2.5. Lagrangian approach to Earth rotation Woolard's theory

This is the classical approach to the rotation of a rigid body and, in particular, to precession and nutation. It also underlies the theory of Woolard (1953) which has been standard in astronomy until recently.

This approach uses Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

The generalized coordinates q_i are the Euler angles:

$$q_1 = \psi \quad , \quad q_2 = \vartheta \quad , \quad q_3 = \varphi$$

which describe the orientation of the body with respect to inertial space, e.g. the orientation of the body frame $x_1 x_2 x_3$ with respect to inertial frame $X_1 X_2 X_3$.

They are defined as follows (Fig. 2.10). The ecliptic (fixed at an epoch t_0) corresponds to the $X_1 X_2$ plane, and the (instantaneous) equator to the $x_1 x_2$ plane. To get the system $x_1 x_2 x_3$ we rotate the system $X_1 X_2 X_3$ first about the X_3 axis by the angle ψ until the X_1 axis coincides with the nodal point N, then about the nodal axis ON by the angle ϑ, so that X_3 goes to x_3, and finally about the x_3 axis by the angle φ.

