquad (3.78)$$

- the equation of motion for a homogeneous isotropic medium,

$$\mathbf{F} + \left(\lambda + \mu \right) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \nabla^2 \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \; ; \qquad\qquad (3.89)$$

- the wave equations for a homogeneous isotropic medium,

$$\nabla^2 \vartheta = \frac{1}{\alpha^2} \frac{\partial^2 \vartheta}{\partial t^2} \; , \quad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \; , \qquad\qquad (3.93, 3.94)$$

$$\nabla^2 \Omega = \frac{1}{\beta^2} \frac{\partial^2 \Omega}{\partial t^2} \; , \quad \beta = \sqrt{\frac{\mu}{\rho}} \; . \qquad\qquad (3.95, 3.96)$$

Chapter 4

Separation of the Elastodynamic Equation in a Homogeneous Isotropic Medium

In these lecture notes we shall restrict ourselves to media which are piecewise homogeneous and isotropic. Nevertheless, the equation of motion (3.89) for their homogeneous and isotropic parts is still rather complicated. This equation is more complicated than the equations which are traditionally solved in the courses of mathematical physics. The standard methods of solving partial differential equations, such as the separation of the individual variables, cannot be immediately applied to solve Eq. (3.89). We shall, therefore, attempt to express its solution as a sum of solutions of simpler equations. This can be accomplished, e.g., by introducing suitable potentials.

Potentials are auxiliary functions which are frequently introduced in mathematics and physics to facilitate the solution of complicated problems. For example, the well-know gravitational and electrostatic potentials enable us to describe the corresponding fields by one scalar function instead of three components of intensity. The velocity potential in hydrodynamics, or the Lagrangian and Hamiltonian in analytical mechanics are examples of analogous auxiliary functions. Electromagnetic potentials make it possible to reduce Maxwell's equations to simpler equations in many problems. Similarly, we shall introduce elastodynamic potentials in order to reduce the equation of motion (3.89) for a homogeneous isotropic medium to two simpler wave equations.

Elastodynamic potentials are frequently used in studying Rayleigh waves, since these waves contain both longitudinal and transverse components of motion. However, Love waves can usually be studied directly in terms of displacements, because these waves are simpler than Rayleigh waves.

4.1 Wave Equations in Terms of Potentials

Consider the equation of motion for a homogeneous isotropic medium without body forces:

$$(\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mu\nabla^2\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2} . \qquad (4.1)$$

Assume that the displacement vector is continuous together with its first derivatives. This vector can then be decomposed into irrotational and solenoidal parts (Arfken, 1970),

$$\mathbf{u} = \operatorname{grad}\varphi + \operatorname{curl}\vec{\psi} , \qquad (4.2)$$

where φ is a scalar potential and $\vec{\psi}$ is a vector potential. By inserting this expression into the equation of motion (4.1) and interchanging the order of some operations, we obtain

$$\text{grad}\left[(\lambda+2\mu)\nabla^2\varphi-\rho\frac{\partial^2\varphi}{\partial t^2}\right]+\text{curl}\left[\mu\nabla^2\vec{\psi}-\rho\frac{\partial^2\vec{\psi}}{\partial t^2}\right]=0 \ .$$

This equation will be satisfied if the expressions in the square brackets are constants. In a special case, when these constants are zero, we arrive at the wave equations

$$\nabla^2\varphi=\frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2}\ , \quad \nabla^2\vec{\psi}=\frac{1}{\beta^2}\frac{\partial^2\vec{\psi}}{\partial t^2}\ , \tag{4.3, 4.4}$$

where

$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}\ ,\quad \beta=\sqrt{\frac{\mu}{\rho}} \tag{4.5}$$

are the longitudinal and transverse wave velocities, respectively. This means that the scalar wave equation (4.3) describes longitudinal waves (compressional waves, P waves), and the vector wave equation (4.4) describes transverse waves (shear waves, S waves).

Note that non-zero constants, which we have omitted in Eqs. (4.3) and (4.4), would describe static deformations of the medium. Since we shall not solve static problems, we shall consider the wave equations without these terms.

4.2 Expressions for the Displacement and Stress in Terms of Potentials

Various coordinate systems are used in the studies of elastic wave propagation. In studying plane waves or waves generated by line horizontal sources, Cartesian coordinates are usually used. Cylindrical coordinates are commonly used to describe waves generated by a point source in a layered medium, and spherical coordinates are used in many problems of wave propagation in spherical models of the Earth.

Let us restrict ourselves to Cartesian coordinates only. We shall usually consider the coordinate axes x and y to be horizontal, the z-axis to be vertical and positive downwards (Fig. 4.1).

Let u, v and w be the Cartesian components of displacement vector $\mathbf{u}$, i.e. $\mathbf{u}=(u, v, w)$. According to (4.2), these components can be expressed in terms of potentials as

$$u=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi_z}{\partial y}-\frac{\partial\psi_y}{\partial z}\ ,\quad v=\frac{\partial\varphi}{\partial y}+\frac{\partial\psi_x}{\partial z}-\frac{\partial\psi_z}{\partial x}\ ,\quad w=\frac{\partial\varphi}{\partial z}+\frac{\partial\psi_y}{\partial x}-\frac{\partial\psi_x}{\partial y}\ .$$

$$\tag{4.6}$$
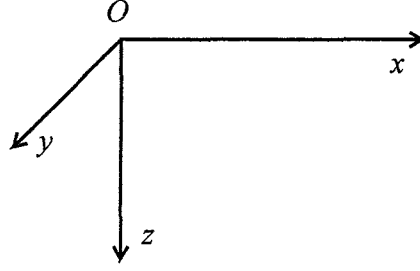
Fig. 4.1. The usual orientation of the Cartesian system.

In formulating boundary conditions, we shall also need the components of the stress tensor, τ_{ij}. For an isotropic medium, they are given by Hooke's law in the form of (3.78), i.e.

$$\tau_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij} \ . \tag{4.7}$$

In the following chapters we shall consider models with horizontal boundaries and interfaces, i.e. with planes which are perpendicular to the z-axis. The stress vector acting at these planes has components τ_{zx}, τ_{zy} and τ_{zz}. For an isotropic medium, using (4.7) and (3.33), we obtain

$$\tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad \tau_{zz} = \lambda \operatorname{div} \mathbf{u} + 2\mu\frac{\partial w}{\partial z} \ . \tag{4.8}$$

These stress components can be expressed in terms of potentials by inserting (4.6) into (4.8). However, we shall not need these general expressions, as we shall solve various special problems only.

4.3 Special Expressions for Wave Fields Which Are Independent of One Cartesian Coordinate

Very often a wave field is independent of one coordinate, say the y-coordinate. This means that, at a given time, the wave parameters are constant along any line which is parallel to the y-axis. The derivatives of all quantities with respect to y are then zero. This situation occurs, for example, if a plane wave with a constant amplitude propagates in an arbitrary direction in the (x, z)-plane.

Let us confine ourselves to this special case of the wave field being independent of the y-coordinate. Displacement vector $\mathbf{u}$ is then a function of the remaining two coordinates, x and z, and of time t, i.e. $\mathbf{u} = \mathbf{u}(x, z, t)$. Putting all derivatives with respect to y equal to zero, displacement components (4.6) simplify to read

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi_y}{\partial z}, \quad v = \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x}, \quad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_y}{\partial x} \ . \tag{4.9}$$

The expressions for the stress components in terms of potentials can now be obtained by inserting (4.9) into (4.8).

Equations (4.9) indicate that the elastodynamic potentials are now separated into two groups. Namely, potentials φ and ψ_y appear only in displacement components u and w, whereas ψ_x and ψ_z appear only in displacement component v. Consequently, the wave motion in the (x, z)-plane, described by components u and w, is now quite independent of the motion which is perpendicular to this plane. Thus, we can decompose the wave field into the corresponding two parts and investigate them separately as two independent wave phenomena.

Now, express vector potential $\vec{\psi}$ as a sum of two vectors,

$$\vec{\psi} = \vec{\psi}_{SV} + \vec{\psi}_{SH} , \qquad (4.10)$$

where

$$\vec{\psi}_{SV} = \left(0, \psi_y, 0\right), \quad \vec{\psi}_{SH} = \left(\psi_x, 0, \psi_z\right) . \qquad (4.11)$$

The displacement vector can then be expressed as

$$\mathbf{u} = \mathbf{u}_{P-SV} + \mathbf{u}_{SH} , \qquad (4.12)$$

where

$$\mathbf{u}_{P-SV} = \left(u, 0, v\right) = \operatorname{grad} \varphi + \operatorname{curl} \vec{\psi}_{SV} ,$$

$$\qquad (4.13)$$

$$\mathbf{u}_{SH} = \left(0, v, 0\right) = \operatorname{curl} \vec{\psi}_{SH} .$$

Vector $\mathbf{u}_{P-SV}$ represents a wave motion polarised in the $\left(x, z\right)$-plane and consisting of a longitudinal wave (P wave, described by potential φ) and a transverse wave polarised in this vertical plane (SV wave, described by potential ψ_y). Vector $\mathbf{u}_{SH}$ represents a transverse wave polarised in the horizontal plane parallel to the y-axis.

Let us summarise the corresponding formulae for the P-SV and SH problems.

4.3.1 P-SV problems

The solutions of P–SV problems are frequently formulated in terms of potentials, i.e. in terms of potentials φ and $\vec{\psi}_{SV} = \left(0, \psi_y, 0\right)$. Omitting the suffices SV and y, we shall express the potential for the SV waves simply as $\vec{\psi} = \left(0, \psi, 0\right)$. Potentials φ and ψ must satisfy the wave equations

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} , \quad \nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} . \qquad (4.14a, b)$$

Since we have put $\psi_x = \psi_z = 0$, displacement components (4.9) simplify further to read

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad v = 0, \quad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial x}.$$ (4.15)

By inserting these expressions into stress components (4.8), we get

$$\tau_{zx} = \mu \left(2 \frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right), \quad \tau_{zy} = 0,$$

(4.16)

$$\tau_{zz} = \lambda \nabla^2 \varphi + 2\mu \left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right) = \frac{\lambda}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} + 2\mu \left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right),$$

where we have substituted for $\nabla^2 \varphi$ from wave equation (4.14a).

4.3.2 *SH* problems

In solving *SH* problems, we shall use displacement component v directly. This will be simpler than to use two potentials ψ_x and ψ_z. Namely, if $\mathbf{u} = (0, v, 0)$ and $\partial v / \partial y = 0$, we get

$$\mathrm{div}\, \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and the equation of motion (4.1) for a homogeneous isotropic medium simplifies to the shear wave equation for component v:

$$\nabla^2 v = \frac{1}{\beta^2} \frac{\partial^2 v}{\partial t^2}.$$ (4.17)

Hence, instead of the vector wave equation (4.4) for shear potential $\vec{\psi}$, it is sufficient to solve the scalar wave equation (4.17) for displacement v.

The stress components now take the form

$$\tau_{zx} = \tau_{zz} = 0, \quad \tau_{zy} = \mu \frac{\partial v}{\partial z}.$$ (4.18)

4.4 Plane Waves

It is well known that a general transient wave can be expressed as a superposition of harmonic waves by means of the Fourier integral. Moreover, cylindrical and spherical waves may be composed of plane waves by means of other integral transforms (Aki and Richards, 1980; Ewing et al., 1957; Pšencík, 1994). Consequently, we shall restrict ourselves to plane harmonic waves only.
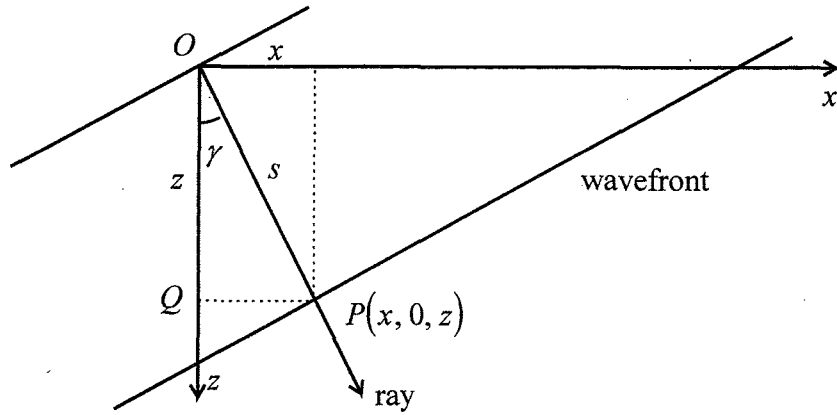
Fig. 4.2. Propagation of a plane wave.

Consider a general expression for a plane harmonic wave,

$$\varphi(x, y, z, t) = A e^{i\omega(t - s/\alpha)} \, ,$$ (4.19)

where φ is a Cartesian component of the wave field (a component of displacement, stress or potential), A is its constant amplitude, ω the angular frequency, t the time, s the distance along the ray from a reference wavefront, α the velocity of propagation.
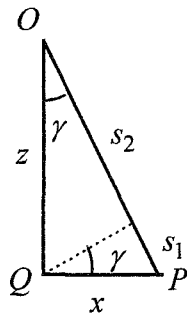
Fig. 4.3 Distance along a ray.

Assume that the rays are parallel to the (x, z)-plane of a Cartesian coordinate system (Fig. 4.2). Consider the ray passing through the coordinate origin, O. Denote by γ the angle between the ray and the positive direction of the z-axis. Consider another point on this ray, P. Its distance from point O can be expressed as

$$s = s_1 + s_2 = x \sin\gamma + z \cos\gamma \; ;$$ (4.20)

see the detail of triangle OPQ in Fig. 4.3. A plane harmonic wave, propagating in the (x, z)-plane obliquely downwards at angle γ, can now be expressed in the well-known form as

$$\varphi = A e^{i\omega\left(t - \frac{x \sin\gamma + z \cos\gamma}{\alpha}\right)} .$$ (4.21)

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It can easily be verified that this function satisfies the scalar wave equation (4.3). Thus, plane waves represent simple solutions of the wave equations.

We have denoted the velocity of propagation, i.e. the velocity along the rays, by α. Denote by c the apparent velocity with which the same wavefront propagates along the x-axis (Fig. 4.4). Let the wavefront pass through origin O at time $t = t_0$, and through points P and R at time $t = t_0 + \Delta t$. It follows from triangle ORP that

$$\sin\gamma = \frac{\overline{OP}}{\overline{OR}} = \frac{\alpha\,\Delta t}{c\,\Delta t} = \frac{\alpha}{c}\,.$$

Consequently,

$$\boxed{\frac{1}{c} = \frac{\sin\gamma}{\alpha}}\,.$$

(4.22)
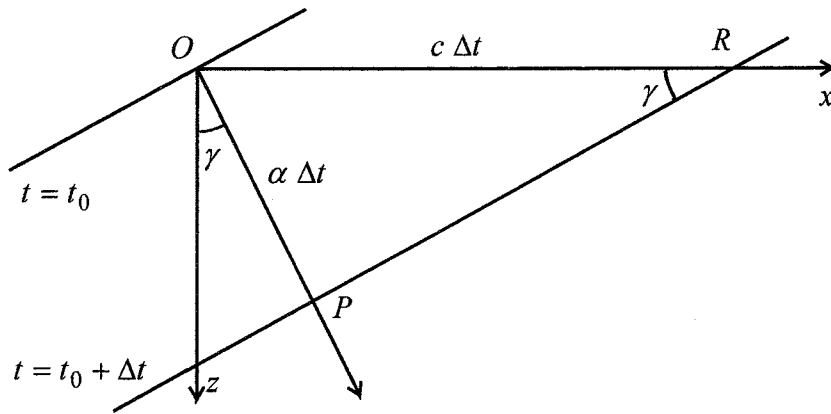
Fig. 4.4. Relation between the apparent velocity, c, and the body wave velocity, α.

Using (4.22), plane wave (4.21) can also be expressed as

$$\varphi = f(z)e^{i\omega(t-x/c)}\,,$$

(4.23)

where $f(z) = A\exp\left(-i\omega\dfrac{z\cos\gamma}{\alpha}\right)$. Expression (4.23) has the form of a wave which propagates in the direction of the x-axis with velocity c, and whose amplitude varies with the z-coordinate. We shall usually express surface waves in this form. We shall also see that surface waves do not represent principally new types of waves, but only interference phenomena of body waves. The apparent velocity c of the body-wave propagation will be the velocity of the corresponding surface waves which propagate along the surface of the medium. Hence, formula (4.22) relates surface wave velocity c to the corresponding body wave velocity α.

4.5 Surface Waves as Superpositions of Body Waves

In the following chapters we shall study surface waves propagating in various types of media. We shall always seek the solution in the form of a surface wave, and substitute this form into the equations of motion and boundary conditions. These approaches are mathematically exact, but rather formal. Many readers would probably require a deeper physical insight into the problems. For this reason, we shall also interpret surface waves as a superposition of body waves. However, it should be noted that these superpositions may be rather complicated, because not only homogeneous body waves, but also inhomogeneous waves have to be usually included (see below).

In this section we shall give a preliminary analysis of the problems which are common to all special cases studied in the following chapters. These remarks should reveal the physical mechanisms which lead to the formation of surface waves.

Consider a layered medium consisting of homogeneous and isotropic layers. Let the surface of the medium coincide with the (x, y)-plane, and the interfaces of the layers be parallel to the surface (for details see Fig. 8.1). According to Snell's law, expression (4.22) remains constant even after the reflection and transmission of the ray at the interfaces:

$$\frac{1}{c} = \frac{\sin\gamma_1}{\alpha_1} = \ldots = \frac{\sin\gamma_m}{\alpha_m} , \qquad (4.24)$$

where the subscripts indicate the ordinal number of the layer.

Consider the whole group of reflected and transmitted waves associated with a given plane wave. As a result of their interference, the wave field in each layer can be expressed as the sum of two body waves, one propagating obliquely downwards, and the other propagating obliquely upwards. The rays of the first wave in the m-th layer make angle γ_m with the positive part of the z-axis, and the rays of the second wave make the same angle with the negative part of the z-axis; see (4.24) and Fig. 4.5. As a generalisation of (4.21), the wave field in the m-th layer can be expressed as

$$\varphi_m = A_m^+ e^{i\omega\left(t - \frac{x\sin\gamma_m + z\cos\gamma_m}{\alpha_m}\right)} + A_m^- e^{i\omega\left(t - \frac{x\sin\gamma_m - z\cos\gamma_m}{\alpha_m}\right)} = f_m(z)e^{i\omega\left(t - \frac{x}{c}\right)} ,(4.25)$$

where

$$f_m(z) = A_m^+ e^{-i\frac{\omega}{\alpha_m}z\cos\gamma_m} + A_m^- e^{i\frac{\omega}{\alpha_m}z\cos\gamma_m} , \qquad (4.26)$$

and c is given by (4.24). We have denoted the amplitude of the downgoing wave by A_m^+, since its direction of propagation has a positive z-component. Analogously, the amplitude of the upgoing wave has been denoted by A_m^-.

From (4.23) and (4.25) we see that a single plane wave, as well as the wave field produced by this wave in a layered medium, are described by similar formulae, which differ only in the depth-dependent amplitudes.
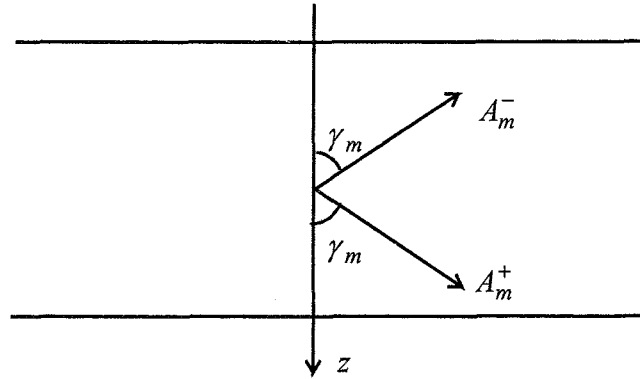
Fig. 4.5. Two systems of waves in the m-th layer.

It follows from (4.25) that the harmonic waves interfering in each layer form a wave which propagates along the x-axis. Since the velocity c along the x-axis is the same in all layers, as a consequence of Snell's law (4.24), we obtain a plane harmonic wave in the whole layered medium, which propagates in the direction of the x-axis. However, the amplitude of the resultant wave varies with depth z, see (4.26). Moreover, if total reflections occur at some of the interfaces, the energy of the waves propagates in a waveguide along the surface. We then speak of a surface wave.

Snell's law (4.24) can be generalised to any vertically inhomogeneous medium, where velocity α is a general function of depth z, $\alpha = \alpha(z)$. Then

$$\frac{1}{c} = \frac{\sin \gamma(z)}{\alpha(z)} ,$$ (4.27)

and the above interpretations of surface waves can also be extended to this type of medium. Note that quantity $p = 1/c$ is called the parameter of a seismic ray.

Chapter 5

Rayleigh Waves in a Homogeneous Isotropic Half-Space

Consider a homogeneous, isotropic and elastic half-space. Let α be the longitudinal wave velocity, β the transverse wave velocity, ρ the density and $\mu = \rho\beta^2$ the shear modulus in this half-space. Assume the surface of the half-space to be free, i.e. without stresses. Introduce a Cartesian coordinate system whose (x, y)-plane coincides with the surface of the medium, and the z-axis is positive downwards (into the medium, Figs. 4.1 and 5.1).

Rayleigh (1887) investigated the problem whether there may be a wave which propagates along the free surface of the half-space, and its amplitude becoming negligible at a distance of few wavelengths from the free surface. He proved theoretically that such a wave can exist. This wave, since called the Rayleigh wave, is polarised in the plane which is determined by the normal to the surface and by the direction of propagation. For historical remarks we refer the reader to Chapter 2 and to the proceedings by Ash and Paige (1985). Here we shall perform the corresponding derivation using elastodynamic potentials.
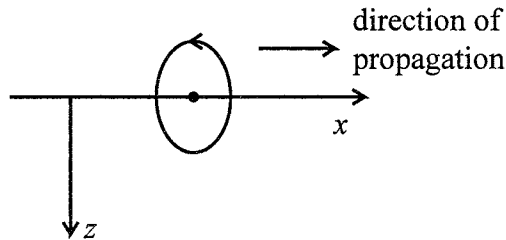
Fig. 5.1. Particle motion in the Rayleigh wave.

The formulae which will be derived in Sections 5.1 and 5.2 have a more general validity; we shall also use them in some sections below. The concrete situation for a half-space will be specified by the boundary conditions in Section 5.3.

5.1 Potentials for a Plane Harmonic Rayleigh Wave

Consider the potential for longitudinal waves, φ, and the potential for transverse waves, $\vec{\psi} = (0, \psi, 0)$, in the form of plane harmonic waves propagating in the x-direction with an identical, but unknown velocity, c:

$$\varphi(x, z, t) = f(z)e^{i\omega(t-x/c)} \ , \quad \psi(x, z, t) = g(z)e^{i\omega(t-x/c)} \ , \tag{5.1}$$

where ω is a given angular frequency, $f(z)$ and $g(z)$ are unknown functions, describing the depth-dependent amplitudes (see Section 4.5).

We do not know in advance whether the solution of our problem may be sought in the form of (5.1). This must be verified by inserting these expressions into the wave equations and boundary conditions. However, if these equations and conditions are satisfied, we shall conclude that the corresponding surface wave can exist.

Potentials (5.1) must satisfy the following wave equations:

$$\nabla^2 \varphi = \frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} \, , \quad \nabla^2 \psi = \frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} \, . \tag{5.2}$$

Inserting expressions (5.1) into these partial differential equations, we obtain ordinary differential equations for unknown functions $f(z)$ and $g(z)$:

$$f''(z) + \frac{\omega^2}{c^2}\left(\frac{c^2}{\alpha^2} - 1\right)f(z) = 0 \, , \quad g''(z) + \frac{\omega^2}{c^2}\left(\frac{c^2}{\beta^2} - 1\right)g(z) = 0 \, . \tag{5.3}$$

These equations resemble the equation of a harmonic oscillator. Hence, their general solutions can be expressed as

$$f(z) = A^- e^{ikrz} + A^+ e^{-ikrz} \, , \quad g(z) = B^- e^{iksz} + B^+ e^{-iksz} \, , \tag{5.4}$$

where $k = \omega/c$ is the wave number of the surface wave,

$$r = \sqrt{\frac{c^2}{\alpha^2} - 1} \, , \quad s = \sqrt{\frac{c^2}{\beta^2} - 1} \, , \tag{5.5}$$

and A^-, A^+, B^-, B^+ are arbitrary constants which must be determined from the boundary conditions. We have thus arrived at the potentials for plane harmonic Rayleigh waves in the form

$$\varphi(x, z, t) = \left(A^- e^{ikrz} + A^+ e^{-ikrz}\right)e^{i\omega(t-x/c)} \, , \tag{5.6a}$$

$$\psi(x, z, t) = \left(B^- e^{ikrz} + B^+ e^{-ikrz}\right)e^{i\omega(t-x/c)} \, . \tag{5.6b}$$

It will also be convenient to express these potential simply as

$$\varphi = \Phi^- + \Phi^+ \, , \quad \psi = \Psi^- + \Psi^+ \, , \tag{5.7a,b}$$

where

$$\Phi^- = A^- e^{ikrz}e^{i(\omega t - kx)} \, , \quad \Psi^- = B^- e^{iksz}e^{i(\omega t - kx)} \, ,$$

$$\Phi^+ = A^+ e^{-ikrz}e^{i(\omega t - kx)} \, , \quad \Psi^+ = B^+ e^{-iksz}e^{i(\omega t - kx)} \, ,$$

$$\tag{5.8}$$

Let us interpret velocity c as the apparent velocity with which the wavefront of a plane longitudinal wave propagates along the x-axis (Fig. 4.4). This body wave propagates (along the ray) with velocity α, and makes angle ξ with the z-axis. According to (4.22), this angle is given by

$$\frac{1}{c} = \frac{\sin \xi}{\alpha} \; .$$

Inserting this expression into r, one gets $r = \cot \xi$. Potential Φ^- of the longitudinal wave can then be expressed as

$$\Phi^- = A^- e^{i\omega\left(t - \frac{x\sin\xi - z\cos\xi}{\alpha}\right)} \; ; \qquad (5.9)$$

see similar expressions (4.25) and Fig. 4.5. This body wave propagates obliquely upwards, so that its direction of propagation has a negative z-component. Consequently, we have denoted its amplitude by A^- and its potential by Φ^-. Potential Φ^+ can be expressed as

$$\Phi^+ = A^+ e^{i\omega\left(t - \frac{x\sin\xi + z\cos\xi}{\alpha}\right)} \; , \qquad (5.10)$$

which describes a wave propagating obliquely downwards. Formula (5.7a) thus represents the decomposition of surface wave (5.6a) into two body waves. A similar interpretation can be given for potential ψ.

5.2 Displacement and Stress Components

By inserting potentials (5.7) and (5.8) into displacement components (4.15), one gets

$$u = -ik\left[\left(\Phi^- + \Phi^+\right) + s\left(\Psi^- - \Psi^+\right)\right], \quad v = 0 \; ,$$

$$\qquad (5.11)$$

$$w = ik\left[r\left(\Phi^- - \Phi^+\right) - \left(\Psi^- + \Psi^+\right)\right] \; .$$

Denote

$$\gamma = 2\left(\beta/c\right)^2 , \quad \delta = \gamma - 1 \; . \qquad (5.12)$$

The following expressions, appearing in the stress components, can then be simplified as follows:

$$2\mu k^2 = \rho\omega^2\gamma \,, \quad \mu\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right) = -\rho\beta^2 k^2\left(1 - s^2\right)\psi = -\rho\omega^2\delta\psi \,,$$

$$\frac{\lambda^2}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2} + 2\mu\frac{\partial^2\varphi}{\partial z^2} = -\left[\frac{\lambda\omega^2}{\alpha^2} + 2\mu k^2\left(\frac{c^2}{\alpha^2} - 1\right)\right]\varphi = \rho\omega^2\delta\varphi \,.$$

Stress components (4.16) can then be expressed as

$$\tau_{zx} = \rho\omega^2\left[\gamma r\left(\varPhi^- - \varPhi^+\right) - \delta\left(\varPsi^- + \varPsi^+\right)\right] \,, \quad \tau_{zy} = 0 \,,$$

$$\tag{5.13}$$

$$\tau_{zz} = \rho\omega^2\left[\delta\left(\varPhi^- + \varPhi^+\right) + \gamma s\left(\varPsi^- - \varPsi^+\right)\right] \,.$$

5.3 Boundary Conditions

We shall require the following boundary conditions to be satisfied:

1) The surface of the medium is free (the stress components vanish there), i.e.

$$\tau_{zx} = \tau_{zy} = \tau_{zz} = 0 \quad \text{for} \quad z = 0 \,. \tag{5.14}$$

Note that $\tau_{zy} = 0$ identically in this case.

2) The amplitudes diminish to zero at large distances from the surface, i.e.

$$f(z) \to 0 \,, \quad g(z) \to 0 \quad \text{for} \quad z \to \infty \,. \tag{5.15}$$

We impose this condition so that the resultant wave has the character of a surface wave.

Let us begin with the analysis of the second boundary condition. If $c > \alpha$ and $c > \beta$, radicals r and s are real, so that amplitudes $f(z)$ and $g(z)$ are oscillating functions for $z \to \infty$. This is inconsistent with boundary condition (5.15). Consequently, we must assume $c < \alpha$ and $c < \beta$, which yields imaginary values of r and s. Let us choose the positive imaginary values of these radicals:

$$r = ir^* \,, \quad s = is^* \,, \quad r^* = \sqrt{1 - (c/\alpha)^2} \,, \quad s^* = \sqrt{1 - (c/\beta)^2} \,. \tag{5.16}$$

Then

$$f(z) = A^- e^{-kr^* z} + A^+ e^{kr^* z} \,, \quad g(z) = B^- e^{-ks^* z} + B^+ e^{ks^* z} \,. \tag{5.17}$$

The terms containing A^- and B^- decrease exponentially as $z \to \infty$, which satisfies conditions (5.15). However, the terms with A^+ and B^+ increase to infinity, which is physically implausible. Thus, we must put $A^+ = B^+ = 0$. Putting simply $A = A^-$ and $B = B^-$, we have

$$\varphi = \Phi^- = Ae^{-kr^*z}e^{i(\omega t - kx)} \; , \quad \psi = \Psi^- = Be^{-ks^*z}e^{i(\omega t - kx)} \; . \qquad (5.18)$$

Hence, we have arrived at the conclusion that the longitudinal and transverse waves forming our surface wave are inhomogeneous (exponentially decreasing with distance from the surface).

Omitting the equations $v = 0$ and $\tau_{zy} = 0$, which are satisfied identically, displacement components (5.11) now become

$$u = -k\left(i\Phi^- - s^*\Psi^-\right), \quad w = -k\left(r^*\Phi^- + i\Psi^-\right), \qquad (5.19)$$

and stress components (5.13) take the form

$$\tau_{zx} = \rho\omega^2\left(i\gamma r^*\Phi^- - \delta\Psi^-\right), \quad \tau_{zz} = \rho\omega^2\left(\delta\Phi^- + i\gamma s^*\Psi\right), \qquad (5.20)$$

where Φ^- and Ψ^- are given by (5.18).

Boundary conditions (5.14) at $z = 0$ yield the equations

$$i\gamma r^* A - \delta B = 0, \quad \delta A + i\gamma s^* B = 0 \; . \qquad (5.21)$$

This is a homogeneous system of equations (their right-hand sides are equal to zero) for the unknown amplitudes A and B.

5.4 Velocity of Rayleigh Waves

The system of Eqs. (5.21) has a non-trivial solution if the corresponding determinant is equal to zero. This yields, after multiplying by $\left(4/\gamma^2\right)$, the Rayleigh equation,

$$\boxed{\left(2 - (c/\beta)^2\right)^2 = 4\sqrt{1 - (c/\alpha)^2}\,\sqrt{1 - (c/\beta)^2}} \; , \qquad (5.22)$$

where we have substituted for r^* and s^* from (5.16). After rationalisation, quantity $(c/\beta)^2$ can be factored out, and the Rayleigh equation takes the form

$$\frac{c^2}{\beta^2}\left[\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + c^2\left(\frac{24}{\beta^2} - \frac{16}{\alpha^2}\right) - 16\left(1 - \frac{\beta^2}{\alpha^2}\right)\right] = 0 \; . \qquad (5.23)$$

This equation has a solution $c = 0$, which describes a static situation. We are not interested in this solution. Thus, the expression in the square brackets in Eq. (5.23) must be equal to zero:

$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + c^2\left(\frac{24}{\beta^2} - \frac{16}{\alpha^2}\right) - 16\left(1 - \frac{\beta^2}{\alpha^2}\right) = 0 . \tag{5.24}$$

Consider a very important special case of Poisson's relation, $\lambda = \mu$. Then $\alpha = \sqrt{3}\beta$, and Eq. (5.24) becomes

$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + \frac{56}{3}\frac{c^2}{\beta^2} - \frac{32}{3} = 0 . \tag{5.25}$$

This equation represents a cubic equation in the unknown c^2/β^2. It could be solved, e.g., by applying Cardan's formulae. However, it can easily be verified that one of the roots is $(c/\beta)^2 = 4$. Consequently, Eq. (5.25) can be expressed as

$$\left(\frac{c^2}{\beta^2} - 4\right)\left(\frac{c^4}{\beta^4} - 4\frac{c^2}{\beta^2} + \frac{8}{3}\right) = 0 . \tag{5.26}$$

This equation has three real roots $(c/\beta)^2 = 4$, $2 + 2/\sqrt{3}$, $2 - 2/\sqrt{3}$. The first two of these roots do not satisfy the condition for the decrease of amplitudes with depth, i.e. the condition $c < \beta$. These roots do not satisfy the original Eq. (5.22), as they are the result of squaring this equation (they satisfy this equation except for a change of sign). The last root yields the velocity

$$\boxed{c = 0.919\beta} , \tag{5.27}$$

which satisfies all requirements.

Hence, the equations of the theory of elasticity (the wave equations and the boundary conditions at the free surface and at infinite depth) admit the existence of a surface wave which propagates along the surface with velocity $c < \beta < \alpha$, and which is polarised in the vertical plane passing through the direction of propagation. This wave is referred to as the *Rayleigh wave*.

5.5 Polarisation

Root $(c/\beta)^2 = 2 - 2/\sqrt{3}$, i.e. velocity (5.27), yields

$$r^* = \sqrt{(1 + 2/\sqrt{3})/3} = 0.847 , \quad s^* = \sqrt{2/\sqrt{3} - 1} = 0.393 ,$$
$$\gamma = 2/(2 - 2/\sqrt{3}) = 2.366 , \tag{5.28}$$

and the first of Eqs. (5.21) becomes

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$$B = iA\sqrt{1 + 2/\sqrt{3}} = 1.468\ iA\ . \qquad (5.29)$$

Introducing a new amplitude factor $D = kA$, displacements (5.19) take the form

$$u = -iD\left(e^{-0.847kz} - 0.577\,e^{-0.393kz}\right)e^{i(\omega t - kx)}\ ,$$

$$\qquad (5.30)$$

$$w = D\left(-0.847\,e^{-0.847kz} + 1.468\,e^{-0.393kz}\right)e^{i(\omega t - kx)}\ .$$

The motion at the free surface $(z = 0)$ is then

$$u = -0.423\ iDe^{i(\omega t - kx)}\ , \quad w = 0.620\ De^{i(\omega t - kx)}\ , \qquad (5.31)$$

which shows that the ratio of the vertical amplitude to the horizontal is approximately $3 : 2$. As mentioned in Chapter 2, this result was not confirmed on the first seismograms, which caused problems in their correct interpretation.

Retaining the real parts of Eqs. (5.31), the displacement components at the free surface are

$$u = 0.423\ D\sin(\omega t - kx)\ , \quad w = 0.620\ D\cos(\omega t - kx)\ , \qquad (5.32)$$

This means that the Rayleigh wave is elliptically polarised, the motion being retrograde, i.e. in the anti-clockwise direction when the wave propagates from left to right (Fig. 5.1). Note that the particle motion in waves on water is opposite, i.e. prograde.

The velocity of Rayleigh waves in a homogeneous isotropic half-space is constant, independent of frequency. We then say that the wave is non-dispersive. From the point of view of dispersion, Rayleigh waves in a homogeneous isotropic half-space represent an exceptional case of surface waves. We shall see below that surface elastic waves in more complicated media are usually dispersive, i.e. their velocity is dependent on frequency.

5.6 Non-Existence of Love Waves in a Homogeneous Half-Space

To complete the discussion of surface elastic waves in a homogeneous half-space, we should verify whether a surface wave of the *SH* type can propagate in this medium. Thus, let us consider the displacement vector in the form

$$\mathbf{u} = (0, v, 0)\ ,$$

where

$$v(x, z, t) = h(z)e^{i(\omega t - kx)}\ .$$

The wave equation (4.17) for transverse waves and the condition at infinite depth then yield

$$v = A e^{-ks^* z} e^{i(\omega t - kx)} \; ;$$

see the analogous expressions (5.18). For the stress component τ_{zy} we get

$$\tau_{zy} = \mu \frac{\partial v}{\partial z} = -\mu k s^* A e^{-ks^* z} e^{i(\omega t - kx)} \; .$$

However, the boundary condition at the free surface, i.e. $\tau_{zy} = 0$ for $z = 0$, yields $A = 0$.

Hence, no surface wave of the *SH* type can propagate in a homogeneous half-space. This was another controversy with real seismograms, as we have mentioned in the historical review in Chapter 2. To explain the presence of Love waves on seismograms, it was necessary to consider more complicated models of the medium.

Chapter 6

Love Waves in a Layer on a Half-Space

Consider a medium which consists of a homogeneous and isotropic layer of a constant thickness, lying on a homogeneous and isotropic half-space. Assume the layer and the half-space to be perfectly elastic and a welded contact to exist between them (Fig. 6.1). Denote by β_1 the velocity of shear waves, ρ_1 the density and $\mu_1 = \rho_1 \beta_1^2$ the shear modulus in the layer, by β_2, ρ_2 and μ_2 the corresponding parameters in the half-space, and by H the thickness of the layer. Assume the velocity in the layer to be lower than that in the half-space, i.e. $\beta_1 < \beta_2$. Introduce again a Cartesian coordinate system whose (x, y)-plane coincides with the surface of the medium, and the z-axis is oriented into the medium (downwards).

We wish to find out whether surface waves of the *SH* type can propagate in this medium. In other words, we are seeking surface waves which are polarised in the horizontal plane perpendicularly to the direction of propagation.
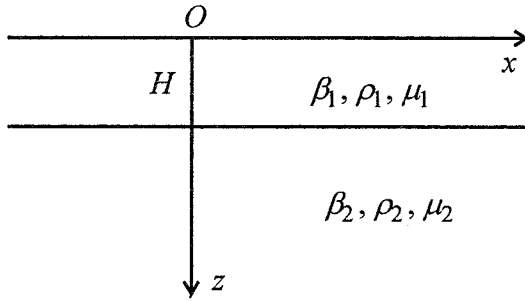
Fig. 6.1. Model of a layer on a half-space.

6.1 Expressions for Displacements

Consider a plane harmonic wave which propagates along the x-axis, is polarised along the y-axis, and its amplitude varies generally with depth z. Denote by ω its angular frequency (a given value), and by c its velocity of propagation along the x-axis. We consider velocity c to be unknown, but the same in the layer and in the half-space; see the discussion in Section 4.5. Thus, assume the displacement vectors in the layer, $\mathbf{u}_1$, and in the half-space, $\mathbf{u}_2$, to be of the form

$$\mathbf{u}_i = \left(u_i, v_i, w_i\right), \quad i = 1, 2 , \tag{6.1}$$

where

$$u_1 = u_2 = w_1 = w_2 = 0 , \tag{6.2}$$

$$v_1(x, z, t) = f(z)e^{i\omega(t - x/c)} , \quad v_2(x, z, t) = g(z)e^{i\omega(t - x/c)} , \tag{6.3, 6.4}$$

These displacements must satisfy the equation of motion (4.1) for a homogeneous and isotropic medium. However, in view of the special form of these displacement vectors, the equation of motion reduces to the wave equations (4.17) for transverse waves:

$$\nabla^2 v_i = \frac{1}{\beta_i^2} \frac{\partial^2 v_i}{\partial t^2} , \quad i = 1, 2 .$$ (6.5)

By inserting displacements (6.3) and (6.4) into these wave equations, we obtain the following ordinary differential equations for the unknown depth-dependent amplitudes $f(z)$ and $g(z)$:

$$f''(z) + \frac{\omega^2}{c^2}\left(\frac{c^2}{\beta_1^2} - 1\right)f(z) = 0 , \quad g''(z) + \frac{\omega^2}{c^2}\left(\frac{c^2}{\beta_2^2} - 1\right)g(z) = 0 ;$$ (6.6)

see analogous equations in Chapter 5. The general solutions of Eqs. (6.6) can be expressed as

$$f(z) = A e^{iks_1 z} + B e^{-iks_1 z} , \quad g(z) = C e^{iks_2 z} + D e^{-iks_2 z} ,$$ (6.7)

where $k = \omega/c$ is again the wave number of the surface wave,

$$s_1 = \sqrt{(c/\beta_1)^2 - 1} , \quad s_2 = \sqrt{(c/\beta_2)^2 - 1} ,$$ (6.8)

and A, B, C, D are arbitrary constants which must be determined from the boundary conditions. Thus,

$$v_1 = \left(A e^{iks_1 z} + B e^{-iks_1 z}\right)e^{i(\omega t - kx)} , \quad v_2 = \left(C e^{iks_2 z} + D e^{-iks_2 z}\right)e^{i(\omega t - kx)} .$$ (6.9)

6.2 Boundary Conditions

Now we shall add boundary conditions, which determine the properties of the wave field at the boundaries. We shall consider the following boundaries and conditions, which must be satisfied at any place of the corresponding boundary and at every time:
1) *Surface of the medium, $z = 0$.* We assume the surface of the medium to be free, i.e. all stress components vanish there, see (5.14). Since now $\tau_{zx} = \tau_{zz} = 0$ identically in view of (4.18), we have just one condition:

$$\left(\tau_{zy}\right)_1 = \mu_1 \frac{\partial v_1}{\partial z} = 0 \quad \text{for} \quad z = 0 ,$$ (6.10)

where $\left(\tau_{zy}\right)_1$ denotes the stress component in the layer.

2) *Interface between the layer and half-space, $z = H$.* We require all displacement and stress components to be continuous across this interface. The continuity of displacements follows from the assumption of the welded contact between these media, and the continuity of stresses follows from the property (3.50) of the stress vector. Since displacements (6.2) are zero, and displacements v_1 and v_2 are independent of y, see (6.3) and (6.4), we need consider only the following two conditions:

$$v_1 = v_2 \quad \text{for} \quad z = H \;, \tag{6.11}$$

and

$$\left(\tau_{zy}\right)_1 = \left(\tau_{zy}\right)_2 \quad \text{for} \quad z = H \;.$$

The latter condition yields

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \quad \text{for} \quad z = H \;. \tag{6.12}$$

3) *Infinite depth, $z \to \infty$.* We shall require the displacement to diminish to zero at large depths, i.e.

$$v_2(z) \to 0 \quad \text{for} \quad z \to \infty \;. \tag{6.13}$$

This condition guarantees that the wave under consideration will have the character of a surface wave.

First, let us consider the last condition (6.13) at infinite depth. If $c > \beta_2$, radical s_2 is real, and $g(z)$ is an oscillating function. This contradicts condition (6.13). Consequently, we must assume $c < \beta_2$, which yields the imaginary value of s_2. Assume this radical to have a positive imaginary part:

$$s_2 = i s_2^* \;, \quad s_2^* = \sqrt{1 - \left(c/\beta_2\right)^2} \;. \tag{6.14}$$

The first term in $g(z)$ then tends to zero for $z \to \infty$, whereas the second term increases exponentially. Therefore, in order to satisfy boundary condition (6.13), we must put $D = 0$. Thus,

$$g(z) = C e^{-k s_2^* z} \;. \tag{6.15}$$

Note that had we chosen the opposite sign in (6.14), i.e. $s_2 = -i\sqrt{1 - \left(c/\beta_2\right)^2}$, we would have had to put $C = 0$, and D would be non-zero.

The boundary condition at the free surface, Eq. (6.10), is satisfied if $df/dz = 0$ for $z = 0$, which yields $A = B$. Consequently,

$$f(z) = 2A\cos\left(ks_1 z\right) = E\cos\left(ks_1 z\right) , \tag{6.16}$$

where we have introduced a new constant $E = 2A$.

The remaining boundary conditions (6.11) and (6.12) now yield

$$E\cos Q - Be^{-ks_2^* H} = 0 , \quad -\mu_1 Eks_1 \sin Q + \mu_2 Bks_2^* e^{-ks_2^* H} = 0 , \tag{6.17}$$

where

$$Q = ks_1 H . \tag{6.18}$$

Equations (6.17) represent two equations for the unknown coefficients E and B. Another quantity, which still remains to be determined, is velocity c.

6.3 Dispersion Equation and Its Solutions

Equations (6.17) represent a system of two homogeneous equations (their right-hand sides are zero). Such a system has a non-trivial solution if the corresponding determinant is zero. This yields the equation

$$\tan Q = \frac{\mu_2 s_2^*}{\mu_1 s_1} . \tag{6.19}$$

By returning to the initial notations, this equation can be expressed as

$$\boxed{\tan\left(\frac{\omega}{c}H\sqrt{\frac{c^2}{\beta_1^2}-1}\right) = \frac{\mu_2\sqrt{1-\dfrac{c^2}{\beta_2^2}}}{\mu_1\sqrt{\dfrac{c^2}{\beta_1^2}-1}} .} \tag{6.20}$$

Hence, the condition of the existence of a non-trivial solution yields the equation for determining velocity c. We have thus proved that surface waves of the SH type can propagate in a medium consisting of a layer on a half-space, and their velocity c is given by Eq. (6.20). These transversally polarised surface waves are called *Love waves*.

Since Eq. (6.20) also contains angular frequency ω, velocity c depends not only on the parameters of the medium, but also on frequency. This means that Love waves propagating in a layer on a half-space are *dispersive*. Equation (6.20) represents the well-known *dispersion equation* (*dispersion relation*, *period equation*) for Love waves in this model of the medium.

The curve showing the dependence of velocity on frequency (or period) is called the *dispersion curve*. Since dispersion equation (6.20) contains a periodic function (function tangent), the number of branches of the

corresponding dispersion curves is infinite. The individual waves are called *modes* of the surface wave.

This situation can be demonstrated better on the equation which is inverse to (6.20):

$$\frac{\omega}{c} H \sqrt{\left(c/\beta_1\right)^2 - 1} = \arctan \frac{\mu_2 \sqrt{1 - \left(c/\beta_2\right)^2}}{\mu_1 \sqrt{\left(c/\beta_1\right)^2 - 1}} + n\pi , \qquad (6.21)$$

where $n = 0, 1, 2, \dots$. Note that values $n < 0$ do not satisfy this equation, because its left-hand side is positive. The branch for $n = 0$ is called the *fundamental mode*, the branch for $n = 1$ is the first *higher mode*, etc. The fundamental and higher modes of surface waves are analogous to the fundamental tone and overtones in acoustics.

Dispersion equation (6.20), resp. (6.21), represents an implicit equation in velocity c. This is a transcendental equation which must be solved, for a given model of the medium and a given angular frequency, by a numerical method.

Velocity c, considered above, is called the *phase velocity*. We shall see later that this velocity is not sufficient to describe the wave propagation in dispersive media. Another velocity, called the *group velocity*, must also be introduced. Namely, if two harmonic waves with different frequencies interfere in a dispersive medium, the carrier wave then propagates at one velocity (phase velocity), and the modulation wave propagates at another velocity (group velocity). The phase velocity thus describes the propagation of the individual peaks and troughs of a wave. It can be shown that the group velocity describes the propagation of energy. We shall discuss these problems in detail in Chapter 11.

The group velocity, U, is given by

$$U = \frac{d\omega}{dk} . \qquad (6.22)$$

Its reciprocal value is

$$\frac{1}{U} = \frac{dk}{d\omega} = \frac{d(\omega/c)}{d\omega} = \frac{1}{c} - \frac{\omega}{c^2} \frac{dc}{d\omega} ,$$

which yields

$$U = \frac{c}{1 - \frac{\omega}{c} \frac{dc}{d\omega}} . \qquad (6.23)$$

In order to compute the group velocity by means of this formula, the derivative $dc/d\omega$ must be determined by numerical differentiation, or by some analytical method. We shall return to this problem at the end of this chapter.

Figure 6.2 shows the dispersion curves of Love waves for a simple model of the Earth's crust and upper mantle with the parameters:

$$\beta_1 = 3.5\,\mathrm{km\,s^{-1}}, \quad \rho_1 = 2.7\,\mathrm{g\,cm^{-3}}, \quad H = 35\,\mathrm{km},$$

$$\beta_2 = 4.5\,\mathrm{km\,s^{-1}}, \quad \rho_2 = 3.3\,\mathrm{g\,cm^{-3}}, \tag{6.24}$$

where ρ_1 and ρ_2 are the densities in the layer and in the half-space, respectively. The corresponding shear moduli are $\mu_i = \rho_i \beta_i^2$, $i = 1, 2$.

It can be shown that dispersion equation (6.20) has real roots c only within the interval $\beta_1 \le c \le \beta_2$; see Fig. 6.2. These modes with real phase velocities are called *normal modes*.

However, the higher modes have continuations also for $|c| > \beta_2$, but the roots are complex. The imaginary part of the corresponding phase velocity describes the leakage of energy from the layer into the half-space, which causes an exponential decrease of amplitudes with increasing distance x. These waves are thus called *leaking modes*; see Section 2.5.
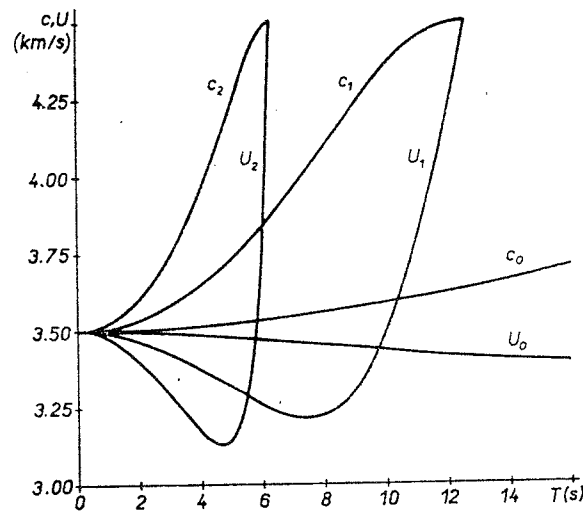
Fig. 6.2. Velocities of Love waves in medium (6.24) as functions of period T. Curves c_0 and U_0 are the phase and group velocities for the fundamental mode, c_1 and U_1 for the first higher mode, and c_2 and U_2 for the second higher mode. (After Novotny (1972)).

6.4 Derivation of the Dispersion Equation from the Condition of Constructive Interference

We shall show that the dispersion equation (6.20) for Love waves in a layer on a half-space can also be derived from the condition of constructive interference of SH waves which propagate in the layer by multiple reflections (see, e.g., Savarensky (1975)).

Consider the same elastic medium as shown in Fig. 6.1, and a plane harmonic SH wave propagating in the layer by multiple reflections (Fig. 6.3). Assume that the angle of incidence, γ_1, is large enough, so that total reflections occur at the bottom of the layer. The wave energy is then confined to the layer.

Consider the wavefront passing through points A and D, and calculate the phase difference between these points when the wave propagates along path $ABCD$. Let the displacement at point A be a harmonic motion of unit amplitude,

$$v(A) = e^{i\omega t} . \tag{6.25}$$

At point D the wave has the amplitude

$$v(D) = V(B) V(C) e^{i\omega(t-L/\beta_1)} , \tag{6.26}$$

where $V(B)$ and $V(C)$ are the reflection coefficients at the corresponding points, and L is the length of path $ABCD$.

We shall first derive the formulae for the reflection coefficients.
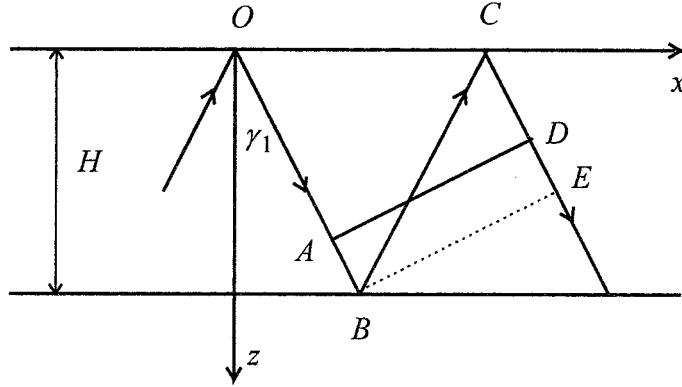
Fig. 6.3. Interference of SH waves in a layer.

6.4.1 Reflection and transmission of SH waves

Let us consider two homogeneous isotropic half-spaces in a welded contact. Let the (x, y)-plane coincide with the interface between the half-spaces (Fig. 6.4). Denote the shear-wave velocity and shear modulus in the first half-space by β_1 and μ_1, respectively. The corresponding parameters in the second half-space are β_2 and μ_2.

Consider a plane harmonic SH wave propagating in the first half-space and incident at the interface at angle γ_1. Without loss of generality, we may put its amplitude equal to unity. The displacement vector for the incident wave is then $\mathbf{u} = (0, v, 0)$, where

$$v = e^{i\omega\left(t - \frac{x\sin\gamma_1 + z\cos\gamma_1}{\beta_1}\right)} , \tag{6.27}$$

ω being the angular frequency, and t the time.
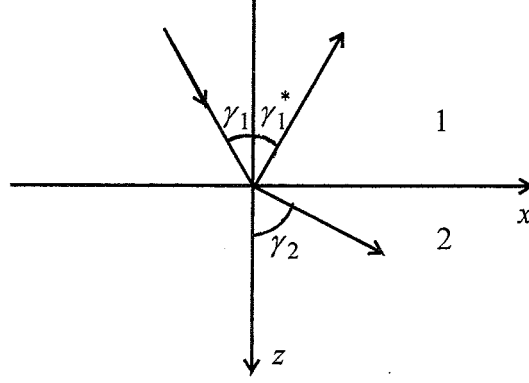
Fig. 6.4. Reflection and transmission of an *SH* wave.

Assume that this incident wave generates a reflected *SH* wave and a transmitted *SH* wave. Denote the angles of reflection and transmission by γ_1^* and γ_2, respectively. The displacement vector for the reflected wave, $\mathbf{u}_R$, and for the transmitted wave, $\mathbf{u}_T$, can then be expressed as

$$\mathbf{u}_R = \left(0, v_R, 0\right), \quad v_R = V e^{i\omega_R\left(t - \frac{x\sin\gamma_1^* - z\cos\gamma_1^*}{\beta_1}\right)} ,$$

(6.28)

$$\mathbf{u}_T = \left(0, v_T, 0\right), \quad v_T = W e^{i\omega_T\left(t - \frac{x\sin\gamma_2 + z\cos\gamma_2}{\beta_2}\right)} ,$$

ω_R and ω_T being the angular frequencies of the corresponding harmonic waves.

The resultant displacement in the first half-space is composed of the incident and reflected waves, $\mathbf{u}_1 = \mathbf{u} + \mathbf{u}_R$, and the displacement in the second half-space is formed by the transmitted wave only, $\mathbf{u}_2 = \mathbf{u}_T$. For their non-zero components, i.e. the y-components, we get

$$v_1 = v + v_R = e^{i\omega\left(t - \frac{x\sin\gamma_1 + z\cos\gamma_1}{\beta_1}\right)} + V e^{i\omega_R\left(t - \frac{x\sin\gamma_1^* - z\cos\gamma_1^*}{\beta_1}\right)} ,$$

(6.29)

$$v_2 = v_T = W e^{i\omega_T\left(t - \frac{x\sin\gamma_2 + z\cos\gamma_2}{\beta_2}\right)} .$$

As boundary conditions, we shall require the continuity of all displacement and stress components at the interface. The continuity of displacements now reduces to the condition

$$v_1 = v_2 \quad \text{for} \quad z = 0 ,$$

(6.30)

which yields

$$e^{i\omega\left(t-\frac{x\sin\gamma_1}{\beta_1}\right)} + Ve^{i\omega_R\left(t-\frac{x\sin\gamma_1^*}{\beta_1}\right)} = We^{i\omega_T\left(t-\frac{x\sin\gamma_2}{\beta_2}\right)} . \tag{6.31}$$

Since this condition must be satisfied at any instant of time, the angular frequencies of all waves must be identical, i.e. $\omega_R = \omega_T = \omega$. Moreover, this condition must be satisfied at any point of the interface, i.e. for all values of x. Consequently, the exponential factors in (6.31) must be identical, which yields Snell's law:

$$\frac{\sin\gamma_1}{\beta_1} = \frac{\sin\gamma_1^*}{\beta_1} = \frac{\sin\gamma_2}{\beta_2} . \tag{6.32}$$

In particular, the angle of reflection must be equal to the angle of incidence, $\gamma_1^* = \gamma_1$. Boundary condition (6.31) then simplifies to

$$1 + V = W . \tag{6.33}$$

The continuity of the stress vector across the interface can be expressed as

$$\mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \quad \text{for} \quad z = 0 . \tag{6.34}$$

By inserting displacements (6.29) into this equation, and taking into account (6.32), we obtain

$$\mu_1 \frac{\cos\gamma_1}{\beta_1}(1 - V) = \mu_2 W \frac{\cos\gamma_2}{\beta_2} . \tag{6.35}$$

Equations (6.33) and (6.35) represent two equations in two unknown coefficients, namely reflection coefficient V and transmission coefficient W. Their solution yields

$$V = \frac{\mu_1\sqrt{1 - \sin^2\gamma_1} - \mu_2\sqrt{n^2 - \sin^2\gamma_1}}{\mu_1\sqrt{1 - \sin^2\gamma_1} + \mu_2\sqrt{n^2 - \sin^2\gamma_1}} , \tag{6.36}$$

$$W = \frac{2\mu_1\sqrt{1 - \sin^2\gamma_1}}{\mu_1\sqrt{1 - \sin^2\gamma_1} + \mu_2\sqrt{n^2 - \sin^2\gamma_1}} , \tag{6.37}$$

where we have substituted for $\cos\gamma_2$ from Snell's law,

$$\cos\gamma_2 = \sqrt{1 - \sin^2\gamma_2} = \sqrt{1 - (\beta_2/\beta_1)^2 \sin^2\gamma_1} ,$$

and introduced the corresponding refraction index, $n = \beta_1/\beta_2$.

In the next subsection we shall need the reflection coefficients for two special cases, namely for total reflection and for the reflection at a free surface. Let us derive the corresponding formulae for these cases.

a) *Total reflection*. Assume the velocity in the second half-space to be higher than that in the first half-space, i.e. $\beta_1 < \beta_2$ and $n < 1$. Let the angle of incidence γ_1 exceeds the critical angle $\gamma_c = \arcsin n$. In this case,

$$\sqrt{n^2 - \sin^2 \gamma_1} = \pm i \sqrt{\sin^2 \gamma_1 - n^2} \ . \tag{6.38}$$

The positive sign on the right-hand side yields a transmitted wave whose amplitude indefinitely increases as $z \to \infty$. Therefore, we must choose the negative sign:

$$\sqrt{n^2 - \sin^2 \gamma_1} = -i \sqrt{\sin^2 \gamma_1 - n^2} \ . \tag{6.39}$$

This yields the transmitted wave which exponentially decreases with depth z. Such waves are called *inhomogeneous waves*, or evanescent waves.

Inserting (6.39) into reflection coefficient (6.36) yields

$$V = \frac{\mu_1 \sqrt{1 - \sin^2 \gamma_1} + i \mu_2 \sqrt{\sin^2 \gamma_1 - n^2}}{\mu_1 \sqrt{1 - \sin^2 \gamma_1} - i \mu_2 \sqrt{\sin \gamma_1 - n^2}} \ . \tag{6.40}$$

This coefficient is of the form

$$V = \frac{a + ib}{a - ib} = \frac{a^2 - b^2 + 2iab}{a^2 + b^2} \ , \tag{6.41}$$

where $a = \mu_1 \sqrt{1 - \sin^2 \gamma_1}$ and $b = \mu_2 \sqrt{\sin^2 \gamma_1 - n^2}$. It is now evident that the absolute value of the reflection coefficient is equal to unity, $|V| = 1$, which indicates that total reflection occurs. The reflection coefficient can now be expressed as

$$V = e^{i2\varphi} \ , \tag{6.42}$$

where the phase shift, denoted by 2φ, satisfies the formula

$$\tan 2\varphi = \frac{2ab}{a^2 - b^2} \ . \tag{6.43}$$

The effect of the phase shift is to increase the time factor from t to $t + 2\varphi/\omega$, regardless of the choice of axes and direction of propagation. Formula (6.43) can be replaced by a simpler one. It follows from trigonometry that

$$\tan 2\varphi = \frac{\sin 2\varphi}{\cos 2\varphi} = \frac{2\sin \varphi \cos \varphi}{\cos^2 \varphi - \sin^2 \varphi}. \qquad (6.44)$$

Formulae (6.43) and (6.44) have a similar structure, so that we may put $a = \cos \varphi$, $b = \sin \varphi$. Angle φ is then given by

$$\tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{b}{a} = \frac{\mu_2 \sqrt{\sin^2 \gamma_1 - n^2}}{\mu_1 \sqrt{1 - \sin^2 \gamma_1}}. \qquad (6.45)$$

Hence, the reflection coefficient of *SH* waves for overcritical reflections $(\gamma_1 > \gamma_c)$ is given by formulae (6.42) and (6.45).

b) *Reflection at a free surface.* If the second medium is the vacuum, then $\mu_2 = 0$ and reflection coefficient (6.36) becomes unity,

$$V = 1 . \qquad (6.46)$$

Now it follows from (6.29) that the amplitude of the resultant motion at the free surface is twice as large as that of the incident wave.

6.4.2 Condition of constructive interference

The reflection coefficient at point B in Fig. 6.3 is given by (6.42) and (6.45), and the reflection coefficient at the free surface is equal to unity. Thus, displacements (6.26) can be expressed as

$$v(D) = e^{i\left[2\varphi + \omega\left(t - L/\beta_1\right)\right]} , \qquad (6.47)$$

where φ is given by (6.45).

The constructive interference at wavefront AD occurs if the interfering waves are shifted in phase by an integer multiple of 2π, i.e. when the phases at points A and D differ by this value. Using (6.25) and (6.47), this condition can be expressed as

$$-2\varphi + \frac{\omega L}{\beta_1} = 2n\pi , \qquad (6.48)$$

n being an integer.

Express length L in terms of the angle of incidence, γ_1, and the thickness of the layer, H. Since the length of path $ABCD$ is equal to the length of BCE (Fig. 6.3),

$$L = \overline{BC} + \overline{CE} . \qquad (6.49)$$

It follows from Fig. 6.3 that

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$$\cos\gamma_1 = \frac{H}{\overline{OB}} = \frac{H}{\overline{BC}} \ .$$

From triangle BCE, since the angle at vertex C is $2\gamma_1$, we get

$$\cos 2\gamma_1 = \frac{\overline{CE}}{\overline{BC}} \ .$$

Consequently,

$$L = \overline{BC}(1 + \cos 2\gamma_1) = \frac{H}{\cos\gamma_1} 2\cos^2\gamma_1 = 2H\cos\gamma_1 \ . \qquad (6.50)$$

Equation (6.48), after dividing through by two, can now be expressed as

$$\frac{\omega}{\beta_1} H\cos\gamma_1 = \varphi + n\pi \ . \qquad (6.51)$$

Taking the tangent of both sides of this equation, and substituting from (6.45), we get

$$\tan\left(\frac{\omega}{\beta_1} H\sqrt{1-\sin^2\gamma_1}\right) = \frac{\mu_2\sqrt{\sin^2\gamma_1 - (\beta_1/\beta_2)^2}}{\mu_1\sqrt{1-\sin^2\gamma_1}} \ . \qquad (6.52)$$

For a given model of the medium and a given angular frequency ω, this equation determines the angle of incidence, γ_1, for which constructive interference occurs.

Introduce the velocity c with which the wavefront propagates along the surface. Analogously with (4.22),

$$\frac{1}{c} = \frac{\sin\gamma_1}{\beta_1} \ . \qquad (6.53)$$

After some simple algebra, Eq. (6.52) yields the dispersion equation (6.20) for Love waves in this medium. If the angle of incidence γ_1 exceeds the critical angle for the total reflection at the bottom of the layer, i.e. if $\sin\gamma_1 > \beta_1/\beta_2$, phase velocity c is real and less than β_2. This yields normal Love modes. If the angle of incidence is less than the critical angle, leakage of energy from the layer into the half-space occurs, and this yields leaking Love modes.

We have arrived at the conclusion that the dispersion equation (6.20) for Love waves represents the condition of constructive interference of SH waves which propagate by multiple reflection in the layer. This proves (for this simple medium) the physical interpretation of *Love waves as interfering SH waves.*

6.5 Methods of Computing the Group Velocity

Assume that phase velocity c is computed, by solving the corresponding dispersion equation, as a function of angular velocity ω. In order to determine group velocity U for a given ω, it is sufficient to determine the derivative $dc/d\omega$ and substitute it into (6.23). If the phase velocity is computed as a function of period, $c = c(T)$, a formula analogous to (6.23) can be used.

Derivative $dc/d\omega$ can be computed by numerical differentiation or by applying analytical methods. Although numerical differentiation is simple in principle, its application is less accurate and also slow, because the transcendental dispersion equation must be solved repeatedly in computing each difference. Therefore, many authors prefer analytical methods, which require only one solution of the transcendental equation per frequency. These methods can be divided into two groups:

- variational methods (Meissner, 1926; Harkrider, 1968, 1970; Takeuchi and Saito, 1972; Novotny, 1976a);
- methods based on the implicit function theorem (Novotny, 1970; Harkrider, 1970; Schwab and Knopoff, 1972).

The variational technique is based on the Rayleigh principle, which states that the kinetic and potential energies, averaged over a cycle, are equal. We shall not describe the details of this method here.

A dispersion equation represents an implicit equation in the unknown phase velocity c. Therefore, derivative $dc/d\omega$ (and also the derivatives of c with respect to the parameters of the medium) can be calculated analytically on the basis of the implicit function theorem (Novotny, 1970; Schwab and Knopoff, 1972). From the theoretical point of view, this method is simpler than the variational method, because the Rayleigh principle is not needed here; only the form of the dispersion equation must be known. The formula for the group velocity for our special case of Love waves in a layer on a half-space, derived by this method, can be found in the paper by Novotny (1971). However, this formula is somewhat "asymmetric". Here we shall derive a better, equivalent formula, using the modified approach proposed by Harkrider (1970).

In studying surface waves in layered media, Harkrider (1970) considered the wave number, $k = \omega/c$, as the unknown quantity instead of phase velocity c. In a similar way, let us replace c by k in Eq. (6.20). This dispersion equation can then be expressed as

$$f\left(\omega, \beta_1, \mu_1, H, \beta_2, \mu_2, k\left(\omega, \beta_1, \mu_1, H, \beta_2, \mu_2\right)\right) = 0 , \qquad (6.54)$$

where

$$f = \mu_1 \tilde{s}_1 \tan Q - \mu_2 \tilde{s}_2 = 0 , $$

$$(6.55)$$

$$\tilde{s}_1 = \sqrt{\left(\omega/\beta_1\right)^2 - k^2} , \quad \tilde{s}_2 = \sqrt{k^2 - \left(\omega/\beta_2\right)^2} , \quad Q = H\tilde{s}_1 .$$

Keeping the parameters of the model fixed, let us differentiate Eq. (6.54) with respect to ω:

$$\frac{\partial f}{\partial \omega} + \frac{\partial f}{\partial k}\frac{\mathrm{d}k}{\mathrm{d}\omega} = 0 \ . \tag{6.56}$$

This yields (the implicit function theorem)

$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = -\frac{\partial f / \partial \omega}{\partial f / \partial k} \ . \tag{6.57}$$

By differentiating the dispersion function f, given by (6.55), one gets

$$\frac{\partial f}{\partial \omega} = \frac{\omega}{\tilde{s}_2 \cos^2 Q}\left(\rho_1 eR + \rho_2 \cos^2 Q\right), \quad \frac{\partial f}{\partial k} = -\frac{k}{\tilde{s}_2 \cos^2 Q}\left(\mu_1 eR + \mu_2 \cos^2 Q\right) , \tag{6.58}$$

where

$$e = 1 + Q^{-1}\sin Q \cos Q , \quad R = H\tilde{s}_2 = \frac{\omega}{c}Hs_2^* . \tag{6.59}$$

Since $U = 1/\left(\mathrm{d}k / \mathrm{d}\omega\right)$, we arrive at the following simple formula for the group velocity of Love waves in our model of the medium (Novotny, 1999):

$$\boxed{U = \frac{\mu_1 eR + \mu_2 \cos^2 Q}{c\left(\rho_1 eR + \rho_2 \cos^2 Q\right)}} \ . \tag{6.60}$$

Hence, the group velocity may be computed in the following two steps:
1) For a given model and a given angular frequency, the phase velocity is determined by solving the dispersion equation (6.20) by some numerical method.
2) This value of the phase velocity is inserted into formula (6.60), which immediately yields the group velocity.

Chapter 7

Rayleigh Waves in a Layer on a Half-Space

Consider the same medium consisting of a layer on a half-space as in Chapter 6; see Fig. 6.1. Denote by α_1 the longitudinal wave velocity, β_1 the transverse wave velocity, ρ_1 the density, λ_1 and μ_1 the Lamé constants, and H the thickness of the layer. Denote the corresponding parameters in the half-space by $\alpha_2, \beta_2, \rho_2, \lambda_2$ and μ_2, respectively. Assume that $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$.

We shall study the properties of harmonic Rayleigh waves propagating in this medium. We shall proceed in a way similar to that in Chapter 6, but instead of one displacement component we shall use two potentials. In view of these analogies, we shall omit some details here. However, in order to simplify the expressions for the boundary conditions, we shall use other forms for the general solutions of the wave equations.

7.1 Expressions for Potentials

Consider a plane harmonic surface wave which propagates along the x-axis, and which is polarised in the (x, z)-plane. Denote again its angular frequency by ω, and its velocity (phase velocity) by c. Assume the displacement vectors in the layer, $\mathbf{u}_1$, and in the half-space, $\mathbf{u}_2$, to be of the form

$$\mathbf{u}_m = (u_m, 0, w_m), \quad m = 1, 2 , \tag{7.1}$$

where u_m and w_m are independent of coordinate y. According to the discussion in Chapter 4, this displacement field can be described by the potentials for longitudinal waves, φ_m, and transverse waves, ψ_m, in the form

$$u_m = \frac{\partial \varphi_m}{\partial x} - \frac{\partial \psi_m}{\partial z} , \quad w_m = \frac{\partial \varphi_m}{\partial z} + \frac{\partial \psi_m}{\partial x} , \tag{7.2}$$

where these potentials must satisfy the wave equations

$$\nabla^2 \varphi_m = \frac{1}{\alpha_m^2} \frac{\partial^2 \varphi_m}{\partial t^2} , \quad \nabla^2 \psi_m = \frac{1}{\beta_m^2} \frac{\partial^2 \psi_m}{\partial t^2} . \tag{7.3}$$

Let us seek the potentials in the form of plane waves which propagate in the direction of the x-axis at velocity c, and their amplitudes decrease to zero at large depths. Assume the potentials to be of the form

$$\varphi_m(x, z, t) = f_m(z) e^{i\omega(t - x/c)} , \quad \psi_m(x, z, t) = g_m(z) e^{i\omega(t - x/c)} , \tag{7.4}$$

where velocity c is the same in both media and for both potentials. By inserting these potentials into wave equations (7.3), we obtain ordinary differential equations for $f_m(z)$ and $g_m(z)$. Their general solutions, in analogy with (5.4), can be expressed as

$$f_m(z) = a_m^- e^{ikr_m z} + a_m^+ e^{-ikr_m z} , \quad g_m(z) = b_m^- e^{iks_m z} + b_m^+ e^{-iks_m z} , \qquad (7.5)$$

where $k = \omega/c$ is the wave number of the surface wave,

$$r_m = \sqrt{(c/\alpha_m)^2 - 1} , \quad s_m = \sqrt{(c/\beta_m)^2 - 1} , \qquad (7.6)$$

$a_m^-, a_m^+, b_m^-, b_m^+$ are arbitrary constants, and $m = 1, 2$. The individual terms in (7.5) could again be interpreted in terms of body waves propagating obliquely upwards and downwards; see the discussion in Section 4.5.

Since we are interested in surface waves only, we must again assume that radicals r_2 and s_2 are imaginary:

$$r_2 = ir_2^* , \quad s_2 = is_2^* , \quad r_2^* = \sqrt{1 - (c/\alpha_2)^2} , \quad s_2^* = \sqrt{1 - (c/\beta_2)^2} . \qquad (7.7)$$

Consequently, the phase velocity must satisfy the condition $c < \beta_2 < \alpha_2$, and we must put $a_2^+ = b_2^+ = 0$ in order to eliminate the exponentially increasing terms in the half-space.

For the sake of brevity of some further expressions, it will be convenient to modify the general solutions (7.5) as follows:
   a) in the layer, replace the exponentials by cosines and sines;
   b) in the half-space, replace coordinate z by $z - H$.
Consequently,

$$f_1(z) = A \cos(kr_1 z) + B \sin(kr_1 z) , \quad f_2(z) = C e^{-kr_2^*(z-H)} ,$$

$$\qquad (7.8)$$

$$g_1(z) = D \cos(ks_1 z) + E \sin(ks_1 z) , \quad g_2(z) = F e^{-ks_2^*(z-H)} .$$

It is evident that the new constants are related to the previous as:

$$A = a_1^- + a_1^+ , \quad B = i(a_1^- - a_1^+) , \quad C = a_2^- e^{-kr_2^* H} , \qquad (7.9)$$

and similarly for the constants in g_1 and g_2. Denoting

$$\xi = kr_1 z , \quad \eta = ks_1 z , \quad \zeta = kr_2^*(z - H) , \quad \chi = ks_2^*(z - H) , \qquad (7.10)$$

we arrive at the following general expressions for the potentials:

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$$\varphi_1 = \left(A\cos\xi + B\sin\xi\right)e^{i(\omega t - kx)} \ , \quad \varphi_2 = Ce^{-\zeta}e^{i(\omega t - kx)} \ ,$$

$$\psi_1 = \left(D\cos\eta + E\sin\eta\right)e^{i(\omega t - kx)} \ , \quad \psi_2 = Fe^{-\chi}e^{i(\omega t - kx)} \ . \tag{7.11}$$

7.2 Displacements and Stresses

The displacements in the layer and in the half-space are given by Eqs. (7.2) and (7.11). After performing the corresponding differentiation and omitting the common factor $e^{i(\omega t - kx)}$, we obtain

$$u_1 = -k\left[i\left(A\cos\xi + B\sin\xi\right) + s_1\left(-D\sin\eta + E\cos\eta\right)\right] \ ,$$

$$w_1 = -k\left[r_1\left(A\sin\xi - B\cos\xi\right) + i\left(D\cos\eta + E\sin\eta\right)\right] \ , \tag{7.12}$$

$$u_2 = -k\left(iCe^{-\zeta} - s_2^{*}Fe^{-\chi}\right) \ , \quad w_2 = -k\left(r_2^{*}Ce^{-\zeta} + iF^{-\chi}\right) \ .$$

These expressions also follow immediately from the analogies with (5.11), if definitions (5.8) and relations (7.9) are taken into account. In particular, the following correspondence between the expressions in (5.11) and the expressions in (7.12) for the layer exist:

$$\Phi^{-} + \Phi^{+} \to A\cos\xi + B\sin\xi \ , \quad \Phi^{-} - \Phi^{+} \to i\left(A\sin\xi - B\cos\xi\right) \ ,$$

$$\tag{7.13}$$

$$\Psi^{-} + \Psi^{+} \to D\cos\eta + E\sin\eta \ , \quad \Psi^{-} - \Psi^{+} \to i\left(D\sin\eta - E\cos\eta\right) \ ,$$

Put

$$\gamma_m = 2\left(\beta_m/c\right)^2 \quad \text{and} \quad \delta_m = \gamma_m - 1 \ . \tag{7.14}$$

The stress components can be obtain by inserting (7.10) and (7.11) into general formulae (4.16). However, we can obtain them directly from (5.13), using relations (7.13). We arrive at

$$\left(\tau_{zx}\right)_1 = \rho_1\omega^2\left[i\gamma_1 r_1\left(A\sin\xi - B\cos\xi\right) - \delta_1\left(D\cos\eta + E\sin\eta\right)\right] \ ,$$

$$\left(\tau_{zz}\right)_1 = \rho_1\omega^2\left[\delta_1\left(A\cos\xi + B\sin\xi\right) + i\gamma_1 s_1\left(D\sin\eta - E\cos\eta\right)\right] \ ,$$

$$\tag{7.15}$$

$$\left(\tau_{zx}\right)_2 = \rho_2\omega^2\left(i\gamma_2 r_2^{*}Ce^{-\zeta} - \delta_2 Fe^{-\chi}\right) \ ,$$

$$\left(\tau_{zz}\right)_2 = \rho_2\omega^2\left(\delta_2 Ce^{-\zeta} + i\gamma_2 s_2^{*}Fe^{-\chi}\right) \ .$$

7.3 Boundary Conditions

We have already considered the conditions that the amplitudes tend to zero as the depth tends to infinity, i.e.

$$f_2(z) \to 0, \quad g_2(z) \to 0 \quad \text{for} \quad z \to \infty . \tag{7.16}$$

These conditions have led to $a_2^+ = b_2^+ = 0$.

In order to determine the remaining six constants, A to F, we need six further boundary conditions. As these conditions, we shall require the stresses to be zero at the free surface, i.e.

$$\left(\tau_{zx}\right)_1 = 0, \quad \left(\tau_{zz}\right)_1 = 0 \quad \text{for} \quad z = 0 , \tag{7.17}$$

and the displacements and stresses to be continuous at the interface between the layer and the half-space, i.e.

$$u_1 = u_2 , \quad w_1 = w_2 , \quad \left(\tau_{zx}\right)_1 = \left(\tau_{zx}\right)_2 , \quad \left(\tau_{zz}\right)_1 = \left(\tau_{zz}\right)_2 \tag{7.18}$$

for $z = H$.

At the free surface, $z = 0$, we have $\xi = \eta = 0$; see (7.10). Using (7.15), boundary conditions (7.17) yield

$$i\gamma_1 r_1 B + \delta_1 D = 0, \quad \delta_1 A - i\gamma_1 s_1 E = 0 . \tag{7.19}$$

Before considering the boundary conditions for $z = H$, let us denote

$$P = k r_1 H , \quad Q = k s_1 H , \tag{7.20}$$

and take into account that $\zeta = \chi = 0$ for $z = H$. Using (7.12) and (7.15), boundary conditions (7.18) yield

$$i\left(A \cos P + B \sin P\right) - s_1\left(D \sin Q - E \cos Q\right) - iC + s_2^* F = 0 ,$$

$$r_1\left(A \sin P - B \cos P\right) + i\left(D \cos Q + E \sin Q\right) - r_2^* C - iF = 0 ,$$

$$\tag{7.21}$$

$$i\gamma_1 r_1\left(A \sin P - B \cos P\right) - \delta_1\left(D \cos Q + E \sin Q\right) - i\rho\gamma_2 r_2^* C + \rho\delta_2 F = 0 ,$$

$$\delta_1\left(A \cos P + B \sin P\right) + i\gamma_1 s_1\left(D \sin Q - E \cos Q\right) - \rho\delta_2 C - i\rho\gamma_2 s_2^* F = 0 ,$$

where $\rho = \rho_2 / \rho_1$.

7.4 Dispersion Equation

Equations (7.19) and (7.21) represent a system of six homogeneous equations in six unknown constants A to F. This system has a non-trivial solution if the corresponding determinant equals zero. This condition represents the dispersion equation for determining the unknown phase velocity c.

Many authors have formulated the dispersion equation for our problem in a similar form, i.e. as the condition that a sixth-order determinant equals zero; for a review see Ewing et al. (1957). A simpler dispersion equation in the form of a third-order determinant was used by Bolt and Butcher (1960), and by Money and Bolt (1966). We shall also attempt to reduce the number of equations before calculating the determinant.

We have replaced the usual expressions (7.5) for amplitudes by expressions (7.8), which contain trigonometric functions. This has simplified the boundary conditions at the free surface, which yielded simple equations (7.19). Consequently, one constant can easily be eliminated from each of these equations. We shall eliminate constants B and E, which stand with the sines in (7.8) and (7.11), i.e.

$$B = i\frac{G}{r_1}D\,, \quad E = -i\frac{G}{s_1}A\,, \tag{7.22}$$

where $G = \delta_1/\gamma_1$. Using these relations, boundary conditions (7.21) can be expressed as

$$iA\left(\cos P - G\cos Q\right) - D\left(Gr_1^{-1}\sin P + s_1\sin Q\right) - iC + s_2^*F = 0\,,$$

$$A\left(r_1\sin P + Gs_1^{-1}\sin Q\right) - iD\left(G\cos P - \cos Q\right) - r_2^*C - iF = 0\,,$$

$$\tag{7.23}$$

$$iA\left(\gamma_1 r_1\sin P + \delta_1 Gs_1^{-1}\sin Q\right) + D\delta_1\left(\cos P - \cos Q\right) - i\rho\gamma_2 r_2^*C + \rho\delta_2 F = 0\,,$$

$$A\delta_1\left(\cos P - \cos Q\right) + iD\left(\delta_1 Gr_1^{-1}\sin P + \gamma_1 s_1\sin Q\right) - \rho\delta_2 C - i\rho\gamma_2 s_2^*F = 0\,.$$

In this way, we have reduced the problem to four equations. Multiply the second of these equations by imaginary unit i, the fourth equation by $(-i)$, and introduce new unknowns $\tilde{A} = iA$ and $\tilde{C} = -iC$. The system of equations for the unknowns $\tilde{A}, D, \tilde{C}$ and F will have a non-trivial solution if the corresponding determinant is equal to zero, i.e.

$$\tag{7.24}$$

$$\begin{vmatrix} \cos P - G\cos Q & -Gr_1^{-1}\sin P - s_1\sin Q & 1 & s_2^* \\ r_1\sin P + Gs_1^{-1}\sin Q & G\cos P - \cos Q & r_2^* & 1 \\ \gamma_1 r_1\sin P + \delta_1 Gs_1^{-1}\sin Q & \delta_1\left(\cos P - \cos Q\right) & \rho\gamma_2 r_2^* & \rho\delta_2 \\ -\delta_1\left(\cos P - \cos Q\right) & \delta_1 Gr_1^{-1}\sin P + \gamma_1 s_1\sin Q & -\rho\delta_2 & -\rho\gamma_2 s_2^* \end{vmatrix} = 0\,.$$

The quantities for the layer appear in the first two columns of the determinant, and the quantities for the half-space in the remaining two columns. It will, therefore, be convenient to develop this determinant, according to Laplace's theorem, in terms of the minors from the first two columns. The corresponding cofactors will be the minors from the remaining two columns with the appropriate signs.

Consider the elements of the matrix in (7.24), and denote the element in the i-th row and j-th column by a_{ij}. Denote the minor of the second order, which is constructed from the element in the i-th and j-th rows and the k-th and l-th columns, by

$$a\begin{vmatrix} ij \\ kl \end{vmatrix} = a_{ik}a_{jl} - a_{il}a_{jk} \ ; \tag{7.25}$$

see Dunkin (1965) and Fuchs (1968).

In constructing the minors, let us consider the ordinal numbers of the rows, i.e. the pairs (i, j), in the following order:

$$12, \quad 13, \quad 14, \quad 23, \quad 24, \quad 34. \tag{7.26}$$

Therefore, there are six minors of the second order which can be constructed from two columns of a fourth-order matrix.

According to Laplace's theorem, Eq. (7.24) can be expressed as

$$L_1 H_1 + L_2 H_2 + L_3 H_3 + L_4 H_4 + L_5 H_5 + L_6 H_6 = 0 \ , \tag{7.27}$$

where L_1 to L_6 are the second-order minors constructed from the first two columns of (7.24), containing the parameters of the layer, and H_1 to H_6 are the corresponding cofactors, constructed from the third and fourth columns and containing the parameters of the half-space.

The layer minors can be expressed as

$$L_1 = a\begin{vmatrix} 12 \\ 12 \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} =$$

$$= 2G - \left(1 + G^2\right)\cos P \cos Q + r_1 \sin P \ s_1 \sin Q + G^2 r_1^{-1} \sin P \ s_1^{-1} \sin Q \ ,$$

$$L_2 = a\begin{vmatrix} 13 \\ 12 \end{vmatrix} =$$

$$= \delta_1 \left[(1 + G)(1 - \cos P \cos Q) + G^2 r_1^{-1} \sin P \ s_1^{-1} \sin Q \right] + \gamma_1 r_1 \sin P \ s_1 \sin Q \ ,$$

$$L_3 = a\begin{vmatrix} 14 \\ 12 \end{vmatrix} = \cos P \ s_1 \sin Q + G^2 \cos Q \ r_1^{-1} \sin P \ , \tag{7.28}$$

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$$L_4 = a\begin{vmatrix}23\\12\end{vmatrix} = G^2 \cos P \ s_1^{-1} \sin Q + \cos Q \ r_1 \sin P \ ,$$

$$L_5 = a\begin{vmatrix}24\\12\end{vmatrix} = L_2 \ ,$$

$$L_6 = a\begin{vmatrix}34\\12\end{vmatrix} =$$

$$= \delta_1^2\left[2(1 - \cos P \cos Q) + G^2 r_1^{-1} \sin P \ s_1^{-1} \sin Q\right] + \gamma_1^2 r_1 \sin P \ s_1 \sin Q \ .$$

The corresponding cofactors are

$$H_1 = a\begin{vmatrix}34\\34\end{vmatrix} = \rho^2\left(\delta_2^2 - \gamma_2^2 r_2^* s_2^*\right) \ , \qquad H_2 = -a\begin{vmatrix}24\\34\end{vmatrix} = -\rho\left(\delta_2 - \gamma_2 r_2^* s_2^*\right) \ ,$$

$$H_3 = a\begin{vmatrix}23\\34\end{vmatrix} = -\rho r_2^* \ , \qquad H_4 = a\begin{vmatrix}14\\34\end{vmatrix} = -\rho s_2^* \ , \qquad (7.29)$$

$$H_5 = -a\begin{vmatrix}13\\34\end{vmatrix} = H_2 \ , \qquad H_6 = a\begin{vmatrix}12\\34\end{vmatrix} = 1 - r_2^* s_2^* \ .$$

Note that in our case, minor $a\begin{vmatrix}ij\\kl\end{vmatrix}$ must be multiplied by $(-1)^{i+j+k+l}$ in order to obtain the corresponding cofactor. For example,

$$H_1 = (-1)^{3+4+3+4} a\begin{vmatrix}34\\34\end{vmatrix} = a\begin{vmatrix}34\\34\end{vmatrix} \ , \quad H_2 = (-1)^{2+4+3+4} a\begin{vmatrix}24\\34\end{vmatrix} = -a\begin{vmatrix}24\\34\end{vmatrix} \ .$$

Finally, let us mention two properties of Eqs. (7.27) to (7.29) which have a more general validity:

Firstly, since we have found that $L_5 = L_2$ and $H_5 = H_2$, dispersion equation (7.27) can be simplified to read

$$L_1 H_1 + 2 L_2 H_2 + L_3 H_3 + L_4 H_4 + L_6 H_6 = 0 \ . \qquad (7.30)$$

A similar property appears also in the general case of Rayleigh waves in a multilayered medium. This enables us to replace the so-called δ-matrices of the sixth order by the reduced δ-matrices of the fifth order; see Chapter 9.

Secondly, in calculating the minors we have also obtained squares $\cos^2 P$, $\sin^2 P$, $\cos^2 Q$, $\sin^2 Q$, but always in the combinations

$$\cos^2 P + \sin^2 P = 1, \quad \cos^2 Q + \sin^2 Q = 1,$$

so that these squares are not contained in the final formulae (7.28). The δ-matrices, mentioned above, do not contain these squares either. This property leads not only to the simplification of the formulae for L_1 to L_6, but is very important from the computational point of view. Namely, at high frequencies, P and Q become imaginary since $c < \beta_1 < \alpha_1$. Then $\cos P = \cos(iP^*) = \cosh P^*$ and $\sin(iP^*) = i \sinh P^*$, and

$$\cos^2 P + \sin^2 P = \cosh^2 P^* - \sinh^2 P^*.$$

For large P^*, the last expression contains the subtraction of large exponential terms, which cancel out analytically, but cause a loss of significant figures when evaluating this term numerically. For example, if determinant (7.24) were computed routinely by being developed in terms of the elements of the first row or first column, these quadratic terms would be present in the development and their subtraction would lead to a loss of accuracy at high frequencies. Many previous formulations of the dispersion equation for Rayleigh waves contain this drawback; see the review by Ewing et al. (1957). The solution in the form of Eqs. (7.27) to (7.29) removes this loss-of-accuracy problem. We encounter the same situation also in the case of Rayleigh waves in a multilayered medium; see Chapter 9.

7.5 Another Form of the Dispersion Equation

Without derivation we shall present another formulation of the corresponding dispersion equation, which is close to that used by Novotny et al. (1996). The formulation derived in the preceding section is a little simpler, but the formulation given below is more convenient for the derivation of analytical formulae for the group velocity.

We consider the same model and notations as at the beginning of this chapter. Denote again by $\rho = \rho_2/\rho_1$ the ratio of densities, and by $k = \omega/c$ the wave number of the surface wave. It can be verified that the dispersion equation (7.35), given below, can be obtained from Eq. (7.30) by multiplying the latter equation by factor $\left[-\omega^8 \beta_1^4 / \left(4c^4 \rho\right)\right]$.

Introduce the following quadratic quantities:

$$\Omega = \omega^2, \quad K = k^2, \quad A_1 = \alpha_1^2, \quad A_2 = \alpha_2^2, \quad B_1 = \beta_1^2, \quad B_2 = \beta_2^2. \tag{7.31}$$

Denote further

$$r_1 = \sqrt{\Omega/A_1 - K}, \quad r_2^* = \sqrt{K - \Omega/A_2}, \quad P = Hr_1,$$

$$s_1 = \sqrt{\Omega/B_1 - K}\,, \quad s_2^* = \sqrt{K - \Omega/B_2}\,, \quad Q = Hs_1\,,$$

$$\gamma_1 = B_1 K\,, \quad \gamma_2 = B_2 K\,, \quad \delta_1 = B_1^2 K\,, \quad \delta_2 = B_2^2 K\,,$$

$$\gamma_{11} = \gamma_1 - \Omega/2\,, \quad \gamma_{21} = \gamma_2 - \Omega/2\,, \quad \sigma = r_2^* s_2^*\,, \tag{7.32}$$

$$p_1 = \cos P\,, \quad p_2 = r_1 \sin P\,, \quad p_3 = (\sin P)/r_1\,,$$

$$q_1 = \cos Q\,, \quad q_2 = s_1 \sin Q\,, \quad q_3 = (\sin Q)/s_1\,,$$

$$w_1 = 1 - p_1 q_1\,, \quad w_2 = p_2 q_2\,, \quad w_3 = p_3 q_3\,.$$

Introduce the following expressions for the half-space,

$$h_1 = \rho\left(\gamma_{21}^2 - \delta_2 \sigma\right)\,, \quad h_2 = r_2^* \Omega^2/4\,, \quad h_3 = \gamma_{21} - B_2 \sigma\,,$$

$$h_4 = s_2^* \Omega^2/4\,, \qquad h_5 = (\sigma - K)/\rho\,, \tag{7.33}$$

and for the layer,

$$y_1 = \Omega^2/4 - \left(\gamma_1^2 + \gamma_{11}^2\right)w_1 - \delta_1 w_2 - \gamma_{11}^2 K w_3\,, \quad y_2 = \delta_1 p_1 q_2 + \gamma_{11}^2 q_1 p_3\,,$$

$$y_3 = \gamma_1 \gamma_{11}\left(\gamma_1 + \gamma_{11}\right)w_1 + \gamma_1 \delta_1 w_2 + \gamma_{11}^3 K w_3\,, \tag{7.34}$$

$$y_4 = \gamma_{11}^2 p_1 q_3 + \delta_1 q_1 p_2\,, \quad y_5 = 2\delta_1 \gamma_{11}^2 w_1 + \delta_1^2 w_2 + \gamma_{11}^4 w_3\,.$$

The dispersion equation for Rayleigh waves in a layer on a half-space can then be expressed as

$$h_1 y_1 + h_2 y_2 + 2 h_3 y_3 + h_4 y_4 + h_5 y_5 = 0\,. \tag{7.35}$$

Since phase velocity c may attain values both higher and lower than the layer velocities α_1 and β_1, quantities r_1 and s_1 may be real or pure imaginary. This fact must be taken into account when writing the computer programme. For pure imaginary values of r_1 or s_1, the trigonometric functions in (7.32) will be replaced by the corresponding hyperbolic functions. Moreover, the quantity p_3 becomes indeterminate if r_1 is close to zero. The same applies to q_3 when s_1 is close to zero. The corresponding expansions should be used in these cases.

Chapter 8

Matrix Methods for Love Waves in a Layered Medium

Matrix methods were used in the theory of elasticity, first in solving static problems; see the review by Pestel and Leckie (1963). Later on, Thomson (1950) and Haskell (1953) introduced these methods into seismology. In this chapter we shall derive a modification of the Thomson-Haskell matrices for Love waves.

8.1 Model of the Medium

Consider a medium consisting of $n-1$ homogeneous and isotropic layers with plane-parallel interfaces lying on a homogeneous and isotropic half-space (Fig. 8.1). Denote the half-space as the n-th layer. All the layers and the half-space are assumed to be perfectly elastic. Assume the surface of the medium to be free, and a welded contact to exist at the individual interfaces.
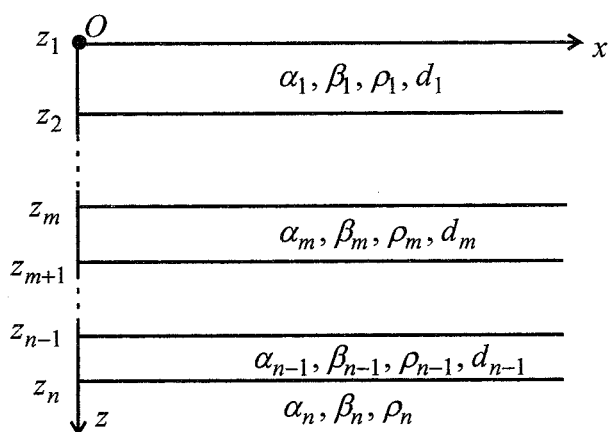
Fig. 8.1. Model of a layered medium.

In the m-th layer, denote by α_m the compressional wave velocity, β_m the shear wave velocity, ρ_m the density and d_m the thickness of the layer. Denote by α_n, β_n and ρ_n the corresponding parameters in the half-space. Denote the depths of the individual interfaces by $z_1 = 0$ (free surface), $z_2, \ldots, z_n$. Thus the m-th layer has the upper boundary at depth z_m, and the lower boundary at depth z_{m+1}. It is evident that $d_m = z_{m+1} - z_m$. Assume that the velocities in the individual layers are lower than the corresponding velocity in the half-space, i.e. $\alpha_m < \alpha_n$, $\beta_m < \beta_n$ for $m = 1, 2, \ldots, n-1$.

Note that longitudinal wave velocities have no influence on Love waves, so that parameters α_1 to α_n will not appear in the formulae of this chapter. However, we shall need them in the next chapter in studying Rayleigh waves.

8.2 Matrix for One Layer

Consider a plane harmonic wave which propagates in the direction of the x-axis, which is polarised along the y-axis, and its amplitude decreases to zero at infinite depth. This surface wave, with transverse polarisation in the horizontal plane, will again be called the harmonic Love wave.

Hence, let us consider the displacement vector in the m-th layer $(m = 1, 2, \ldots, n)$ in the form

$$\mathbf{u} = (0, v_m, 0) , \tag{8.1}$$

where

$$v_m = f_m(z)e^{i\omega(t-x/c)} , \tag{8.2}$$

ω being the angular frequency, c the velocity of the surface wave along the x-axis, and $f_m(z)$ the depth-dependent amplitude. We assume that ω is given, but c and $f_m(z)$ are unknown.

As mentioned in Section 6.5, simpler analytical formulae for the group velocity can be derived if the wave number is used as the unknown function instead of the phase velocity. Therefore, as opposed to the explanation in Chapter 6, we shall formulate the theory in this chapter in terms of the wave number (although we shall not deal with the group velocity). Thus, express displacements (8.2) as

$$v_m = f_m(z)e^{i(\omega t - kx)} , \tag{8.3}$$

where $k = \omega/c$ is the wave number of the surface wave. These shear displacements must satisfy the wave equations for shear waves

$$\nabla^2 v_m = \frac{1}{\beta_m^2} \frac{\partial^2 v_m}{\partial t^2} , \tag{8.4}$$

that is

$$\frac{\partial^2 v_m}{\partial x^2} + \frac{\partial^2 v_m}{\partial y^2} + \frac{\partial^2 v_m}{\partial z^2} = \frac{1}{\beta_m^2} \frac{\partial^2 v_m}{\partial t^2} ; \tag{8.5}$$

see Chapter 4. By inserting (8.3) into this equation, one gets the following ordinary differential equation for $f_m(z)$, which is analogous to Eqs. (6.6):

$$f_m''(z) + \left(\frac{\omega^2}{\beta_m^2} - k^2 \right) f_m(z) = 0 . \tag{8.6}$$

Its general solution can be expressed in any of the following forms

$$f_m(z) = C_m e^{i s_m z} + D_m e^{-i s_m z} = \qquad (8.7a)$$

$$= E_m e^{i s_m (z - z_m)} + F_m e^{-i s_m (z - z_m)} = \qquad (8.7b)$$

$$= G_m \cos\left[s_m (z - z_m) \right] + H_m \sin\left[s_m (z - z_m) \right] , \qquad (8.7c)$$

where

$$s_m = \begin{cases} \sqrt{\left(\omega / \beta_m \right)^2 - k^2} & \text{if} \quad k \le \dfrac{\omega}{\beta_m}, \quad \text{i.e.} \quad c \ge \beta_m, \\[2mm] i\sqrt{k^2 - \left(\omega / \beta_m \right)^2} & \text{if} \quad k > \dfrac{\omega}{\beta_m}, \quad \text{i.e.} \quad c < \beta_m. \end{cases} \qquad (8.8)$$

Moreover, introduce

$$Q_m = d_m s_m . \qquad (8.9)$$

Note that quantities s_m defined by (8.8) and by (6.8) differ by factor k.

The individual general forms of (8.7) differ only in the choice of the arbitrary constants. For example,

$$C_m = E_m \exp\left(-i s_m z_m \right), \quad G_m = E_m + F_m , \quad \text{etc.}$$

The general form (8.7a) is simple, but not convenient in formulating boundary conditions. For this purpose, it is convenient to shift the coordinate origin always to the top of the particular layer (Haskell, 1953). Fuchs (1968) achieved the same effect by using the general solutions in form (8.7b) and keeping the coordinate origin fixed at the free surface. Although this form of the general solutions is frequently used in the literature, we shall use it only in the half-space, i.e. for $m = n$. In the layers we shall use the trigonometric form (8.7c).

The constants in (8.7) are different in the individual layers, and must be determined from boundary conditions. Thus, these constants play the role of auxiliary parameters which must be determined (eliminated) by these conditions. In the matrix methods we try to eliminate these auxiliary parameters at the very beginning, and to express them in terms of the "physical" quantities which are used in boundary conditions (which are continuous at the interfaces). This simplifies the formulation of the boundary conditions and the derivation of the dispersion equation. In our case, we shall replace these constants by the displacement and stress at the upper boundary of the corresponding layer.

Since the expressions for displacements v_m and stresses $\left(\tau_{zy} \right)_m$ in all layers and in the half-space contain the same exponential term $\exp\left[i(\omega t - kx) \right]$, we shall omit this term hereafter. Consequently,

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$$v_m = G_m \cos\!\left[s_m\!\left(z - z_m\right)\right] + H_m \sin\!\left[s_m\!\left(z - z_m\right)\right] ,$$

$$(8.10)$$

$$\tau_m = \mu_m \frac{\partial v_m}{\partial z} = \mu_m s_m \left\{ -G_m \sin\!\left[s_m\!\left(z - z_m\right)\right] + H_m \cos\!\left[s_m\!\left(z - z_m\right)\right] \right\} ,$$

where we have simply put τ_m instead of $\left(\tau_{zy}\right)_m$.

Now we are seeking the relation between the displacement and stress at the upper and lower boundaries of the m-th layer. At the upper boundary $\left(z = z_m\right)$,

$$\left(v_m\right)_{z_m} = G_m , \quad \left(\tau_m\right)_{z_m} = \mu_m s_m H_m .$$

$$(8.11)$$

At the lower boundary $\left(z = z_{m+1}\right)$,

$$\left(v_m\right)_{z_{m+1}} = G_m \cos Q_m + H_m \sin Q_m ,$$

$$(8.12)$$

$$\left(\tau_m\right)_{z_{m+1}} = \mu_m s_m \left[-G_m \sin Q_m + H_m \cos Q_m \right] .$$

By expressing G_m and H_m from (8.11) and inserting them into (8.12), we arrive at

$$\left(v_m\right)_{z_{m+1}} = \left(v_m\right)_{z_m} \cos Q_m + \frac{\left(\tau_m\right)_{z_m}}{\mu_m s_m} \sin Q_m ,$$

$$(8.13)$$

$$\left(\tau_m\right)_{z_{m+1}} = \left(v_m\right)_{z_m}\left(-\mu_m s_m \sin Q_m\right) + \left(\tau_m\right)_{z_m} \cos Q_m .$$

We shall rearrange these relations in matrix form, as is usual in linear algebra. Let us remind the reader that the product of a matrix **b** with a column vector **a** yields a column vector **c**, the elements of which are the scalar products of the rows of matrix **b** with vector **a**, i.e.

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} ,$$

where

$$c_1 = b_{11} a_1 + b_{12} a_2 , \quad c_2 = b_{21} a_1 + b_{22} a_2 .$$

Relations (8.13) may then be expressed in the following matrix form:

$$\begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_{m+1}} = \begin{pmatrix} \cos Q_m , & \dfrac{1}{\mu_m s_m} \sin Q_m \\ -\mu_m s_m \sin Q_m , & \cos Q_m \end{pmatrix} \begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_m} .$$

$$(8.14)$$

The column vector

$$\begin{pmatrix} v_m \\ \tau_m \end{pmatrix} , \tag{8.15}$$

the elements of which are the displacement and stress, will be called the displacement-stress vector. Denoting the corresponding matrix by

$$\mathbf{a}_m = \begin{pmatrix} \cos Q_m , & \dfrac{1}{\mu_m s_m} \sin Q_m \\ -\mu_m s_m \sin Q_m , & \cos Q_m \end{pmatrix} , \tag{8.16}$$

relation (8.14) may be expressed as

$$\begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_{m+1}} = \mathbf{a}_m \begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_m} . \tag{8.17}$$

This expresses the required relation between the displacement and stress at the upper and lower boundaries of the corresponding layer. The quantities at the lower boundary are obtained from the quantities at the upper boundary by multiplying by matrix $\mathbf{a}_m$.

Two properties of matrix $\mathbf{a}_m$ should be noted:

• this matrix is symmetric with respect to the secondary diagonal, i.e. $(a_m)_{22} = (a_m)_{11}$;

• its determinant is equal to unity, $\det \mathbf{a}_m = 1$.

We shall also use the inverse relation to (8.17). According to the general rule, for a second-order matrix

$$\mathbf{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} , \tag{8.18}$$

its inverse matrix is

$$\mathbf{m}^{-1} = \dfrac{1}{\det \mathbf{m}} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} . \tag{8.19}$$

The reader can check that the product of matrices (8.18) and (8.19) indeed yields the unit matrix.

By applying this rule to matrix $\mathbf{a}_m$, we arrive at

$$\begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_m} = \mathbf{b}_m \begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_{m+1}} , \tag{8.20}$$

where the inverse matrix $\mathbf{b}_m = \mathbf{a}_m^{-1}$ is

$$\mathbf{b}_m = \begin{pmatrix} \cos Q_m \, , & -\dfrac{1}{\mu_m s_m} \sin Q_m \\ \mu_m s_m \sin Q_m \, , & \cos Q_m \end{pmatrix} . \qquad (8.21)$$

The striking similarity between matrix $\mathbf{a}_m$ and its inverse $\mathbf{b}_m$ has deeper physical causes. Namely, the transition to the inverse relation for the displacements and stresses actually means a change in the orientation of the z-axis. In this case, thickness d_m must be replaced by $(-d_m)$ and Q_m by $(-Q_m)$. Consequently, the cosines in the principal diagonal of $\mathbf{a}_m$ will not change, and the sines in the secondary diagonal will change their signs.

8.3 Matrix for a Stack of Layers

The condition of the continuity of the displacement and stress at the interface of the m-th and $(m-1)$-st layers may be expressed in vector form as

$$\begin{pmatrix} v_m \\ \tau_m \end{pmatrix}_{z_m} = \begin{pmatrix} v_{m-1} \\ \tau_{m-1} \end{pmatrix}_{z_m} . \qquad (8.22)$$

By alternately using (8.22) and (8.17), we can express the relation between the displacement and stress at the boundary of the half-space and at the surface:

$$\begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} = \begin{pmatrix} v_{n-1} \\ \tau_{n-1} \end{pmatrix}_{z_n} = \mathbf{a}_{n-1} \begin{pmatrix} v_{n-1} \\ \tau_{n-1} \end{pmatrix}_{z_{n-1}} = \ldots = \mathbf{a}_{n-1} \ldots \mathbf{a}_1 \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} . \qquad (8.23)$$

Introducing the product of the matrices

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{a}_{n-1} \ldots \mathbf{a}_2 \mathbf{a}_1 \, , \qquad (8.24)$$

one gets

$$\begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} = \begin{pmatrix} v_{n-1} \\ \tau_{n-1} \end{pmatrix}_{z_n} = \mathbf{A} \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} . \qquad (8.25)$$

Hence, the matrix for the stack of layers is obtained as the product of the matrices for the individual layers.

For the inverse relation we obtain

$$\begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} = \mathbf{B} \begin{pmatrix} v_{n-1} \\ \tau_{n-1} \end{pmatrix}_{z_n} = \mathbf{B} \begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} , \qquad (8.26)$$

where

$$\mathbf{B} = \mathbf{b}_1 \mathbf{b}_2 \ldots \mathbf{b}_{n-1} . \qquad (8.27)$$

8.4 Expressions for the Half-Space

For $c > \beta_n$, amplitude $f_n(z)$ in the half-space would be an oscillating function of depth z. Such a wave would not have the character of a surface wave. Therefore, we shall assume that $c < \beta_n$. The corresponding radical (8.8) then becomes

$$s_n = i s_n^*, \quad s_n^* = \sqrt{k^2 - \left(\omega/\beta_n\right)^2} . \tag{8.28}$$

It then follows from (8.7b) and (8.3) that the displacement in the half-space, if term $\exp\left[i(\omega t - kx)\right]$ is again omitted, takes the form

$$v_n = E_n e^{-s_n^*(z - z_n)} + F_n e^{s_n^*(z - z_n)} . \tag{8.29}$$

The first exponential term in this expression tends to zero for $z \to \infty$, but the second term tends to infinity. Therefore, we must put $F_n = 0$, so that

$$v_n = E_n e^{-s_n^*(z - z_n)} . \tag{8.30}$$

The stress in the half-space is then

$$\tau_n = \mu_n \frac{\partial v_n}{\partial z} = -\mu_n s_n^* E_n e^{-s_n^*(z - z_n)} . \tag{8.31}$$

In particular, at the top of the half-space $\left(z = z_n\right)$,

$$\begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} = \begin{pmatrix} E_n \\ -\mu_n s_n^* E_n \end{pmatrix} . \tag{8.32}$$

8.5 Dispersion Equation

8.5.1 Traditional formulation

By inserting (8.32) into (8.23), one gets

$$\begin{pmatrix} E_n \\ -\mu_n s_n^* E_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} . \tag{8.33}$$

Finally, using the boundary condition at the free surface, $\left(\tau_1\right)_{z_1} = 0$, the latter equation may be expressed as the system of equations

$$E_n - A_{11}(v_1)_{z_1} = 0, \quad -\mu_n s_n^* E_n - A_{21}(v_1)_{z_1} = 0 \qquad (8.34)$$

for the unknown quantities E_n and $(v_1)_{z_1}$. This system of homogeneous equations has a non-trivial solution if the corresponding determinant is equal to zero, i.e.

$$\boxed{-A_{21} - \mu_n s_n^* A_{11} = 0} . \qquad (8.35)$$

This is the required dispersion equation for Love waves in a layered medium. For a given model of the medium and a given value of angular velocity ω, this equation serves to compute wave number k or phase velocity c of Love waves. Since this is a transcendental equation, some numerical method must be used to solve it; see Section 8.6.

By inserting for s_n^*, dispersion equation (8.35) takes the form

$$-A_{21} - \mu_n \sqrt{k^2 - (\omega/\beta_n)^2} A_{11} = 0 . \qquad (8.36)$$

We keep the negative signs on the left-hand side of the dispersion equation, so that the left-hand side is negative for the values of c below the fundamental mode, and positive between the fundamental mode and the first higher mode, etc.

Note that the left-hand side of the dispersion equation is usually called the *dispersion function*. In our case, the dispersion function is

$$f = -A_{21} - \mu_n s_n^* A_{11} . \qquad (8.37)$$

As a special case, let us derive the dispersion equation for Love waves propagating in one layer on a half-space, i.e. let us consider the special case of $n = 2$. In this case, matrix $\mathbf{A}$ is equal to the matrix for the layer, $\mathbf{A} = \mathbf{a}_1$. Consequently, it follows from matrix (8.16) that

$$A_{11} = (a_1)_{11} = \cos Q_1 , \quad A_{21} = (a_1)_{21} = -\mu_1 s_1 \sin Q_1 . \qquad (8.38)$$

Dispersion equation (8.35) then takes the form

$$\mu_1 s_1 \sin Q_1 - \mu_2 s_2^* \cos Q_1 = 0 , \qquad (8.39)$$

which agrees (after dividing by ω^2) with the dispersion equation which we have derived in Chapter 6.

8.5.2 Formulation in terms of the inverse matrices

If the dispersion equation is satisfied, the system of homogeneous equations (8.34) has an infinite number of solutions. We can chose one unknown arbitrarily, and calculate the other unknown.

Therefore, let us normalise the solution by putting the amplitude at the top of the half-space to unity, i.e. $E_n = 1$. By inserting this value into (8.32) and (8.26), we obtain

$$\begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 1 \\ -\mu_n s_n^* \end{pmatrix} , \tag{8.40}$$

where matrix $\mathbf{B}$ is given by (8.27) and (8.21). The boundary condition at the free surface, i.e. $\left(\tau_1\right)_{z_1} = 0$, then immediately yields the dispersion equation in the form

$$B_{21} - \mu_n s_n^* B_{22} = 0 . \tag{8.41}$$

This formulation of the dispersion equation has the following advantages:
1) Its derivation is simple, no determinant is computed. The dispersion equation is simply the condition of zero stress at the surface.
2) The matrix multiplication from the bottom to the surface is more stable numerically in computing the amplitudes at higher frequencies (when $c < \beta_m$ in some deeper layers). We shall not prove this here. This property may be important in computing energy integrals and synthetic seismograms of Love waves.

There is, of course, a close relationship between dispersion equations (8.41) and (8.35). Since matrix $\mathbf{B}$ is inverse to $\mathbf{A}$, it holds that

$$\mathbf{B} = \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{vmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{vmatrix} ;$$

see (8.18) and (8.19). However, the determinant of a matrix product is equal to the product of the individual determinants, so that

$$\det \mathbf{A} = \left(\det \mathbf{a}_{n-1}\right) \dots \left(\det \mathbf{a}_2\right)\left(\det \mathbf{a}_1\right) .$$

Since all $\det \mathbf{a}_m = 1$, we also get $\det \mathbf{A} = 1$. Consequently,

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} . \tag{8.42}$$

By putting $B_{21} = -A_{21}$ and $B_{22} = A_{11}$ into Eq. (8.41), we immediately arrive at Eq. (8.35). We have thus proved that these two dispersion equations are

analytically equivalent. Nevertheless, they may differ in some numerical properties.

8.6 Comments on Some Numerical Problems

For a given model of the medium and a given frequency, we may start the computation of the dispersion curve for the fundamental mode by choosing a small value of c, e.g., a value which is slightly above the minimum shear velocity in the model (usually slightly above β_1). For the selected value of c we compute the quantities k, s_m, Q_m and matrix $\mathbf{a}_m$ in each layer, and the matrix product $\mathbf{a}_m \mathbf{a}_{m-1} \ldots \mathbf{a}_1$ in a loop from the first to the $(n-1)$-st layer. The value of the dispersion function f is then given by (8.37). If f is negative, we increase the value of c by a given step, and repeat the calculation until the dispersion function changes sign. We then refine the position of the root, e.g., by halving the intervals, or by another numerical method, until the required accuracy is achieved. The value of the phase velocity, found in this way, is then used as the starting value for the next frequency, etc. When the dispersion curve for the fundamental mode is determined, we can proceed to computing the first higher mode, etc.

Two situations must be distinguished in computing the elements of the layer matrix $\mathbf{a}_m$. First, if $c \geq \beta_m$ $\left(k \leq \omega / \beta_m\right)$, then the elements contain trigonometric functions, see (8.16). Second, for $c < \beta_m$, quantity s_m becomes purely imaginary, and the trigonometric functions must be replaced by hyperbolic functions according to the formulae $\cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$. Note that the elements of matrix $\mathbf{a}_m$ remain real even in this case, because function $\sin Q_m$ occurs there in the combinations $\left(\sin Q_m\right) / s_m$ and $s_m \sin Q_m$.

Moreover, for small absolute values of s_m, expression $\left(\sin Q_m\right) / s_m$ must be replaced by its Taylor series.

If $c < \beta_m$ and the frequency or thickness d_m are large, overflow may occur in computing the hyperbolic functions. Several approaches have been proposed to avoid this problem. A simple method consists in reducing the matrix by dividing it by a large exponential term. This reduction changes the value of the dispersion function, but not its sign and the position of its root.

The listing of the corresponding computer subroutine can be found in Proskuryakova et al. (1981).

We have described the calculations which are based on matrices $\mathbf{a}_m$. Inverse matrices $\mathbf{b}_m$ can be used in an analogous way.

8.7 Other Forms of the Dispersion Equation. Thomson-Haskell Matrices

In the original paper by Haskell (1953), the layer matrix is introduced by the relation

$$\begin{pmatrix} \dot{v}_m/c \\ \tau_m \end{pmatrix}_{z_{m+1}} = \tilde{\mathbf{a}}_m \begin{pmatrix} \dot{v}_m/c \\ \tau_m \end{pmatrix}_{z_m} , \qquad (8.43)$$

where $\dot{v}_m$ is the velocity of the displacement, i.e. $\dot{v}_m = \partial v_m/\partial t = i\omega v_m$. Multiplying the first of Eqs. (8.13) by ik, and introducing the dimensionless quantities

$$r_{\beta m} = \frac{s_m}{k} = \begin{cases} \sqrt{(c/\beta_m)^2 - 1} & \text{for} \quad c \geq \beta_m , \\[2ex] i\sqrt{1 - (c/\beta_m)^2} & \text{for} \quad c < \beta_m , \end{cases}$$

we arrive at the Thomson-Haskell matrix,

$$\tilde{\mathbf{a}}_m = \begin{pmatrix} \cos Q_m , & \dfrac{i}{\mu_m r_{\beta m}} \sin Q_m \\[2ex] i\mu_m r_{\beta m} Q_m , & \cos Q_m \end{pmatrix} . \qquad (8.44)$$

The dispersion equation then takes the form

$$\tilde{A}_{21} - i\mu_n \sqrt{1 - (c/\beta_n)^2}\, \tilde{A}_{11} = 0 , \qquad (8.45)$$

where

$$\tilde{\mathbf{A}} = \tilde{\mathbf{a}}_{n-1} \ldots \tilde{\mathbf{a}}_2 \tilde{\mathbf{a}}_1 . \qquad (8.46)$$

Matrix $\tilde{\mathbf{a}}_m$, as opposed to matrix $\mathbf{a}_m$, is more complicated, because it contains both real and purely imaginary elements. This can be expressed schematically as

$$\tilde{a}_m = \begin{pmatrix} R & I \\ I & R \end{pmatrix} .$$

It can easily be verified that the product of matrices of this type again yields a matrix of the same type (Haskell, 1953). Consequently, all computations based on the Thomson-Haskell matrices can be performed in real arithmetic only (the imaginary units may be ignored), but some formulae for the matrix multiplication must be modified.

The appearance of the purely imaginary elements in the Thomson-Haskell matrices is surprising, but has a very simple physical explanation (Novotny, 1973). Namely, displacement v_m and stress τ_m are functions with the same

phase, which yields real matrix $\mathbf{a}_m$. However, velocity $\dot{v}_m$ is shifted in phase by 90° with respect to v_m and τ_m, which represents the multiplication by the imaginary unit. Consequently, purely imaginary elements appear in the corresponding matrix.

Knopoff (1964) used another approach. He expressed all the boundary conditions as one system of equations, and required the corresponding large determinant to be equal to zero. By applying Laplace's theorem, he expressed this determinant in terms of matrices of the second order. However, the matrices for odd and even layers have different forms, which is the disadvantage of this method. The physical interpretation of Knopoff's matrices in terms of modified displacement-stress vectors was given by Novotny (1974).

Chapter 9

Matrix Methods for Rayleigh Waves in a Layered Medium

Following the main ideas from the preceding chapter, in this chapter we shall derive the basic matrices for Rayleigh waves. These matrices will be more complicated than those for Love waves, because we must now consider four boundary conditions at each interface.

We shall consider the same model of the medium as in Section 8.1. However, we shall not formulate the problem for the unknown wave number, k, but we shall return to the usual formulation in terms of the phase velocity, c; see Proskuryakova et al. (1981). In fact, we shall generalise the formulae for Rayleigh waves in a single surface layer (Chapter 7) to a multilayered medium. The reformulation of the method for the unknown wave number will be the subject of a future study.

9.1 Basic Notations and Formulae

Consider a plane harmonic Rayleigh wave of angular velocity ω, propagating in a layered medium in the direction of the x-axis at velocity c. Denote by $k = \omega/c$ the corresponding wave number, and by $\mathbf{u}_m = (u_m, 0, w_m)$ the displacement vectors in the m-th layer, $m = 1, 2, \ldots, n$.

In analogy with (7.6) and (7.7), introduce the quantities

$$r_m = \sqrt{(c/\alpha_m)^2 - 1}, \quad s_m = \sqrt{(c/\beta_m)^2 - 1} \tag{9.1}$$

in the layers, i.e. for $m = 1, 2, \ldots, n-1$, and

$$r_n^* = \sqrt{1 - (c/\alpha_n)^2}, \quad s_n^* = \sqrt{1 - (c/\beta_n)^2} \tag{9.2}$$

in the half-space. It will be more convenient to replace coordinate z by $z - z_m$ in quantities (7.10), i.e. to introduce

$$\xi_m = kr_m(z - z_m), \quad \eta_m = ks_m(z - z_m) \tag{9.3}$$

for $m = 1, 2, \ldots, n-1$, and

$$\zeta_n = kr_n^*(z - z_n), \quad \chi_n = ks_n^*(z - z_n). \tag{9.4}$$

For the sake of brevity, denote the stress components in the m-th layer by $\tau_m = (\tau_{zx})_m$ and $\sigma_m = (\tau_{zz})_m$. In view of (7.12), (7.14) and (7.15), the displacements and stresses in the layers can be expressed as

$$u_m = -k\left[i\left(A_m \cos\xi_m + B_m \sin\xi_m\right) + s_m\left(-D_m \sin\eta_m + E_m \cos\eta_m\right)\right] \ ,$$

$$w_m = -k\left[r_m\left(A_m \sin\xi_m - B_m \cos\xi_m\right) + i\left(D_m \cos\eta_m + E_m \sin\eta_m\right)\right] \ ,$$

$$\tau_m = \rho_m\omega^2\left[i\gamma_m r_m\left(A_m \sin\xi_m - B_m \cos\xi_m\right) - \delta_m\left(D_m \cos\eta_m + E_m \sin\eta_m\right)\right] \ ,$$

$$\sigma_m = \rho_m\omega^2\left[\delta_m\left(A_m \cos\xi_m + B_m \sin\xi_m\right) + i\gamma_m s_m\left(D_m \sin\eta_m - E_m \cos\eta_m\right)\right] \ ,$$

$$(9.5)$$

and in the half-space as

$$u_n = -k\left(iC_n e^{-\zeta_n} - s_n^* F_n e^{-\chi_n}\right) \ ,$$

$$w_n = -k\left(r_n^* C_n e^{-\zeta_n} + iF_n e^{-\chi_n}\right) \ ,$$

$$\tau_n = \rho_n\omega^2\left(i\gamma_n r_n^* C_n e^{-\zeta_n} - \delta_n F_n e^{-\chi_n}\right) \ ,$$

$$\sigma_n = \rho_n\omega^2\left(\delta_n C_n e^{-\zeta_n} + i\gamma_n s_n^* F_n e^{-\chi_n}\right) \ ,$$

$$(9.6)$$

where we have again omitted the common factors $e^{i(\omega t - kx)}$, and put

$$\gamma_m = 2\left(\beta_m/c\right)^2 \quad \text{and} \quad \delta_m = \gamma_m - 1 \ . \qquad (9.7)$$

9.2 Matrix for One Layer

At the top of the m-th layer, i.e. for $z = z_m$, $\xi_m = \eta_m = 0$, and formulae (9.5) simplify to read

$$\left(u_m\right)_{z_m} = -k\left(iA_m + s_m E_m\right) \ ,$$

$$\left(w_m\right)_{z_m} = -k\left(-r_m B_m + iD_m\right) \ ,$$

$$\left(\tau_m\right)_{z_m} = \rho_m\omega^2\left(-i\gamma_m r_m B_m - \delta_m D_m\right) \ ,$$

$$(9.8)$$

$$\left(\sigma_m\right)_{z_m} = \rho_m\omega^2\left(\delta_m A_m - i\gamma_m s_m E_m\right) \ .$$

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At the bottom of the m-th layer, i.e. for $z = z_{m+1}$, formulae (9.5) take the form

$$\left(u_m\right)_{z_{m+1}} = -k\left[i\left(A_m \cos P_m + B_m \sin P_m\right) + s_m\left(-D_m \sin Q_m + E_m \cos Q_m\right)\right] ,$$

$$\left(w_m\right)_{z_{m+1}} = -k\left[r_m\left(A_m \sin P_m - B_m \cos P_m\right) + i\left(D_m \cos Q_m + E_m \sin Q_m\right)\right] ,$$

(9.9)

$$\left(\tau_m\right)_{z_{m+1}} =$$

$$= \rho_m\omega^2\left[i\gamma_m r_m\left(A_m \sin P_m - B_m \cos P_m\right) - \delta_m\left(D_m \cos Q_m + E_m \sin Q_m\right)\right] ,$$

$$\left(\sigma_m\right)_{z_{m+1}} =$$

$$= \rho_m\omega^2\left[\delta_m\left(A_m \cos P_m + B_m \sin P_m\right) + i\gamma_m s_m\left(D_m \sin Q_m - E_m \cos Q_m\right)\right] ,$$

where

$$P_m = kr_m d_m , \quad Q_m = ks_m d_m , \tag{9.10}$$

$d_m = z_{m+1} - z_m$ being the thickness of the layer.

Eliminate the coefficients A_m and E_m from the first and fourth equations in (9.8), i.e. express them in terms of displacement u_m and stress σ_m at the top of the layer:

$$A_m = \frac{i\gamma_m\left(u_m\right)_{z_m}}{k} - \frac{\left(\sigma_m\right)_{z_m}}{\rho_m\omega^2} , \quad E_m = \frac{\delta_m\left(u_m\right)_{z_m}}{s_m k} + \frac{i\left(\sigma_m\right)_{z_m}}{\rho_m s_m\omega^2} . \tag{9.11}$$

Analogously, express B_m and D_m from the second and third equations in (9.8):

$$B_m = -\frac{\delta_m\left(w_m\right)_{z_m}}{r_m k} + \frac{i\left(\tau_m\right)_{z_m}}{\rho_m r_m\omega^2} , \quad D_m = \frac{i\gamma_m\left(w_m\right)_{z_m}}{k} + \frac{\left(\tau_m\right)_{z_m}}{\rho_m\omega^2} . \tag{9.12}$$

By inserting these coefficients into (9.9), we obtain the relations between the displacements and stresses at the bottom and at the top of the layer. However, instead of the displacements and stresses themselves, it will be convenient to consider, e.g., the following quantities:

$$\frac{iu_m}{k} , \quad -\frac{w_m}{k} , \quad \frac{\sigma_m}{\omega^2} , \quad \frac{i\tau_m}{\omega^2} ; \tag{9.13}$$

see Proskuryakova et al. (1981). After some simple algebra, the relations between these quantities at the bottom and top of the layer can be expressed in the following matrix form:

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$$\begin{pmatrix} iu_m/k \\ -w_m/k \\ \sigma_m/\omega^2 \\ i\tau_m/\omega^2 \end{pmatrix}_{z_{m+1}} = \mathbf{a}_m \begin{pmatrix} iu_m/k \\ -w_m/k \\ \sigma_m/\omega^2 \\ i\tau_m/\omega^2 \end{pmatrix}_{z_m} , \tag{9.14}$$

where matrix $\mathbf{a}_m$ is of the fourth order and its elements are (Proskuryakova et al., 1981):

$$(a_m)_{11} = (a_m)_{44} = \gamma_m \cos P_m - \delta_m \cos Q_m ,$$

$$(a_m)_{12} = (a_m)_{34} = \delta_m r_m^{-1} \sin P_m + \gamma_m s_m \sin Q_m ,$$

$$(a_m)_{13} = (a_m)_{24} = -\rho_m^{-1}(\cos P_m - \cos Q_m) ,$$

$$(a_m)_{14} = \rho_m^{-1}(r_m^{-1} \sin P_m + s_m \sin Q_m) ,$$

$$(a_m)_{21} = (a_m)_{43} = \gamma_m r_m \sin P_m + \delta_m s_m^{-1} \sin Q_m ,$$

$$(a_m)_{22} = (a_m)_{33} = -\delta_m \cos P_m + \gamma_m \cos Q_m , \tag{9.15}$$

$$(a_m)_{23} = -\rho_m^{-1}(r_m \sin P_m + s_m^{-1} \sin Q_m) ,$$

$$(a_m)_{31} = (a_m)_{42} = \rho_m \gamma_m \delta_m (\cos P_m - \cos Q_m) ,$$

$$(a_m)_{32} = \rho_m(\delta_m^2 r_m^{-1} \sin P_m + \gamma_m^2 s_m \sin Q_m) ,$$

$$(a_m)_{41} = -\rho_m(\gamma_m^2 r_m \sin P_m + \delta_m^2 s_m^{-1} \sin Q_m) .$$

Note that matrix $\mathbf{a}_m$ is symmetric with respect to the secondary diagonal, and all its elements are real (for real values of ω and c) even if $c < \alpha_m$ or $c < \beta_m$. In this case, r_m becomes imaginary, $r_m = ir_m^*$, and $P_m = ikr_m^* d_m = iP_m^*$, $\cos P_m = \cosh P_m^*$, $r_m \sin P_m = -r_m^* \sinh P_m^*$, $r_m^{-1} \sin P_m = (\sinh P_m^*)/r_m^*$. Similar formulae hold true for $\cos Q_m$, $s_m \sin Q_m$ and $s_m^{-1} \sin Q_m$ if $c < \beta_m$.

Rayleigh waves are elliptically polarised, so that displacement u and stress $\tau = \tau_{zx}$ are in the same phase, but shifted by 90° with respect to w and $\sigma = \tau_{zz}$. This fact is taken into account in (9.14), where u_m and τ_m are multiplied by the imaginary unit i. Consequently, all elements of matrix $\mathbf{a}_m$ are real. If we considered the displacements and stresses alone, the corresponding matrix would contain both real and purely imaginary elements. Also the original Thomson-Haskell matrices contain both real and purely imaginary elements (Haskell, 1953).

Finally, let us add a comment on the choice of the general solutions of wave equations. In the preceding chapter, we considered three equivalent general forms, given by formulae (8.7a, b, c). We then selected the trigonometric form (8.7c). Analogously, in this chapter we have chosen the general expressions for displacements and stresses in terms of trigonometric functions; see (9.5). This form is very convenient, because we have $\sin \xi_m = \sin \eta_m = 0$ at the top of the layer, so that the right-hand sides of (9.8) always contain only two of the

unknowns A_m, B_m, D_m, E_m. Consequently, the system (9.8) of four equations can be reduced to two systems of two equations, which is much easier to solve. If exponential terms were used in (9.5), all four unknowns would appear on the right-hand sides of (9.8), and an inverse matrix of the fourth order would have to be calculated (Haskell, 1953; Fuchs, 1968; Proskuryakova et al., 1981). In this chapter we have succeeded in simplifying the problem by a convenient choice of general solutions (9.5).

9.3 Boundary Conditions and the Matrix for a Stack of Layers

We shall require the displacements and stresses to be continuous at the individual interfaces. Consequently, also quantities $ik^{-1}u$, $-kw$, $\omega^{-2}\sigma$, $i\omega^{-2}\tau$ must be continuous; see (9.14). The relation between these quantities at the top of the half-space and at the free surface can then be expressed by the matrix product

$$\mathbf{A} = \mathbf{a}_{n-1} \dots \mathbf{a}_2 \mathbf{a}_1 ; \tag{9.16}$$

see a similar discussion in Section 8.3.

9.4 Expressions for the Half-Space and the Dispersion Equation

Specify quantities (9.6) for the top of the half-space, where $z = z_n$, and $\zeta_n = \chi_n = 0$. The continuity of components (9.13) at the top of the half-space can then be expressed in the matrix form as

$$\begin{pmatrix} C_n + is_n^* F_n \\ r_n^* C_n + iF_n \\ \rho_n\left(\delta_n C_n + i\gamma_n s_n^* F_n\right) \\ -\rho_n\left(\gamma_n r_n^* C_n + i\delta_n F_n\right) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} iu_1/k \\ -w_1/k \\ \sigma_1/\omega^2 \\ i\tau_1/\omega^2 \end{pmatrix}_{z_1} , \tag{9.17}$$

where matrix $\mathbf{A}$ is given by (9.16). However, we assume the surface of the medium to be free, so that we must put $\sigma_1 = \tau_1 = 0$ for $z = z_1$. Moreover, introduce new unknowns $\tilde{u}_1 = i(u_1)_{z_1}/k$, $\tilde{w}_1 = -(w_1)_{z_1}/k$, $G_n = -C_n$ and $H_n = -iF_n$. Equations (9.17) can then be expressed as

$$A_{11}\tilde{u}_1 + A_{12}\tilde{w}_1 + G_n + s_n^* H_n = 0 ,$$
$$A_{21}\tilde{u}_1 + A_{22}\tilde{w}_1 + r_n^* G_n + H_n = 0 , \tag{9.18}$$
$$A_{31}\tilde{u}_1 + A_{32}\tilde{w}_1 + \rho_n\delta_n G_n + \rho_n\gamma_n s_n^* H_n = 0 ,$$
$$A_{41}\tilde{u}_1 + A_{42}\tilde{w}_1 - \rho_n\gamma_n r_n^* G_n - \rho_n\delta_n H_n = 0 .$$

This is a system of homogeneous equations in the unknowns $\tilde{u}_1$, $\tilde{w}_1$, G_n and H_n. This system has a non-trivial solution if the corresponding determinant is zero:

$$\begin{vmatrix} A_{11} & A_{12} & 1 & s_n^* \\ A_{21} & A_{22} & r_n^* & 1 \\ A_{31} & A_{32} & \rho_n \delta_n & \rho_n \gamma_n s_n^* \\ A_{41} & A_{42} & -\rho_n \gamma_n r_n^* & -\rho_n \delta_n \end{vmatrix} = 0 \ . \tag{9.19}$$

This is the desired *dispersion equation* for the unknown phase velocity, c.

It will be convenient to decompose the determinant in (9.19) with respect to the first two columns according to Laplace's theorem. We shall proceed in a way analogous to that in Section 7.4, because dispersion equation (9.19) is similar to the special dispersion equation (7.24). Denote the minors of matrix **A** according to (7.25), and arrange them in the order given by (7.26). Equation (9.19) can then be expressed formally in the same form as Eq. (7.27), i.e.

$$L_1 H_1 + L_2 H_2 + L_3 H_3 + L_4 H_4 + L_5 H_5 + L_6 H_6 = 0 \ , \tag{9.20}$$

where L_1 to L_6 are now the second-order minors constructed from the first two columns of (9.19), and H_1 to H_6 are the corresponding cofactors, constructed from the third and fourth columns and containing the parameters of the half-space.

Minors L_1 to L_6, characterising the contributions of the stack of layers, are given by

$$L_1 = A\begin{vmatrix} 12 \\ 12 \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21} \ , \quad L_2 = A\begin{vmatrix} 13 \\ 12 \end{vmatrix} = A_{11}A_{32} - A_{12}A_{31}$$

$$\tag{9.21}$$

$$L_3 = A\begin{vmatrix} 14 \\ 12 \end{vmatrix} , \qquad L_4 = A\begin{vmatrix} 23 \\ 12 \end{vmatrix} , \qquad L_5 = A\begin{vmatrix} 24 \\ 12 \end{vmatrix} , \qquad L_6 = A\begin{vmatrix} 34 \\ 12 \end{vmatrix} ;$$

see notation (7.25).

The corresponding cofactors are similar to cofactors (7.29), but in (9.19) we have interchanged the third and fourth rows and their signs. Thus,

$$H_1 = \rho_n \left(\gamma_n^2 r_n^* s_n^* - \delta_n^2 \right) \ , \qquad\qquad H_2 = -\rho_n r_n^* \ ,$$

$$H_3 = H_4 = \rho_n \left(\gamma_n r_n^* s_n^* - \delta_n \right) \ , \tag{9.22}$$

$$H_5 = -\rho_n s_n^* \ , \qquad\qquad H_6 = 1 - r_n^* s_n^* \ .$$

The computation of the phase velocity, therefore, proceeds in the following way. For a given value of ω and a trial value of phase velocity c, we compute

layer matrices (9.15) in a loop from the first to the $(n-1)$-st layer. Some problems of programming the layer matrices have already been mentioned in Section 8.6, so that we shall not repeat them here. In the same loop we also compute the partial products $\mathbf{a}_m \mathbf{a}_{m-1} \ldots \mathbf{a}_1$ in (9.16). When matrix $\mathbf{A}$ for the stack of layers is obtained (only the first and second columns are needed), we compute expressions (9.21), (9.22) and the left-hand side of Eq. (9.20). The procedure is then repeated with a modified value of c, until the required accuracy is achieved.

The matrix method, just described, is relatively simple, but leads to loss-of-accuracy problems at high frequencies. Consequently, other forms of the dispersion equation for Rayleigh waves are desirable.

9.5 Matrices of the Sixth Order

Dispersion equation (9.20) contains the second-order minors of matrix $\mathbf{A}$. In principle, we can determine these minors in two ways:

a) We can determine matrix $\mathbf{A}$ as the product of layer matrices $\mathbf{a}_m$, and then compute the minors of $\mathbf{A}$ using (9.21). The original Thomson-Haskell method for Rayleigh waves was also formulated in this form (Haskell, 1953). However, this approach leads to the computational problems mentioned above.

b) We can determine the minors for all layer matrices $\mathbf{a}_m$, and then use these minors to construct the minors of matrix $\mathbf{A}$ directly. This method is based on the application of *associated matrices*, which are studied in advanced courses of the matrix theory; see, e.g., Gantmakher (1953).These matrices were also described in detail by Dunkin (1965).

Although both methods are equivalent theoretically, they differ substantially in their numerical properties. It was found that method b) yields more accurate results; see the historical remarks in Section 9.6. Consequently, methods a) are not in use now.

9.5.1 Associated matrices

Minors L_1 to L_6, given by (9.21), were constructed from the first and second columns of matrix $\mathbf{A}$. These minors may be considered as the first column of a new matrix, $\widetilde{\mathbf{A}}$, composed of all the minors of $\mathbf{A}$. The second column of $\widetilde{\mathbf{A}}$ will be composed of the minors from the first and third columns of $\mathbf{A}$, etc. Six pairs can be formed from the ordinal numbers of the rows and columns of matrix $\mathbf{A}$, i.e. combinations 12, 13, 14, 23, 24 and 34. Consequently, matrix $\widetilde{\mathbf{A}}$, composed of these minors, is of the sixth order (6 rows and 6 columns).

Consider a general matrix $\mathbf{A}$ of order n, and construct its minors of order m, where $m < n$. The matrix composed of these minors is referred to as the *associated matrix*. We shall denote this associated matrix by $\widetilde{\mathbf{A}}$. Note that the associated matrices, composed of the second-order minors $(m = 2)$, are also referred to as *δ-matrices, delta matrices, determinant matrices* (Pestel and Leckie, 1963), or *second compound matrices* (Thrower, 1965).

The usefulness of associated matrices is closely related to their properties in matrix multiplication. Namely, the following important theorem holds true.

Let matrix $\mathbf{C}$ be the product of matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \ . \tag{9.23}$$

The same relation then holds between the corresponding associated matrices, i.e.

$$\widetilde{\mathbf{C}} = \widetilde{\mathbf{A}} \cdot \widetilde{\mathbf{B}} \ . \tag{9.24}$$

We shall not prove this theorem here. The reader may verify it, e.g., for our special case of $n = 4$ and $m = 2$.

9.5.2 Associated matrices in the Rayleigh-wave problem

Again consider matrix $\mathbf{A}$ for the stack of layers, given by (9.16). The corresponding δ-matrix can be expressed, according to the above-mentioned theorem, as

$$\widetilde{\mathbf{A}} = \widetilde{\mathbf{a}}_{n-1} \ldots \widetilde{\mathbf{a}}_2 \widetilde{\mathbf{a}}_1 \ , \tag{9.25}$$

where $\widetilde{\mathbf{a}}_m$ denotes the δ-matrix associated with layer matrix $\mathbf{a}_m$.

Omit subscript m in the symbol $\left.\left(a_m\right)\right|_{kl}^{ij}$ for the minors of layer matrix $\mathbf{a}_m$, given by (9.15), i.e. denote

$$a\Big|_{kl}^{ij} = \left(a_m\right)_{ik}\left(a_m\right)_{jl} - \left(a_m\right)_{il}\left(a_m\right)_{jk} \ . \tag{9.26}$$

These minors are arranged in the associated matrix $\widetilde{\mathbf{a}}_m$ as follows:

$$\widetilde{\mathbf{a}}_m = \begin{pmatrix} a\Big|_{12}^{12}, & a\Big|_{13}^{12}, & a\Big|_{14}^{12}, & a\Big|_{23}^{12}, & a\Big|_{24}^{12}, & a\Big|_{34}^{12} \\ a\Big|_{12}^{13}, & a\Big|_{13}^{13}, & \cdots & & & a\Big|_{34}^{13} \\ \vdots & & & & & \\ a\Big|_{12}^{34}, & a\Big|_{13}^{34}, & \cdots & & & a\Big|_{34}^{34} \end{pmatrix} \ . \tag{9.27}$$

For a moment, also omit subscripts m in elements $\left(\widetilde{a}_m\right)_{ij}$, and introduce the following notations (see the similar abbreviations in (7.32)):

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$$\gamma = \gamma_m \ , \quad \delta = \delta_m \ , \quad \rho = \rho_m \ ,$$

$$p_1 = \cos P_m \ , \quad p_2 = r_m \sin P_m \ , \quad p_3 = r_m^{-1} \sin P_m \ ,$$

$$q_1 = \cos Q_m \ , \quad q_2 = s_m \sin Q_m \ , \quad q_3 = s_m^{-1} \sin Q_m \ ,$$

$$w_1 = 1 - p_1 q_1 \ , \quad w_2 = p_2 q_2 \ , \quad w_3 = p_3 q_3 \ .$$

(9.28)

Inserting elements (9.15) into (9.26) and (9.27), after some algebra we arrive at

$$\tilde{a}_{11} = \tilde{a}_{66} = -2\gamma\delta + \left(2\gamma^2 - 2\gamma + 1\right)p_1 q_1 - \gamma^2 w_2 - \delta^2 w_3 \ ,$$

$$\tilde{a}_{12} = \tilde{a}_{56} = -\rho^{-1}\left(p_1 q_3 + q_1 p_2\right) \ ,$$

$$\tilde{a}_{13} = \tilde{a}_{14} = \tilde{a}_{36} = \tilde{a}_{46} = -\rho^{-1}\left[(\gamma + \delta)w_1 + \gamma w_2 + \delta w_3\right] \ ,$$

$$\tilde{a}_{15} = \tilde{a}_{26} = -\rho^{-1}\left(p_1 q_2 + q_1 p_3\right) \ ,$$

$$\tilde{a}_{16} = \rho^{-2}\left(2w_1 + w_2 + w_3\right) \ ,$$

$$\tilde{a}_{21} = \tilde{a}_{65} = \rho\left(\gamma^2 p_1 q_2 + \delta^2 q_1 p_3\right) \ ,$$

$$\tilde{a}_{22} = \tilde{a}_{55} = p_1 q_1 \ ,$$

$$\tilde{a}_{23} = \tilde{a}_{24} = \tilde{a}_{35} = \tilde{a}_{45} = \gamma p_1 q_2 + \delta q_1 p_3 \ ,$$

(9.29)

$$\tilde{a}_{25} = -p_3 q_2 \ ,$$

$$\tilde{a}_{31} = \tilde{a}_{41} = \tilde{a}_{63} = \tilde{a}_{64} = \rho\left[\gamma\delta(\gamma + \delta)w_1 + \gamma^3 w_2 + \delta^3 w_3\right] \ ,$$

$$\tilde{a}_{32} = \tilde{a}_{42} = \tilde{a}_{53} = \tilde{a}_{54} = \delta p_1 q_3 + \gamma q_1 p_2 \ ,$$

$$\tilde{a}_{33} = \tilde{a}_{44} = 1 + 2\gamma\delta w_1 + \gamma^2 w_2 + \delta^2 w_3 \ ,$$

$$\tilde{a}_{34} = \tilde{a}_{43} = \tilde{a}_{33} - 1 \ ,$$

$$\tilde{a}_{51} = \tilde{a}_{62} = \rho\left(\delta^2 p_1 q_3 + \gamma^2 q_1 p_2\right) \ ,$$

$$\tilde{a}_{52} = -p_2 q_3 \ ,$$

$$\tilde{a}_{61} = \rho^2\left(2\gamma^2\delta^2 w_1 + \gamma^4 w_2 + \delta^4 w_3\right) \ ;$$

see Proskuryakova et al. (1981).

By multiplying these associated matrices we obtain immediately associated matrix $\tilde{\mathbf{A}}$, given by (9.25). In fact, only the first column of this resultant matrix is needed. This first column contains minors L_1 to L_6, i.e. $L_i = A_{i1}$, $i = 1$ to 6, which are needed in dispersion equation (9.20).

This method of computing minors L_1 to L_6 is more accurate than their determination from the elements of matrix $\mathbf{A}$. The numerical advantage of associated matrices $\tilde{\mathbf{a}}_m$ over the original layer matrices $\mathbf{a}_m$ is in that the associated matrices do not contain quadratic terms $\cos^2 P_m$, $\sin^2 P_m$, $\cos^2 Q_m$ and $\sin^2 Q_m$; see the discussion in Section 7.4.

Since the third and fourth rows and columns of matrix $\tilde{\mathbf{a}}_m$ are similar to each other, this delta matrix of the sixth order may be replaced by a *reduced*

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delta matrix of the fifth order (Watson, 1970). The listing of a subroutine, which is based on the reduced delta matrices, can be found in Proskuryakova et al. (1981). Since the passage from matrices 6×6 to matrices 5×5 is of no principal importance from the computational point of view, we shall not present these reduced matrices here.

9.6 Historical Remarks and Other Formulations of the Dispersion Equation

Various methods of computing the dispersion curves of Rayleigh waves in a layered medium have been proposed. They can be divided into several groups which we shall briefly comment on below.

9.6.1 Thomson-Haskell matrices and their modifications

The present matrix methods of computing theoretical dispersion curves of surface waves began with the paper by Thomson (1950), dealing with body waves in layered media. Haskell (1953) introduced a new notation and worked out matrix formulations of the dispersion equations for Rayleigh and Love waves in layered media. However, it soon became evident that the Thomson-Haskell matrices for Rayleigh waves lead to loss-of-accuracy problems at high frequencies (Dorman et al., 1960; Press et al., 1961). At first, these problems were solved, e.g., by simplifying the model of the medium at high frequencies (by including some deeper layers into the half-space), or by computing with double accuracy. However, these approaches did not solve the problem in principle.

Similar numerical problems were also known from solutions of some engineering problems, e.g., from computations of high eigenfrequencies of technical structures; see the review in Pestel and Leckie (1963). One possibility of solving these problems was passing from the original matrices to the associated matrices, which were composed of second-order minors of the original matrices (minors 2×2). This experience was used by Thrower (1965) to solve the problem of Rayleigh waves. Instead of the original 4×4 matrices, he formulated the problem in terms of the associated 6×6 matrices, and obtained better numerical results.

Dunkin (1965) came independently to the same conclusion that the numerical problems in the Thomson-Haskell matrices could be overcome by applying associated matrices. He found the sources of numerical instabilities in a more concrete form, namely in subtracting squares of exponential terms, which cancel out analytically, but lead to a loss of accuracy when the minors are evaluated numerically; see the discussion in section 7.4. Since these subtractions are eliminated analytically in the associated matrices, these matrices yield more accurate results.

Watson (1970) reduced Dunkin's 6×6 matrices to 5×5 matrices; see again Section 7.4.

9.6.2 Knopoff's method

Molotkov (1961) derived recurrent formulae which made it possible to calculate the minors of the system of boundary conditions immediately. He applied this method to a body-wave problem, but possible applications to Rayleigh-wave problems were evident. It was recognised only later that Molotkov's method solves the computational problems mentioned above, and is equivalent to the multiplication of matrices of the fifth order (e.g., the matrices proposed by Watson (1970)).

Another method of computing the determinant in the system of boundary conditions was proposed by Knopoff (1964); for further details we refer the reader to Schwab (1970), and Schwab and Knopoff (1972). Knopoff succeeded in expressing the corresponding determinants for Rayleigh and Love problems by means of the products of matrices for individual layers. Knopoff himself considered his methods only as an alternative to the method of Thomson and Haskell. However, Dunkin (1965) recognised that Knopoff's technique removed the computational problems contained in the Thomson-Haskell matrices. Abo-Zena (1979) further modified Knopoff's for computing at high frequencies, but this modification requires much longer computer time.

Knopoff's matrices have different forms for odd and even layers, which represents a small disadvantage. Nevertheless, many authors prefer Knopoff's method, e.g., in computing synthetic seismograms by the modal summation method (Panza, 1985; Panza and Suhadolc, 1987; Urban et al., 1993).

9.6.3 Computing reflection and transmission coefficients

Another effective approach to calculating the dispersion curves of surface waves is founded on the use of reflection and transmission coefficients. This approach was first applied to studies of electromagnetic waves in the ionosphere. Kennett and his co-workers modified and further developed this method with regard to the problems of elastic waves in layered media. These methods not only efficiently solve the dispersion problem of surface waves, but also explicitly display the physical mechanism of surface wave formation, i.e. constructive interference. For further details we refer the reader to Kennett (1983), Luco and Apsel (1983), and Chen (1993).

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Chapter 10

Matrix Formulations of Some Body-Wave Problems

In Chapters 8 and 9 we used matrix methods to study the propagation of surface waves in layered media. However, matrix methods may be used to study many other problems of wave propagation in layered media, such as the propagation of elastic body waves, temperature waves, or electromagnetic waves. Even the paper by Thomson (1950), the first paper dealing with matrix methods in the theory of elastic waves, was devoted to a problem of body waves.

In this chapter, we shall demonstrate the application of matrix methods to two simple problems of *SH* waves, and we shall then solve the analogous problems for *P* waves.

10.1 Motion of the Surface of a Layered Medium Caused by an Incident *SH* Wave

Consider the same layered medium as in Chapter 8. We shall again study the propagation of plane harmonic waves of the *SH* type in this medium but, as opposed to the case of Love waves, we shall consider the homogeneous waves in the half-space.

Consider a plane, harmonic *SH* wave which propagates in the half-space obliquely upwards and is incident at the stack of layers from below (Fig. 10.1). Again denote its angular frequency by ω, and its angle of incidence in the half-space by γ_n. We are interested in the motion at the free surface, which is produced by this incident wave.
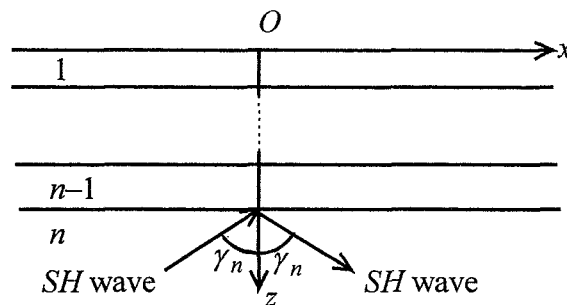

Fig. 10.1. Incidence of an *SH* wave at a stack of layers.

The solution of the wave equation in terms of plane, harmonic *SH* waves may be expressed in several forms, see (8.7). As the general expression for the displacement vector in the half-space we shall use form (8.7b), i.e.

$$\mathbf{u}_n = \left(0, v_n, 0\right), \quad v_n = V_n^- + V_n^+ , \tag{10.1}$$

where

$$V_n^- = A_n^- e^{is_n(z-z_n)} e^{i(\omega t - kx)} , \quad V_n^+ = A_n^+ e^{-is_n(z-z_n)} e^{i(\omega t - kx)} , \tag{10.2}$$

z_n being the depth of the boundary of the half-space, c the horizontal velocity, which is related to the angle of incidence by Snell's law,

$$\frac{1}{c} = \frac{\sin \gamma_n}{\beta_n} , \tag{10.3}$$

$k = \omega/c$ the horizontal wave number, and

$$s_n = \sqrt{\left(\omega/\beta_n\right)^2 - k^2} . \tag{10.4}$$

It follows from the discussion in Section 4.5 that V_n^- represents a wave propagating obliquely upwards (incident wave), and V_n^+ represents a wave propagating obliquely downwards (reflected wave). Thus, coefficient A_n^- is the amplitude of the incident wave (a given value, e.g., unity), and A_n^+ is the unknown amplitude of the reflected wave.

The corresponding shear stress in the half-space is

$$\tau_n = \mu_n \frac{\partial v_n}{\partial z} = i\mu_n s_n\left(V_n^- - V_n^+\right) . \tag{10.5}$$

The displacement-stress vector in the half-space may then be expressed in matrix form as

$$\begin{pmatrix} v_n \\ \tau_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1, & 1 \\ i\mu_n s_n, & -i\mu_n s_n \end{pmatrix}}_{\mathbf{T}_n} \begin{pmatrix} V_n^- \\ V_n^+ \end{pmatrix} . \tag{10.6}$$

The inverse matrix to $\mathbf{T}_n$ is

$$\mathbf{T}_n^{-1} = \frac{-1}{2i\mu_n s_n} \begin{pmatrix} -i\mu_n s_n, & -1 \\ -i\mu_n s_n, & 1 \end{pmatrix} . \tag{10.7}$$

Hereafter we shall again omit the common exponential terms $e^{i(\omega t - kx)}$. At the top of the half-space, i.e. for $z = z_n$, we then get $V_n^- = A_n^-$, $V_n^+ = A_n^+$, and the inverse relation to (10.6) takes the form

$$\begin{pmatrix} A_n^- \\ A_n^+ \end{pmatrix} = \mathbf{T}_n^{-1} \begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} . \tag{10.8}$$

However, the displacement-stress vector at the top of the half-space is related to that at the free surface by (8.25), i.e. by

$$
\begin{pmatrix} v_n \\ \tau_n \end{pmatrix}_{z_n} = \mathbf{A} \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} ,
$$

(10.9)

where matrix $\mathbf{A}$ is the product of the layer matrices, given by (8.24). Equation (10.8) then becomes

$$
\begin{pmatrix} A_n^- \\ A_n^+ \end{pmatrix} = \underbrace{\mathbf{T}_n^{-1} \mathbf{A}}_{\mathbf{D}} \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix}_{z_1} .
$$

(10.10)

Note that matrix $\mathbf{A}$ is real, but matrices $\mathbf{T}_n^{-1}$ and $\mathbf{D}$ are complex.

Assuming the surface of the medium to be free, $\left(\tau_1 \right)_{z_1} = 0$, the first equation in (10.10) becomes

$$
A_n^- = D_{11} \left(v_1 \right)_{z_1} .
$$

(10.11)

Putting the amplitude of the incident wave equal to unity, $A_n^- = 1$, the desired displacement at the free surface is simply

$$
\boxed{\left(v_1 \right)_{z_1} = \frac{1}{D_{11}}} .
$$

(10.12)

This solves our problem.

Note that this displacement refers to the point at the surface which lies above the point of incidence in the half-space (Fig. 10.1); both points have the same horizontal coordinate x, because the same exponential term $e^{i(\omega t - kx)}$ has been omitted in all displacements and stresses. Note further that the motion at the surface depends not only on the angle of incidence, but also on frequency, because angular frequency ω enters matrices $\mathbf{a}_m$.

Let us recapitulate the computational process. For a given model of the medium, and given values of ω and γ_n, we compute c from (10.3), $k = \omega/c$, matrices $\mathbf{a}_m$, given by (8.8), (8.9) and (8.16), then matrices $\mathbf{A}$, $\mathbf{T}_n^{-1}$ and $\mathbf{D}$. The result is given by (10.12).

The formulae given above cannot be applied immediately to the important special case of normal incidence, when $\gamma_n = 0$ and horizontal velocity c becomes infinite. However, it is easy to reformulate the problem in terms of the parameter of the seismic ray, $p = 1/c$, or wave number k. It is, therefore, more convenient to replace formula (10.3) by

$$k = \frac{\omega \sin \gamma_n}{\beta_n} ,$$

(10.13)

retaining all the remaining formulae without change.

10.2 Reflection and Transmission Coefficients of *SH* Waves for a Transition Zone

The reflection and transmission of plane waves at a plane interface of two homogeneous and isotropic half-spaces belongs to the basic problems of wave propagation in many branches of physics. Here we shall restrict ourselves again to the case of *SH* waves. However, instead of a simple interface of two half-spaces, we shall consider the more complicated case of a transition zone between these half-space.

We shall assume that the transition zone is formed by $(n-1)$ homogeneous and isotropic layers, separated by parallel interfaces; see Fig. 10.2 and the notation in Fig. 8.1. The transition zone is sandwiched between two homogeneous isotropic half-spaces. The upper half-space is denoted as the zero-th layer, the lower as the n-th layer. Denote the shear modulus and shear wave velocity in the upper half-space by μ_0 and β_0, respectively.
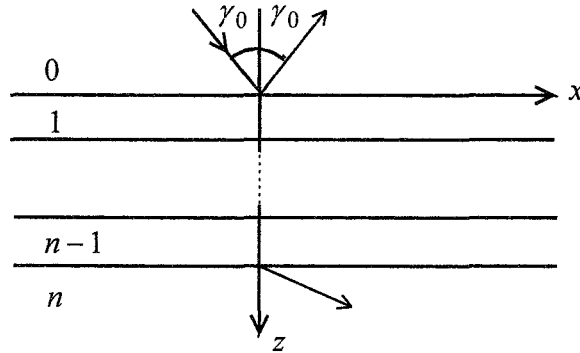
Fig. 10.2. Reflection and transmission of an *SH* wave at the transition zone.

Here we shall study a problem similar to that in the preceding section, but instead of the vacuum above the free surface in Fig. 10.1, we now consider an elastic half-space. Moreover, we shall assume that the incident wave propagates in the upper half-space, i.e. that it is incident at the stack of layers from above.

Thus, assume the incident *SH* wave to be plane, harmonic and propagating in the upper half-space Denote its angular frequency by ω and the angle of incidence by γ_0. Analogously to (10.13), introduce the horizontal wave number, k, by the relation

$$k = \frac{\omega \sin \gamma_0}{\beta_0} .$$

(10.14)

We shall express the displacement vector in the upper half-space, analogously to (10.1) and (10.2) for the lower half-space, as

$$\mathbf{u}_0 = (0, v_0, 0), \quad v_0 = V_0^- + V_0^+ , \tag{10.15}$$

where

$$V_0^- = A_0^- e^{is_0 z} e^{i(\omega t - kx)} , \quad V_0^+ = A_0^+ e^{-is_0 z} e^{i(\omega t - kx)} , \tag{10.16}$$

and

$$s_0 = \sqrt{(\omega/\beta_0)^2 - k^2} . \tag{10.17}$$

Displacement V_0^+ describes the incident wave, and V_0^- the reflected wave.

The displacement-stress vector for the upper half-space is analogous to (10.6) for the lower half-space:

$$\begin{pmatrix} v_0 \\ \tau_0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1, & 1 \\ i\mu_0 s_0, & -i\mu_0 s_0 \end{pmatrix}}_{\mathbf{T}_0} \begin{pmatrix} V_0^- \\ V_0^+ \end{pmatrix} . \tag{10.18}$$

At the boundary of the upper half-space $(z = z_1 = 0)$, this vector takes the form

$$\begin{pmatrix} v_0 \\ \tau_0 \end{pmatrix}_{z_1} = \mathbf{T}_0 \begin{pmatrix} A_0^- \\ A_0^+ \end{pmatrix} . \tag{10.19}$$

By using the continuity of this vector at the interface $z = z_1$, and inserting (10.19) into Eq. (10.10), one gets

$$\begin{pmatrix} A_n^- \\ A_n^+ \end{pmatrix} = \mathbf{T}_n^{-1} \mathbf{A} \begin{pmatrix} v_0 \\ \tau_0 \end{pmatrix}_{z_1} = \underbrace{\mathbf{T}_n^{-1} \mathbf{A} \mathbf{T}_0}_{\mathbf{E}} \begin{pmatrix} A_0^- \\ A_0^+ \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} A_0^- \\ A_0^+ \end{pmatrix} . \tag{10.20}$$

Assume that no waves propagate upwards in the lower half-space (Fig. 10.2), i.e. put $A_n^- = 0$. Moreover, put the amplitude of the incident wave equal to unity, $A_0^+ = 1$. Equation (10.20) then becomes

$$\begin{pmatrix} 0 \\ A_n^+ \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} A_0^- \\ 1 \end{pmatrix} , \tag{10.21}$$

which represents two scalar equations,

$$0 = E_{11} A_0^- + E_{12} , \quad A_n^+ = E_{21} A_0^- + E_{22} . \tag{10.22}$$

This yields the reflection coefficient for the transition zone in the form

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$$\boxed{A_0^- = -\frac{E_{12}}{E_{11}}}\,, \qquad\qquad (10.23)$$

and the transmission coefficient

$$A_n^+ = E_{22} - E_{21}\frac{E_{12}}{E_{11}}\,. \qquad\qquad (10.24)$$

These formulae solve our problem.

The latter formula can also be expressed as

$$A_n^+ = \frac{\det \mathbf{E}}{E_{11}}\,, \qquad\qquad (10.25)$$

where $\det \mathbf{E}$ is the product of the determinants of matrices $\mathbf{T}_n^{-1}$, $\mathbf{A}$ and $\mathbf{T}_0$. Since $\det \mathbf{T}_n^{-1} = -1/(2i\mu_n s_n)$, $\det \mathbf{A} = 1$, $\det \mathbf{T}_0 = -2i\mu_0 s_0$,

$$\boxed{A_n^+ = \frac{\mu_0 s_0}{\mu_n s_n}\frac{1}{E_{11}}}\,. \qquad\qquad (10.26)$$

Other simple formulae for the reflection and transmission coefficients can be obtained from the inverse relation to (10.21):

$$\begin{pmatrix} A_0^- \\ 1 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\begin{pmatrix} 0 \\ A_n^+ \end{pmatrix}\,, \qquad\qquad (10.27)$$

where matrix

$$\mathbf{F} = \mathbf{E}^{-1} = \mathbf{T}_0^{-1}\mathbf{B}\mathbf{T}_n\,. \qquad\qquad (10.28)$$

Note that matrix $\mathbf{B} = \mathbf{A}^{-1}$ is given by (8.27) and (8.21), and matrix $\mathbf{T}_0^{-1}$ is similar to (10.7). Equation (10.27) yields the following very simple formulae

$$\boxed{A_n^+ = \frac{1}{F_{22}}\,, \qquad A_0^- = \frac{F_{12}}{F_{22}}}\,. \qquad\qquad (10.29)$$

It is easy to verify that formulae (10.23) to (10.25) are equivalent to formulae (10.29). Namely, according to (8.19), the inverse matrix to matrix $\mathbf{E}$ is

$$\mathbf{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \mathbf{E}^{-1} = \frac{1}{\det \mathbf{E}}\begin{pmatrix} E_{22} & -E_{12} \\ -E_{21} & E_{11} \end{pmatrix}\,. \qquad (10.30)$$

Thus,

$$F_{12} = -\frac{E_{12}}{\det \mathbf{E}}\,, \qquad F_{22} = \frac{E_{11}}{\det \mathbf{E}}\,.$$

By inserting these relations into (10.29) we obtain (10.25) and (10.23).

Although the above-mentioned formulae are equivalent theoretically, they may yield different numerical results. In particular, if total reflection occurs in the transition zone, the trigonometric functions in the corresponding layer matrix $\mathbf{a}_m$ are replaced by hyperbolic functions, and the elements of this matrix may become large in absolute value. Consequently, the resultant matrix $\mathbf{E}$ may also contain large elements. The computation of the transmission coefficient A_n^+ by means of (10.24) then requires the subtraction of large numbers, which may be accompanied by a considerable loss of significant figures. The computation of A_n^+ by means of (10.26) or (10.29) is more accurate.

10.3 Spectral Ratio of the Horizontal and Vertical Components of P Waves

Now we shall study a problem similar to that in Section 10.1, but for an incident P wave. We again assume that this wave is plane and harmonic.

Express the potentials of longitudinal and transverse waves in the half-space as

$$\varphi_n = \Phi_n^- + \Phi_n^+ , \quad \psi_n = \Psi_n^- + \Psi_n^+ , \tag{10.31}$$

where, according to (5.7) and (5.8),

$$\Phi_n^- = A_n^- e^{ikr_n(z-z_n)} , \quad \Phi_n^+ = A_n^+ e^{-ikr_n(z-z_n)} ,$$

$$\Psi_n^- = B_n^- e^{iks_n(z-z_n)} , \quad \Psi_n^+ = B_n^+ e^{-iks_n(z-z_n)} , \tag{10.32}$$

but we have omitted the common term $e^{i(\omega t-kx)}$ in these potentials. Here we have put

$$c = \frac{\alpha_n}{\sin \gamma_n} , \quad k = \frac{\omega}{c} , \quad r_n = \sqrt{(c/\alpha_n)^2 - 1} , \quad s_n = \sqrt{(c/\beta_n)^2 - 1} , \tag{10.33}$$

where ω is the angular frequency and γ_n the angle of incidence of the P wave.

Introduce the motion-stress vector in the half-space with elements (9.13). According to (5.11) and (5.13), this vector may be expressed in terms of potentials (10.32) as

$$\begin{pmatrix} iu_n/k \\ -w_n/k \\ \sigma_n/\omega^2 \\ i\tau_n/\omega^2 \end{pmatrix} = \mathbf{T}_n \begin{pmatrix} \Phi_n^- \\ \Psi_n^- \\ \Phi_n^+ \\ \Psi_n^+ \end{pmatrix} , \tag{10.34}$$

where $\sigma_n = (\tau_{zz})_n$, $\tau_n = (\tau_{zx})_n$, and matrix $\mathbf{T}_n$ is

$$
\mathbf{T}_n = \begin{pmatrix} 1 & s_n & 1 & -s_n \\ -ir_n & i & ir_n & i \\ \rho_n \delta_n & \rho_n \gamma_n s_n & \rho_n \delta_n & -\rho_n \gamma_n s_n \\ i\rho_n \gamma_n r_n & -i\rho_n \delta_n & -i\rho_n \gamma_n r_n & -i\rho_n \delta_n \end{pmatrix} .
\tag{10.35}
$$

The inverse matrix is (Proskuryakova et al., 1981)

$$
\mathbf{T}_n^{-1} = \frac{1}{2} \begin{pmatrix} \gamma_n & -i\delta_n r_n^{-1} & -\rho_n^{-1} & -i\rho_n^{-1} r_n^{-1} \\ -\delta_n s_n^{-1} & -i\gamma_n & \rho_n^{-1} s_n^{-1} & -i\rho_n^{-1} \\ \gamma_n & i\delta_n r_n^{-1} & -\rho_n^{-1} & i\rho_n^{-1} r_n^{-1} \\ \delta_n s_n^{-1} & -i\gamma_n & -\rho_n^{-1} s_n^{-1} & -i\rho_n^{-1} \end{pmatrix} .
\tag{10.36}
$$

The inverse relation to (10.34) at the top of the half-space $\left(z = z_n\right)$ can be expressed as

$$
\begin{pmatrix} A_n^- \\ B_n^- \\ A_n^+ \\ B_n^+ \end{pmatrix} = \mathbf{T}_n^{-1} \begin{pmatrix} iu_n/k \\ -w_n/k \\ \sigma_n/\omega^2 \\ i\tau_n/\omega^2 \end{pmatrix}_{z_n} = \underbrace{\mathbf{T}_n^{-1}\mathbf{A}}_{\mathbf{D}} \begin{pmatrix} iu_1/k \\ -w_1/k \\ \sigma_1/\omega^2 \\ i\tau_1/\omega^2 \end{pmatrix}_{z_1} ,
\tag{10.37}
$$

where we have used (9.16).

Assuming that only a P wave is incident in the half-space, i.e. no S wave propagates upwards in the half-space, we put $B_n^- = 0$. Moreover, we again assume the surface of the medium to be free, i.e. $\sigma_1 = \tau_1 = 0$ for $z = z_1$. Equation (10.37) then takes the form

$$
\begin{pmatrix} A_n^- \\ 0 \\ A_n^+ \\ B_n^+ \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \begin{pmatrix} iu_0/k \\ -w_0/k \\ 0 \\ 0 \end{pmatrix} ,
\tag{10.38}
$$

where we have put $u_0 = \left(u_1\right)_{z_1}$ and $w_0 = \left(w_1\right)_{z_1}$. For a given angular frequency ω, angle of incidence γ_n and amplitude A_n^- for the potential of the incident P wave, Eq. (10.38) represents four scalar equations in the unknown displacements at the surface, u_0 and w_0, and the amplitudes of the reflected P and SV waves in the half-space, A_n^+ and B_n^+. This solves the problem which is analogous to that in Section 10.1.

However, we can solve yet another problem, when the angle of incidence is again known, but the amplitude of the incident wave is unknown. To eliminate the unknown A_n^-, consider only the second scalar equation in (10.38):

$$0 = D_{21}iu_0 - D_{22}w_0 \ . \qquad\qquad (10.39)$$

This yields the following simple formula for the ratio of the horizontal and vertical displacements at the free surface:

$$\boxed{\frac{iu_0}{w_0} = \frac{D_{22}}{D_{21}}} \ . \qquad\qquad (10.40)$$

This ratio depends on ω, γ_n and the parameters of the medium, but not on the amplitude of the incident wave.

Many authors have used this approach to study the structure of the Earth's crust under seismic stations. If the epicentral distance and depth of an earthquake are known, the angle of incidence γ_n below the seismic station can be estimated from the travel-time curves (Richter, 1958). For a chosen model, ratio (10.40) can then be computed as a function of ω, and compared with the measured spectral ratio. The parameters of the model are then modified unless a satisfactory fit with observations is achieved. This enables us to determine a layered structure under a seismic station from the observed spectral ratio.

Note that deep earthquakes have been recommended for this purpose, in order to reduce the effects of possible layered structures in the vicinity of the source. The same method, using seismic sources in boreholes, may be applied to study shallow structures in seismic prospecting.

10.4 Reflection and Transmission Coefficients of P and SV Waves for a Transition Zone

The reflection and transmission of P and SV waves in layered structures has been studied, e.g., by Thomson (1950), Molotkov (1961), Ratnikova and Levshin (1967), Fuchs (1968), Cerveny (1974), Kind (1976), Kennett (1983), Luco and Apsel (1983), Molotkov (1984), Chen (1993).
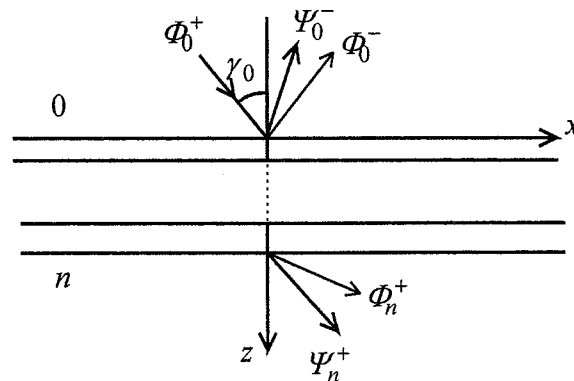

Fig. 10.3. Reflection and transmission of a P wave at the transition zone.

In this section we shall solve a problem similar to that in Section 10.2, but for an incident P wave. As opposed to Fig. 10.2, the incident P wave generates

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two reflected waves in the upper half-space, and two analogous transmitted waves in the lower half-space (Fig. 10.3).

In addition to potentials (10.31) and (10.32) for the lower half-space, denote the analogous potentials of longitudinal and transverse waves in the upper half-space by

$$\varphi_0 = \Phi_0^- + \Phi_0^+ \ , \quad \psi_0 = \Psi_0^- + \Psi_0^+ \ , \qquad (10.41)$$

where

$$\Phi_0^- = A_0^- e^{ik r_0 z} \ , \quad \Phi_0^+ = A_0^+ e^{-ik r_0 z} \ ,$$

$$\Psi_0^- = B_0^- e^{ik s_0 z} \ , \quad \Psi_0^+ = B_0^+ e^{-ik s_0 z} \ . \qquad (10.42)$$

Since we now assume that the incident wave propagates in the upper half-space, subscripts n in (10.33) must be replaced by subscripts 0:

$$c = \frac{\alpha_0}{\sin\gamma_0} \ , \quad k = \frac{\omega}{c} \ , \quad r_0 = \sqrt{\left(c/\alpha_0\right)^2 - 1} \ , \quad s_0 = \sqrt{\left(c/\beta_0\right)^2 - 1} \ , \quad (10.43)$$

where ω is again the angular frequency and γ_0 the angle of incidence of the P wave.

The motion-stress vector in the upper half-space is analogous to (10.34). At the boundary of this half-space, i.e. at depth $z = z_1 = 0$, it takes the form

$$\begin{pmatrix} iu_0/k \\ -w_0/k \\ \sigma_0/\omega^2 \\ i\tau_0/\omega^2 \end{pmatrix}_{z_1} = \mathbf{T}_0 \begin{pmatrix} A_0^- \\ B_0^- \\ A_0^+ \\ B_0^+ \end{pmatrix} , \qquad (10.44)$$

where

$$\mathbf{T}_0 = \begin{pmatrix} 1 & s_0 & 1 & -s_0 \\ -ir_0 & i & ir_0 & i \\ \rho_0\delta_0 & \rho_0\gamma_0 s_0 & \rho_0\delta_0 & -\rho_0\gamma_0 s_0 \\ i\rho_0\gamma_0 r_0 & -i\rho_0\delta_0 & -i\rho_0\gamma_0 r_0 & -i\rho_0\delta_0 \end{pmatrix} . \qquad (10.45)$$

Considering the continuity of the motion-stress vector at depth z_1 and inserting (10.44) into (10.37), we obtain the following relation between the amplitudes in the lower and upper half-spaces:

$$\begin{pmatrix} A_n^- \\ B_n^- \\ A_n^+ \\ B_n^+ \end{pmatrix} = \mathbf{E} \begin{pmatrix} A_0^- \\ B_0^- \\ A_0^+ \\ B_0^+ \end{pmatrix} , \qquad (10.46)$$

where matrix $\mathbf{E}$ is

$$\mathbf{E} = \mathbf{T}_n^{-1}\mathbf{A}\mathbf{T}_0 , \tag{10.47}$$

$\mathbf{T}_n^{-1}$ being given by (10.36).

Assuming that only a P wave is incident in the upper half-space, we put its amplitude $A_0^+ = 1$, and $B_0^+ = 0$ for an incident SV wave. Moreover, we assume that no waves propagate upwards in the lower half-space, i.e. $A_n^- = B_n^- = 0$. Equation (10.46) then becomes

$$\begin{pmatrix} 0 \\ 0 \\ A_n^+ \\ B_n^+ \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix} \begin{pmatrix} A_0^- \\ B_0^- \\ 1 \\ 0 \end{pmatrix} . \tag{10.48}$$

This represents four scalar equations:

$$\begin{aligned}
0 &= E_{11}A_0^- + E_{12}B_0^- + E_{13} , \\
0 &= E_{21}A_0^- + E_{22}B_0^- + E_{23} , \\
A_n^+ &= E_{31}A_0^- + E_{32}B_0^- + E_{33} , \\
B_n^+ &= E_{41}A_0^- + E_{42}B_0^- + E_{43} .
\end{aligned} \tag{10.49}$$

The first two equations yield the reflection coefficients for P and SV waves:

$$A_0^- = \frac{E_{12}E_{23} - E_{13}E_{22}}{E_{11}E_{22} - E_{12}E_{21}} = \frac{E\left|\begin{smallmatrix} 12 \\ 23 \end{smallmatrix}\right.}{E\left|\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right.} , \quad B_0^- = -\frac{E\left|\begin{smallmatrix} 12 \\ 13 \end{smallmatrix}\right.}{E\left|\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right.} , \tag{10.50}$$

where $E\left|\begin{smallmatrix} ij \\ kl \end{smallmatrix}\right.$ denotes the second-order minor of matrix $\mathbf{E}$ containing rows i, j, and columns k, l; see notation (7.25). The transmission coefficients, after substituting (10.50) into (10.49), can be expressed as

$$A_n^+ = \frac{E\left|\begin{smallmatrix} 123 \\ 123 \end{smallmatrix}\right.}{E\left|\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right.} , \quad B_n^+ = \frac{E\left|\begin{smallmatrix} 124 \\ 123 \end{smallmatrix}\right.}{E\left|\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right.} , \tag{10.51}$$

where $E\left|\begin{smallmatrix} ijk \\ lmn \end{smallmatrix}\right.$ is the third-order minor of matrix $\mathbf{E}$ composed of rows i, j, k, and columns l, m, n.

Since the reflection coefficients are expressed in terms of the second-order minors of matrix $\mathbf{E}$, they can be computed effectively by means of the corresponding delta matrix,

$$\widetilde{\mathbf{E}} = \widetilde{\mathbf{T}}_n^{-1} \widetilde{\mathbf{A}} \widetilde{\mathbf{T}}_0 , \qquad (10.52)$$

where delta matrix $\widetilde{\mathbf{A}}$ is described in Subsection 9.5.2, and delta matrices $\widetilde{\mathbf{T}}_n^{-1}$ and $\widetilde{\mathbf{T}}_0$ can be derived from matrices (10.36) and (10.45) in a similar way.

The formulae for the transmission coefficients are more complicated. The computation of the corresponding third-order minors was analysed in detail, e.g., by Cerveny (1974). Some other formulations of this problem can be found in the papers and books mentioned at the beginning of this section. Nevertheless, all these approaches yield rather complicated formulae. However, in the previous chapters we have found several times that some problems can be formulated effectively in terms of inverse matrices, i.e. by multiplying the corresponding matrices from the lower half-space upwards.

Thus, let us apply inverse matrices also to our problem. The inverse relation to (10.48) is

$$
\begin{pmatrix} A_0^- \\ B_0^- \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ A_n^+ \\ B_n^+ \end{pmatrix} , \qquad (10.53)
$$

where matrix

$$\mathbf{F} = \mathbf{E}^{-1} = \mathbf{T}_0^{-1} \mathbf{A}^{-1} \mathbf{T}_n = \mathbf{T}_0^{-1} \mathbf{a}_1^{-1} \ldots \mathbf{a}_{n-1}^{-1} \mathbf{T}_n ; \qquad (10.54)$$

note that matrix $\mathbf{T}_0^{-1}$ is analogous to (10.36), and matrices $\mathbf{a}_m^{-1}$ can be derived from matrices $\mathbf{a}_m$, given by (9.15). The last two equations in (10.53) yield the transmission coefficients in the form

$$A_n^+ = \frac{F_{44}}{F\left|\begin{smallmatrix} 34 \\ 34 \end{smallmatrix}\right.} , \quad B_n^+ = -\frac{F_{43}}{F\left|\begin{smallmatrix} 34 \\ 34 \end{smallmatrix}\right.} . \qquad (10.55)$$

For the reflection coefficients we then obtain

$$A_0^- = \frac{F\left|\begin{smallmatrix} 14 \\ 34 \end{smallmatrix}\right.}{F\left|\begin{smallmatrix} 34 \\ 34 \end{smallmatrix}\right.} , \quad B_0^- = \frac{F\left|\begin{smallmatrix} 24 \\ 34 \end{smallmatrix}\right.}{F\left|\begin{smallmatrix} 34 \\ 34 \end{smallmatrix}\right.} . \qquad (10.56)$$

Formulae (10.55) simplify considerably the computation of the transmission coefficients in comparison with formulae (10.51). It is now sufficient to compute matrix $\mathbf{F}$ and the corresponding delta matrix $\widetilde{\mathbf{F}}$. The convenient

formulae (10.55) and (10.56) for the transmission and reflection coefficients have been derived here for the first time.

The derivation of the reflection and transmission coefficients for an incident *SV* wave would be similar. Only the horizontal velocity in (10.43) would be given by $c = \beta_0/\sin\gamma_0$, and in (10.46) we would put $A_0^+ = 0$ and $B_0^+ = 1$.

Note that the method described above cannot be applied to the important case of normal incidence, $\gamma_0 = 0$, because velocity c becomes infinite, as well as some elements of the layer matrices and of matrix $\mathbf{T}_0$. Therefore, the problem should be reformulated to include the normal incidence, too.

10.5 Some Other Studies

In these lecture notes we were unable to solve many other theoretical problems, such as the propagation of elastic waves generated by a point source in a layered medium (Ewing et al., 1957; Takeuchi and Saito, 1972; Brekhovskikh, 1973; Aki and Richards, 1980; Ben-Menahem and Singh, 1981; Kennett, 1983; Molotkov, 1984), computation of the partial derivatives of the phase and group velocities with respect to the parameters of the medium (Brune and Dorman, 1963; Novotny, 1970; Rodi et al., 1975; Urban et al., 1993), computation of synthetic seismograms using the reflectivity method (Fuchs and Müller, 1971; Kind, 1978; Kind and Odom, 1983) and with the modal summation method (Kennett, 1983; Panza, 1985; Panza and Suhadolc, 1987; Florsch et al., 1991), surface waves in general vertically inhomogeneous media (Levshin, 1973), surface waves in anisotropic media (Crampin, 1970; Crampin and Taylor, 1971; Martin and Thomson, 1997; Thomson, 1997), or surface waves in laterally inhomogeneous media (Keilis-Borok, 1989).

Chapter 11

Wave Propagation in Dispersive Media

In this chapter we shall study the interference of waves in several special cases, and then derive the basic formulae for determining the phase and group velocities from observed waves.

11.1 Superposition of Two Plane Harmonic Waves in a Non-Dispersive Medium

Consider two plane harmonic waves of angular frequencies ω_1 and ω_2 propagating along the Cartesian axis x at velocity c which is the same for both waves. Denote by u_1 and u_2 the displacements or other parameters of these waves (particle velocities, accelerations, stresses, electrical intensities, etc.). For simplicity, assume both waves to be of the same amplitude, A. Then, for example,

$$u_1 = A \sin\left[\omega_1(t - x/c)\right], \quad u_2 = A \sin\left[\omega_2(t - x/c)\right] . \tag{11.1}$$

Since

$$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2} , \tag{11.2}$$

the superposition of the two waves may by expressed as

$$u = u_1 + u_2 = 2A \sin\left[\frac{\omega_1 + \omega_2}{2}\left(t - \frac{x}{c}\right)\right]\cos\left[\frac{\omega_1 - \omega_2}{2}\left(t - \frac{x}{c}\right)\right] . \tag{11.3}$$

Assuming the frequencies to be close to each other, i.e.

$$\omega_1 = \omega + \Delta\omega , \quad \omega_2 = \omega - \Delta\omega , \tag{11.4}$$

one gets

$$u = 2A \underbrace{\sin\left[\omega(t - x/c)\right]}_{\text{"carrier wave"}}\underbrace{\cos\left[\Delta\omega(t - x/c)\right]}_{\substack{\text{amplitude-modulated} \\ \text{travelling wave}}} . \tag{11.5}$$

The resulting wave has the character of biases. In this case, the carrier wave and the modulating wave propagate at the same velocity, c.

11.2 Superposition of Two Plane Harmonic Waves in a Dispersive Medium

Assume the velocities of the two waves to be different, and denote them by c_1 and c_2, respectively. Instead of (11.1) we now have

$$u_1 = A\sin\left[\omega_1\left(t - x/c_1\right)\right], \quad u_2 = A\sin\left[\omega_2\left(t - x/c_2\right)\right]. \tag{11.6}$$

Introducing the wave numbers

$$k_1 = \omega_1/c_1, \quad k_2 = \omega_2/c_2, \tag{11.7}$$

the individual waves can be expressed as

$$u_1 = A\sin\left(\omega_1 t - k_1 x\right), \quad u_2 = A\sin\left(\omega_2 t - k_2 x\right), \tag{11.8}$$

and the resulting wave is

$$u = u_1 + u_2 = 2A\sin\left(\frac{\omega_1 + \omega_2}{2}t - \frac{k_1 + k_2}{2}x\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t - \frac{k_1 - k_2}{2}x\right). \tag{11.9}$$

Again assume close frequencies and wave numbers:

$$\omega_1 = \omega + \Delta\omega, \quad k_1 = k + \Delta k,$$

$$\omega_2 = \omega - \Delta\omega, \quad k_2 = k - \Delta k. \tag{11.10}$$

We then obtain

$$u = 2A\sin\left(\omega t - kx\right)\cos\left(\Delta\omega\, t - \Delta k\, x\right) = 2A\underbrace{\cos\left[\Delta\omega\left(t - \frac{\Delta k}{\Delta\omega}x\right)\right]}_{\text{envelope}}\sin\left(\omega t - kx\right).$$

$$\tag{11.11}$$

The envelope propagates at velocity

$$U = \frac{\Delta\omega}{\Delta k}. \tag{11.12}$$

In the limiting case for small $\Delta\omega$ and Δk, we arrive at the important formula

$$\boxed{U = \frac{d\omega}{dk}}. \tag{11.13}$$

Velocity U is called the *group velocity*. In our case, this is the velocity of propagation of the envelope. This velocity is generally different from velocity $c = \omega/k$, which is called the *phase velocity*.

We can summarise these results as follows: The individual peaks and troughs of the resulting wave propagate at the phase velocity, whereas their envelope propagates at the group velocity.

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Hence, the propagation of waves in a dispersive medium cannot be described by one velocity only, but we must use two velocities. Generally, the *group velocity* is the velocity of propagation of *energy* (Brillouin, 1960).

Let us add several other formulae for the group velocity. Inserting $\omega = kc$ and $k = 2\pi/\lambda$ into (11.13), we get

$$U = c + k\frac{\mathrm{d}c}{\mathrm{d}k} = c - \lambda\frac{\mathrm{d}c}{\mathrm{d}\lambda} \; , \qquad (11.14)$$

where $\lambda = 2\pi/k$ is the wavelength. Using the reciprocal relation to (11.13),

$$\frac{1}{U} = \frac{\mathrm{d}k}{\mathrm{d}\omega} \; , \qquad (11.15)$$

and inserting $k = \omega/c$, we obtain other important formulae:

$$U = \frac{c}{1 - \dfrac{\omega}{c}\dfrac{\mathrm{d}c}{\mathrm{d}\omega}} = \frac{c}{1 - \dfrac{f}{c}\dfrac{\mathrm{d}c}{\mathrm{d}f}} = \frac{c}{1 + \dfrac{T}{c}\dfrac{\mathrm{d}c}{\mathrm{d}T}} \; , \qquad (11.16)$$

where $T = 2\pi/\omega$ is the period and $f = 1/T$ the frequency. These formulae can be used to determine the dispersion curve of the group velocity if the dispersion curve of the phase velocity is known.

11.3 Propagation of a Plane Wave with a Narrow Spectrum

Again consider the propagation of plane waves along the x-axis. Let $f(t)$ be the time function of the corresponding wave characteristics (displacement, pressure, etc.) at the origin $x = 0$. Under certain general conditions, this function may be expressed as the Fourier integral,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) e^{i\omega t}\, \mathrm{d}\omega \; , \qquad (11.17)$$

where spectrum $S(\omega)$ is defined by

$$S(\omega) = \int_{-\infty}^{+\infty} f(\tau) e^{-i\omega\tau}\, \mathrm{d}\tau \; . \qquad (11.18)$$

If function $f(t)$ is real, which we shall assume here, its spectrum for negative ω is complex conjugate, i.e. $S(-\omega) = \bar{S}(\omega)$. Function (11.17) can then be expressed as

$$f(t) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty S(\omega) e^{i\omega t} \, \mathrm{d}\omega \ . \tag{11.19}$$

This represents the decomposition of function f into harmonic components. Each of these components propagates at its own velocity $c = c(\omega)$, and at distance $x \neq 0$ takes the form

$$S(x, \omega) = S(\omega) e^{i\omega(t - x/c)} \ .$$

We shall assume that the wave propagation obeys linear equations, i.e. the individual harmonic components propagate independently of each other. The superposition principle then holds true, and the resulting wave is given by the summation (integration) of the individual components:

$$f(x, t) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty S(\omega) e^{i\omega(t - x/c)} \, \mathrm{d}\omega = \frac{1}{\pi} \operatorname{Re} \int_0^\infty S(\omega) e^{i(\omega t - kx)} \, \mathrm{d}\omega \ , \tag{11.20}$$

$k = \omega/c$ being the wave number.

If the medium is non-dispersive, i.e. $c = \text{const.}$, it follows from (11.20) that the shape of the wave does not change during propagation:

$$f(x, t) = f(t - x/c) \ . \tag{11.21}$$

11.3.1 Form of a wave with a narrow spectrum

Now let us consider wave propagation in a dispersive medium, assuming the signal under consideration to have a non-zero spectrum only in a narrow neighbourhood of angular frequency ω_0. Wave (11.20) can then be expressed as

$$f(x, t) = \frac{1}{\pi} \operatorname{Re} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} S(\omega) e^{i(\omega t - kx)} \, \mathrm{d}\omega \ . \tag{11.22}$$

Replace the wave number by the first two terms of its Taylor series in the neighbourhood of ω_0:

$$k(\omega) = k(\omega_0) + \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)_{\omega_0} (\omega - \omega_0) + \dots \ . \tag{11.23}$$

For the sake of brevity, denote $k(\omega_0)$ by k_0, and derivative $(\mathrm{d}k/\mathrm{d}\omega)_{\omega_0}$ by $1/U_0$. Moreover, assume that spectrum $S(\omega)$ varies only slowly in the

neighbourhood of ω_0, so that it may be approximated by $S(\omega_0)$. Function (11.22) can then be expressed approximately as

$$f(x,t) \doteq \frac{1}{\pi} \operatorname{Re}\left[S(\omega_0) e^{i(\omega_0 t - k_0 x)} \int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} e^{i(\omega - \omega_0)(t - x/U_0)} \, d\omega \right]. \quad (11.24)$$

Denote the integral in (11.24) by A. Since ω_0 and U_0 are constant on integrating over ω, this integral can be easily calculated. We obtain

$$A = 2\frac{\sin[\Delta\omega(t - x/U_0)]}{t - x/U_0} = 2\Delta\omega \operatorname{sinc}\left[\Delta\omega\left(t - \frac{x}{U_0}\right)\right], \quad (11.25)$$

where the function sinc is defined as

$$\operatorname{sinc} x = \frac{\sin x}{x}. \quad (11.26)$$

This function is frequently used in the theory of signal propagation. Note that $\operatorname{sinc} 0 = 1$, and for $|x| > 0$ this function is oscillating with decreasing amplitudes.

Thus, the resulting wave (11.24) represents a carrier harmonic wave of angular frequency ω_0 which is modulated by function sinc. The carrier wave propagates at velocity $c_0 = \omega_0/k_0$, whereas its envelope propagates at velocity U_0.

Note that instead of the Fourier integral in form (11.19) we could use the general form (11.17). However, if a spectrum is non-zero in the neighbourhood of ω_0, it is also non-zero in the corresponding neighbourhood of $(-\omega_0)$. Consequently, we would have to consider two integrals, namely one from the vicinity of ω_0 and the other from the vicinity of $(-\omega_0)$. The sum of these integrals would yield the same result as (11.24).

11.3.2 Simple methods of determining the phase and group velocities from observations

As a practical result of the theoretical considerations given above, we can propose suitable methods for determining the phase and group velocity from observations. Therefore, assume that a source of waves, located at the origin $x = 0$, has generated a wave with a narrow spectrum. (Note that this case is not typical of seismic waves, because earthquakes usually generate seismic waves with broad spectra; see below).

The phase velocity, as defined above, is the velocity with which the individual peaks propagate. However, the amplitudes of the individual peaks vary during the propagation. Therefore, we must observe the wave at two

distances x_1 and x_2, and correlate their records, i.e. for each peak on one record we must find the corresponding peak on the second record. Denote the arrival times of a selected peak at the two distances by t_1 and t_2, respectively. The phase velocity can then be determined as

$$c(\omega_0) = \frac{x_2 - x_1}{t_2 - t_1} \,.$$ (11.27)

The determination of the group velocity is even simpler, because only one point of observation is required. Namely, function (11.24) attains its maximum values if the argument of function sinc is zero, which yields

$$U(\omega_0) = \frac{x}{t} \,.$$ (11.28)

Thus, in order to apply this simple formula to determine the group velocity, we must know epicentral distance x, and determine travel time t when the maximum amplitudes appear on the record.

The simple formulae (11.27) and (11.28) for the practical determination of the phase and group velocities follow from expressions (11.24) and (11.25), which are valid only for waves with narrow spectra. Nevertheless, we shall see below that formulae (11.27) and (11.28) have a more general validity.

11.4 Propagation of a Plane Wave with a Broad Spectrum

In the previous section we were able to estimate the wave-form, given by integral (11.20), analytically assuming the spectrum to be concentrated in a narrow frequency interval. Now we shall consider the opposite extreme when the spectrum is very broad. This case describes the typical situations in seismology better, because the motions at a seismic source are usually of short duration and, consequently, display a broad spectrum. We shall show that, under certain assumptions, it will again be possible to derive an approximate analytical expression for the corresponding integral (11.20). In particular, if either the distance x, or time t is large, the estimate of integral (11.20) can be obtained by applying the asymptotic *method of stationary phase*.

11.4.1 Asymptotic expressions for large distances

Let us perform the asymptotic estimation of integral (11.20) for large x. For this purpose, express this integral as

$$f(x,t) = \frac{1}{\pi} \text{Re} \int_0^\infty S(\omega) e^{ix\Phi} \, d\omega \,,$$ (11.29)

where

$$\Phi = \frac{\omega t}{x} - k \ . \tag{11.30}$$

Assume that functions $S(\omega)$ and $\Phi(\omega)$ do not vary very rapidly with ω, i.e. $S(\omega)$ is approximately constant, and $\Phi(\omega)$ is approximately a linear function of ω. For large x, the integrand in (11.29) is then a rapidly oscillating function, whose amplitude varies only slowly with ω. Consequently, the contribution to the integral from one half-period of this function is nearly compensated by the contribution of the opposite sign from the next half-period, etc. These contributions approximately cancel each other and, therefore, may be neglected. This compensation is not sufficiently complete only at places where the above-mentioned assumptions are not satisfied.

Consider the Taylor expansion of function Φ in the neighbourhood of angular frequently ω_0:

$$\Phi(\omega) = \Phi(\omega_0) + \Phi'(\omega_0)(\omega - \omega_0) + \frac{1}{2}\Phi''(\omega_0)(\omega - \omega_0)^2 + \dots \ , \tag{11.31}$$

where

$$\Phi'(\omega) = \frac{d\Phi}{d\omega} = \frac{t}{x} - \frac{dk}{d\omega} = \frac{t}{x} - \frac{1}{U(\omega)} \ , \tag{11.32}$$

$$\Phi''(\omega) = -\frac{d^2 k}{d\omega^2} = \frac{1}{U^2}\frac{dU}{d\omega} \ , \tag{11.33}$$

U being the group velocity, defined by (11.13). The approximately linear character of function Φ, assumed above, is perturbed at places where the linear term in (11.31) is not large enough in comparison with the higher-order terms. The largest deviation from linearity occurs at points where this linear term vanishes, i.e. where the first derivative of Φ is zero. These points are referred to as *points of stationary phase*.

Assume that function Φ, for given values of x and t, has only one point of stationary phase, and specify expansion (11.31) for this point. Then $\Phi'(\omega_0) = 0$, and (11.32) yields

$$\boxed{U(\omega_0) = \frac{x}{t}} \ ; \tag{11.34}$$

see the formally identical formula (11.28). Hence, we have arrived at a very simple and important formula for estimating the group velocity. Namely, according to (11.34), the ratio of epicentral distance x to travel time t yields the group velocity for the angular frequency of the stationary phase, $U(\omega_0)$. However, we do not know yet, how to determine this frequency from the seismogram. Let us deal with this problem.

A significant contribution to integral (11.29) comes only from the neighbourhood of stationary point ω_0, where expansion (11.31) takes the form

$$\Phi(\omega) = \Phi(\omega_0) + \frac{1}{2}\Phi''(\omega_0)(\omega - \omega_0)^2 + \dots . \qquad (11.35)$$

Neglecting the higher-order terms in this expansion, and inserting it into (11.29), one gets approximately

$$f(x,t) \doteq \frac{1}{\pi}\text{Re}\left[S(\omega_0)e^{i(\omega_0 t - k_0 x)}\int\limits_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} e^{i\frac{1}{2}x\Phi''(\omega_0)(\omega - \omega_0)^2}\,d\omega\right], \qquad (11.36)$$

where $k_0 = k(\omega_0)$ and $(\omega_0 - \Delta\omega, \omega_0 + \Delta\omega)$ is the interval where expansion (11.35) can be used. Using the substitution

$$u = (\omega - \omega_0)\sqrt{\frac{1}{2}x|\Phi''(\omega_0)|} , \qquad (11.37)$$

we get

$$f(x,t) \doteq \frac{1}{\pi}\sqrt{\frac{2}{x|\Phi''(\omega_0)|}}\,\text{Re}\left[S(\omega_0)e^{i(\omega_0 t - k_0 x)}\int\limits_{-\Delta u}^{+\Delta u} e^{\pm iu^2}\,du\right], \qquad (11.38)$$

where

$$\Delta u = \Delta\omega\sqrt{\frac{1}{2}x|\Phi''(\omega_0)|} , \qquad (11.39)$$

and the positive sign in the argument of the exponential applies if $\Phi''(\omega_0) > 0$, and the negative if $\Phi''(\omega_0) < 0$. If epicentral distance x is large, integration limit Δu is also large (assuming $\Phi''(\omega_0) \neq 0$). For large u, the integrand in (11.38) is a rapidly oscillating function. Thus, the corresponding integral will not change significantly if the integration limits are extended to $\pm\infty$. This yields Poisson's integral

$$\int\limits_{-\infty}^{+\infty} e^{\pm iu^2}\,du = \sqrt{\pi}e^{\pm i\frac{\pi}{4}} . \qquad (11.40)$$

The derivation of this integral may be found in the textbooks of mathematical analysis. Using (11.33), we finally arrive at

$$f(x,t) \doteq \sqrt{\frac{2}{\pi x\left|\dfrac{1}{U^2}\dfrac{dU}{d\omega}\right|_{\omega_0}}}\,\text{Re}\left[S(\omega_0)e^{i\left(\omega_0 t - k_0 x \pm \frac{\pi}{4}\right)}\right]. \qquad (11.41)$$

11.4.2 Properties of the asymptotic solution

The expression in the square brackets in (11.41) represents a harmonic wave of angular frequency ω_0. Thus, we have arrived at the result that, at large distances x from the source and at any time t, the wave may be approximated by a harmonic wave. When we proceed to another time t (or another distance x), angular frequency ω_0 will change. Thus, the seismogram at a large distance has the character of a quasi-harmonic wave whose period gradually varies. This agrees rather well with observations; see the examples of seismograms in the next chapter.

The amplitude of the wave also varies, which is described by the square root in (11.41). This formula indicates that large amplitudes may be expected at periods which are close to the extremes of the group-velocity dispersion curve, since $dU/d\omega = 0$ there. However, formula (11.41) itself cannot be used at these points, because it yields infinite amplitudes.

At these extremes of the group velocity, not only $\Phi'(\omega_0) = 0$, but also $\Phi''(\omega_0) = 0$. Consequently, the next term, containing the third derivative $\Phi'''(\omega_0)$, must be taken into account in (11.35). An analytical estimate of integral (11.29) can be obtained even in this more complicated case, but the resulting formula contains the Airy function. The large-amplitude waves corresponding to these extremes are thus referred to as Airy phases.

Yet another property should be mentioned. Namely, as the wave packet extends in time during propagation, its amplitudes must diminish. This is expressed by the factor $1/\sqrt{x}$ in (11.41). This factor appears here although only the propagation of plane waves is being considered. Note that further factors leading to a reduction of amplitudes with distance, which we do not consider here, are the geometrical spreading of the wave-front, reflections and attenuation.

The asymptotic solution (11.41) allows us to derive the classical, graphico-numerical methods of determining group and phase velocities from observations; see the following Sections 11.5 and 11.6.

11.5 The Peak and Trough Technique for Estimating Group Velocities from Observations

We have already derived the simple formula (11.34) for determining the group velocity. Now we already know that ω_0 represents the angular velocity of the harmonic wave which approximates the seismogram at the chosen time. This means that ω_0 represents the instantaneous angular frequency of the dispersed wave train. Thus, by determining the travel time and instantaneous period for a selected point on the seismogram, we obtain one point of the group-velocity dispersion curve.

The practical procedure, referred to as the *peak and trough technique* (Ewing and Press, 1952), may be as follows. After smoothing the seismogram,

in order to suppress the noise and other waves which perturb the wave train, we measure the travel times of the peaks, troughs, or zero-crossings. The successive peaks, troughs, or zero-crossings are indexed and their arrival times are plotted as a function of index number. Periods (half-periods) are then estimated from the slope of this curve.

Note that it is usually difficult to determine the arrival times of the peaks and troughs accurately. Since the times of the zero-crossings are usually better defined, these times are used more frequently.

Since time t in formula (11.34) is the travel time between source and receiver, the dispersion curve determined in this way characterises the structure of the medium between these points. In other words, by interpreting such dispersion curve we obtain a *mean model* of the medium *between source and receiver*.

Examples of group-velocity dispersion curves, obtained by this simple method, will be given in Chapter 12.

11.6 The Peak and Trough Technique for Estimating Phase Velocities from Observations

Denote by $A(\omega_0)$ the amplitude and by $\varphi(\omega_0)$ the phase of the complex number $S(\omega_0)$ in (11.41):

$$S(\omega_0) = A(\omega_0)e^{i\varphi(\omega_0)} .$$
(11.42)

The complex function in the square brackets of (11.41) can then be expressed as

$$A(\omega_0)e^{i\omega_0\left[t-t_0-x/c(\omega_0)\right]} ,$$
(11.43)

where $c(\omega_0) = \omega_0/k_0$ is the phase velocity for angular frequency ω_0, and

$$t_0 = -\frac{1}{\omega_0}\left[\varphi(\omega_0) \pm \frac{\pi}{4}\right]$$
(11.44)

is the time shift. According to (11.43), the phase velocity can be estimated by using the formula

$$c(\omega_0) = \frac{x}{t - t_0} .$$
(11.45)

However, in this case the phase shift at the source, $\varphi(\omega_0)$, must be known. This shift can be computed if the earthquake mechanism is known, but this information is available only for some strong earthquakes.

Therefore, the usual method of determining the phase velocity is not based on observations at one station, but seismograms of two stations are used. These

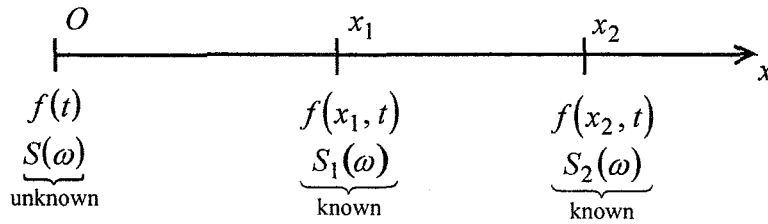stations must be located along a profile which passes through the epicentre (Fig. 11.1).

Fig. 11.1. Position of the source, O, and two seismic stations.

Let the seismograms at the source, first station and second station be described by functions $f(t)$, $f(x_1, t)$ and $f(x_2, t)$, respectively. The seismograms at the stations are schematically shown in Fig. 11.2.
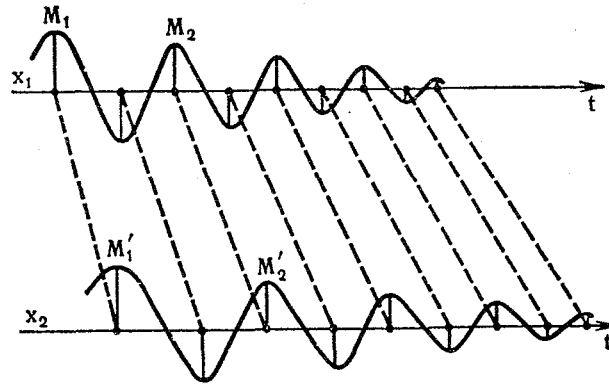

Fig. 11.2 Seismograms at two epicentral distances x_1 and x_2, where $x_2 > x_1$. (After Savarensky (1975)).

Consider, e.g., peak M_1 on the first seismogram and the corresponding peak M_1' on the second seismogram. Denote the instantaneous periods at these peaks by T_1 and T_1', respectively. As opposed to the case in Section 11.3, these periods are now slightly different $(T_1' > T_1)$. According to (11.41) and (11.42), the condition of the same phases at points M_1 and M_1' can be expressed as

$$\varphi(\omega_1) + \omega_1\left[t_1 - x_1/c(\omega_1)\right] = \varphi(\omega_1') + \omega_1'\left[t_1' - x_2/c(\omega_1)\right] . \qquad (11.46)$$

where we have omitted identical factors $\pm \pi/4$ on both sides, $\varphi(\omega_1)$ and $\varphi(\omega_1')$ are the phase shifts at the source, $t_1 = t(M_1)$ and $t_1' = t(M_1')$ the instants of the corresponding peaks, $\omega_1 = 2\pi/T_1$, and $\omega_1' = 2\pi/T_1'$. If the distance between the stations is relatively small (not exceeding several wavelengths), we may expect angular frequencies ω_1, ω_1', and periods T_1, T_1'

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to be close to each other, and replace them by the corresponding mean values. Equation (11.46) then simplifies and yields the following approximate formula for estimating the phase velocity:

$$c\left(\frac{T_1 + T_1'}{2}\right) = \frac{x_2 - x_1}{t(M_1') - t(M_1)} .$$

(11.47)

Note that this formula is very similar to (11.27), but now we assign the velocity to the mean period.

As mentioned above, the group velocities computed by using formula (11.34) characterise the structure between the source and seismic station. As opposed to this, the phase velocities computed by using formula (11.47) characterise the *structure between the seismic stations*, because the differences of their epicentral distances and of travel times are used. Examples of dispersion curves, determined by means of formula (11.47), will be given in Chapter 12.

Note that the records at two stations can also be used to estimate the group velocities between the stations, but graphical methods are not suitable for this purpose. Namely, for a point on one seismogram, we must find the point on the other seismogram with the same instantaneous period. If the point on the first seismogram coincides with a peak or trough, the corresponding point on the second seismogram generally does not coincide with a peak or trough, so that its position cannot be determined accurately by graphical methods. More convenient methods of estimating group velocities from seismograms of two stations will be mentioned in Section 11.8.

11.7 Determination of Phase Velocities from Fourier Spectra

We shall again consider observations at two stations, as in the previous Section 11.6, but instead of seismograms we shall use the corresponding Fourier spectra (Fig. 11.1). Denote the Fourier spectra at the source, at the first and second stations by $S(\omega)$, $S_1(\omega)$ and $S_2(\omega)$, respectively. The seismograms at the stations are related to the source spectrum by formula (11.20) if propagation of plane waves is being assumed. Thus,

$$f(x_k, t) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty S(\omega) e^{i\omega[t - x_k/c(\omega)]} \, d\omega = \frac{1}{\pi} \mathrm{Re} \int_0^\infty S_k(\omega) e^{i\omega t} \, d\omega , \quad (11.48)$$

where $k = 1, 2$, and the relation between the spectra is

$$S_k(\omega) = S(\omega) e^{-i\omega \frac{x_k}{c(\omega)}} .$$

(11.49)

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Express the latter formula explicitly for $k = 1$ and $k = 2$, and eliminate the unknown source spectrum $S(\omega)$. This yields

$$S_2(\omega) = S_1(\omega)e^{-i\omega\frac{x_2-x_1}{c(\omega)}}.$$ (11.50)

Hence, the amplitude spectra at the stations are identical, $|S_2(\omega)| = |S_1(\omega)|$, since we have used the plane-wave approximation, but the phase spectra differ by $\left[-\omega\frac{x_2-x_1}{c(\omega)}\right]$. This difference in the phase spectra may be used to determine the phase velocity.

Compute spectra $S_1(\omega)$ and $S_2(\omega)$ from the known seismograms $f(x_1,t)$ and $f(x_2,t)$, respectively:

$$S_k(\omega) = \int_{-\infty}^{+\infty} f(x_k,\tau)e^{-i\omega\tau}\, d\tau = p_k(\omega) + iq_k(\omega) ,$$ (11.51)

where again $k = 1,2$, $p_k(\omega)$ and $q_k(\omega)$ are the real and imaginary parts of the spectra. These functions are thus known at both stations. By comparing the phases in (11.50), one gets

$$\arctan\frac{q_2(\omega)}{p_2(\omega)} = \arctan\frac{q_1(\omega)}{p_1(\omega)} - \omega\frac{x_2-x_1}{c(\omega)} + 2m\pi ,$$ (11.52)

where m is an integer. This yields the final formula for the phase velocity in the form

$$c(\omega) = (x_2 - x_1)\frac{\omega}{\arctan\dfrac{q_1(\omega)}{p_1(\omega)} - \arctan\dfrac{q_2(\omega)}{p_2(\omega)} + 2m\pi} .$$ (11.53)

The unknown integer in (11.53) represents the number of wavelengths which can be laid between the stations. In practice, we usually compute the phase velocities for $m = 0, \pm 1, \pm 2$, etc., and select the curve which seems to be most realistic. The correct value of m can best be determined at long periods, where the distances between the individual curves are large. Supplementary data, such as the velocities of body waves or group velocities of surface waves, can also facilitate the selection of the correct value of m.

11.8 Time-Frequency Analysis

Observed waves are frequently composed of several components, and the problem of their separation arises. If observed waves are well separated in time, e.g., P and S waves on seismograms, they can be analysed in the time domain.

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If a wave is composed of two or several harmonic components, the best separation and analysis can be performed in the frequency domain. However, we also encounter mixed situations, when the frequency content of a wave varies with time. A combined analysis, i.e. a time-frequency analysis, must then be applied. Seismic surface waves are typical examples of waves displaying this time-frequency dependence.

The peak and trough technique, described above, is the classical time-frequency analysis technique for estimating surface-wave velocities. It is evident that this technique is designed exclusively for estimating the dominant frequency at a given time. This technique fails if the wave does not exhibit a simple quasi-sinusoidal character, which may be caused by seismic noise, interference of modes, interference in the neighbourhood of group velocity minima or maxima, etc. In these cases, a spectral estimation needs to be applied. Moreover, the dispersion curves determined by the peak and trough method usually display a scatter, due to the inaccurate determination of periods by numerical differentiation. Hence, some smoothing procedure is also required.

Many contemporary time-frequency analysis techniques are based on the concept of dynamic spectra. The *dynamic spectrum* of a function $f(t)$ is defined as a linear transformation of the form

$$S(\omega, t) = \int_{-\infty}^{+\infty} K(\omega, t, \tau) f(\tau) \, d\tau \, , \tag{11.54}$$

where function K determines the character of the transformation.

Two special cases of this transformation should be mentioned, namely:

a) the Fourier transform if $K(\omega, t, \tau) = e^{-i\omega\tau}$;

b) the identical transformation if $K(\omega, t, \tau) = \delta(t - \tau)$, which yields the original function, $S(\omega, t) = f(t)$.

The dependence of function K on t and τ is frequently assumed in the form of the difference $(t - \tau)$, e.g.,

$$K(\omega, t, \tau) = g(t - \tau) e^{i\omega(t-\tau)} \, . \tag{11.55}$$

Many authors have used function g in the Gaussian form, which yields the optimum resolving power (Dziewonski et al., 1969; Dobeš, 1981). A suitable form of function K can then be, for example,

$$K(\omega, t, \tau) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha^2(t-\tau)^2} e^{i\omega(t-\tau)} \, , \tag{11.56}$$

α being the parameter controlling the width of the filter.

Assume that we have computed and plotted the absolute value of the dynamic spectrum, $\left| S(\omega, t) \right|$, as a function of ω and t. For a given t, the maximum value of this function along the ω-axis determines the dominant angular frequency ω_0. By inserting these values into (11.34), we obtain one point of the group-velocity dispersion curve.

Instead of time t, we can use a new variable, $u = x/t$, where x is the epicentral distance. The ridge of the (u, ω)-plot then yields directly the dispersion curve of group velocity, $U = U(\omega)$.

In comparison with the peak and trough techniques, the method just described enables us to determine the observed dispersion curves with higher accuracy and in broader frequency intervals, and also to separate and analyse higher modes.

Another variant of the above-mentioned method is the autoregressive spectral analysis, also known as maximum entropy spectral analysis, which usually provides higher spectral resolution. Recently, a new category of time-frequency analysis techniques, called time-frequency distributions, have appeared in the literature. In particular, the Wigner and Choi-Williams distributions have become very popular. For details we refer the reader to the review by Kocaoglu and Long (1993).

Chapter 12

Examples of Structural Studies by Surface Waves

The most detailed information on the structure of the Earth's crust can be obtained from studies of seismic body waves, e.g., by means of seismic reflection methods or by deep seismic soundings. Studies of seismic surface waves can only be used to construct mean structural models between epicentre and station, or between two station. However, if surface waves from many earthquakes are recorded at several stations, tomographic methods make it possible to reveal also many structural details (Levshin and Ritzwoller, 1995; Rial et al., 1997).

Surface waves can be used to study extended regions of continents and oceans, including high mountains, polar regions, or thick forests. Even in the regions where the crustal structure is known from body-wave studies, the surface-wave method may bring independent structural information, in particular (Novotny, 1997):

a) As opposed to deep seismic soundings, dispersion curves contain information predominantly on the distribution of shear-wave velocities.

b) The surface-wave method is an efficient and inexpensive method of studying the upper mantle structure if seismograms of distant earthquakes are used.

c) High-quality observations of surface waves can be used to study some structural details, such as low-velocity zones, or anisotropy.

d) Shallow structures to depths of several hundred metres can be studied by short-period surface waves generated by explosions.

We shall demonstrate briefly some of these applications on the following examples.

12.1 Short-Period Surface Waves Generated by Explosions and Their Interpretation

Short-period surface waves have been used by many authors to study the uppermost crustal structure; see the papers by Åström and Lund (1993), or Ruud et al. (1993), where further references can be found. Here we shall reproduce some results of studying short-period surface waves in the West Carpathians by Holub and Novotny (1997).
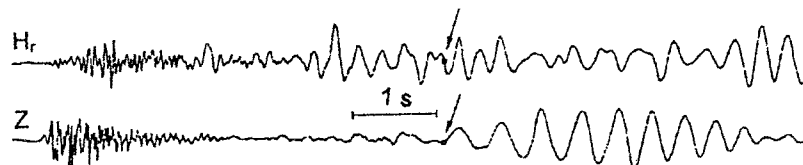

Fig. 12.1. Vertical (Z) and radial (H_r) seismograms recorded at an epicental distance of 3.6 km from one of the shot points in the West Carpathians.

During deep seismic soundings (DSS) along the international profile VI, which crossed the boundary of the Czech Republic and Slovakia, seismic waves were recorded also by broad-band seismographs. In the regions of sedimentary basins, these seismographs recorded, apart from body waves, slow short-period Rayleigh waves (Fig. 12.1). The observed group-velocity dispersion curves, determined by the peak and trough method, are shown in Fig. 12.2. Simple models, consisting of one or two layers on a half-space, were sufficient to explain the observed dispersion. The velocities in the individual models are rather different, but all the models exhibit a significant velocity discontinuity at depths of 50–70 m, which has not been recognised by the previous body-wave studies.
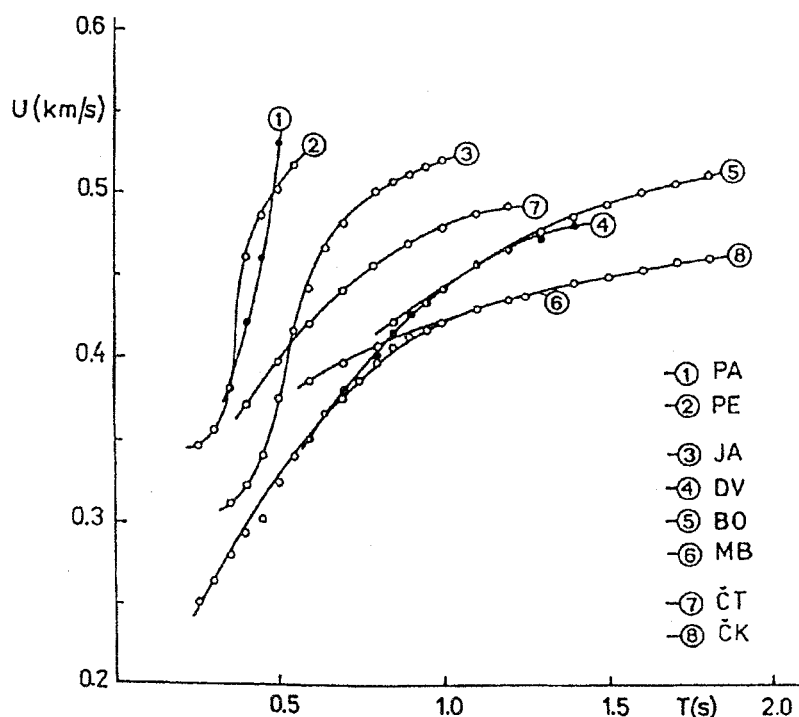

Fig. 12.2. Dispersion curves of short-period Rayleigh waves for several segments of the DSS profile VI: U is the group velocity, T the period, the points denote the observed values, and the lines represent the theoretical dispersion curves.

12.2 Surface Waves Generated by Earthquakes and Their Application in Studies of the Earth Crust and Upper Mantle

Hundreds of papers have dealt with studies of the crustal and upper mantle structure by surface waves. Of the classical papers which considerably influenced further development, we would mention, e.g., the papers by Dorman et al. (1960), Brune and Dorman (1963), Anderson and Toksöz (1963), or the review by Kovach (1965). Here we shall present only two recent examples.
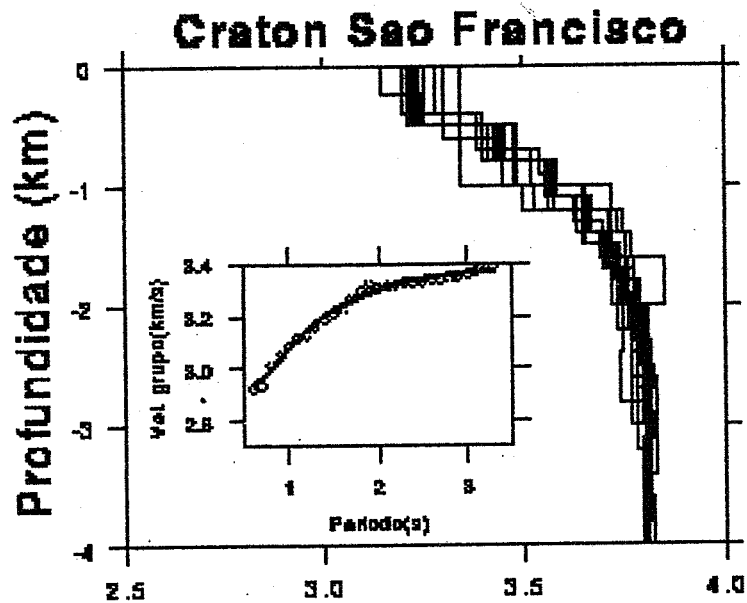
Craton Sao Francisco

Fig. 12.3. The shear-wave velocity cross-sections for the São Francisco craton in south-eastern Brazil, derived from the group-velocity dispersion curves (the figure inside). (After Marchioreto and Assumpção (1997)).

Earthquakes from regional distances have been used to study the upper crustal structure in a region of south-eastern Brazil by Marchioreto and Assumpção (1997). The results for the São Francisco craton are reproduced in Fig. 12.3. The dispersion curves in the period range from 0.6 to 3.4 s yielded structural models to a depth of about 4 km.
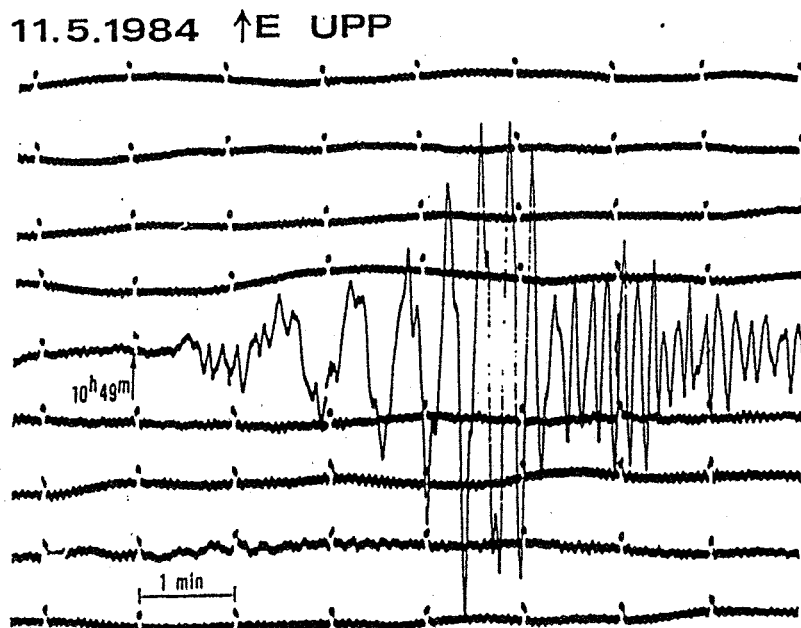

11.5.1984 ↑E UPP

Fig. 12.4. A record of an earthquake in Southern Italy, obtained at Uppsala with the long-period EW-components seismograph (Press-Ewing). The epicentral distance is 18.3°. The waves of large amplitudes are Love waves.

Several Italian earthquakes were used to study surface waves propagating along profile Prague (Czech Republic) – Uppsala (Sweden) by Novotny et al. (1997); see Figs. 12.4 and 12.5. The surface-wave dispersion was used to verify the structure of the Earth's crust, known from nearby profiles of deep seismic soundings, and to study the uppermost mantle.
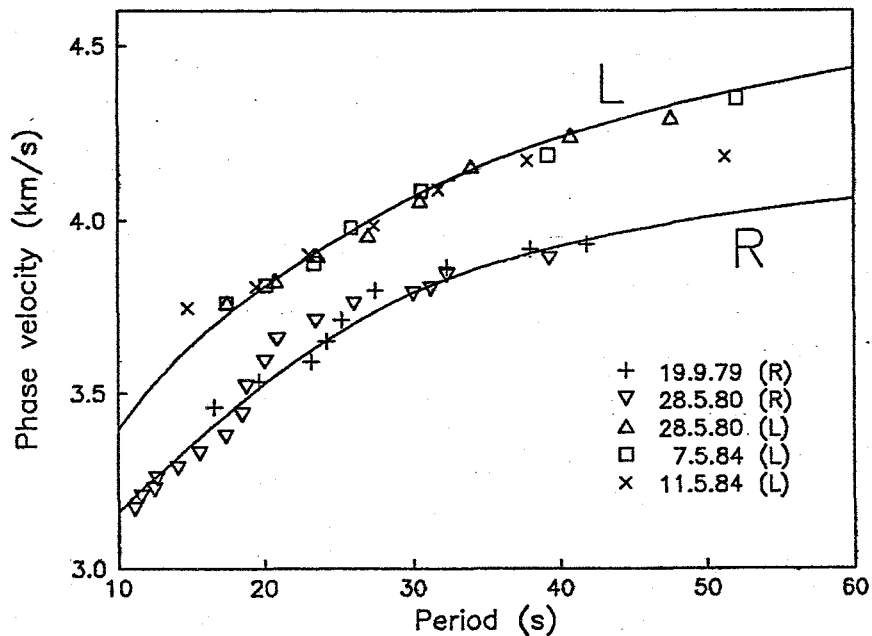

Fig. 12.5. Phase velocity dispersion of Rayleigh waves (R) and Love waves (L) for the Prague–Uppsala profile.

We hope that these few examples have demonstrated sufficiently the possibilities and limitations of the surface-wave method in structural studies. In these lecture notes, we have not discussed other seismological applications of surface waves, e.g., in studies of earthquake mechanisms, in studies of lateral inhomogeneities, such as vertical faults, and some others. Nevertheless, the increasing number of broad-band instruments, used at seismic observatories and in field measurements, and the progress in the theory of surface waves indicate extending possibilities of the future research of seismic surface waves.

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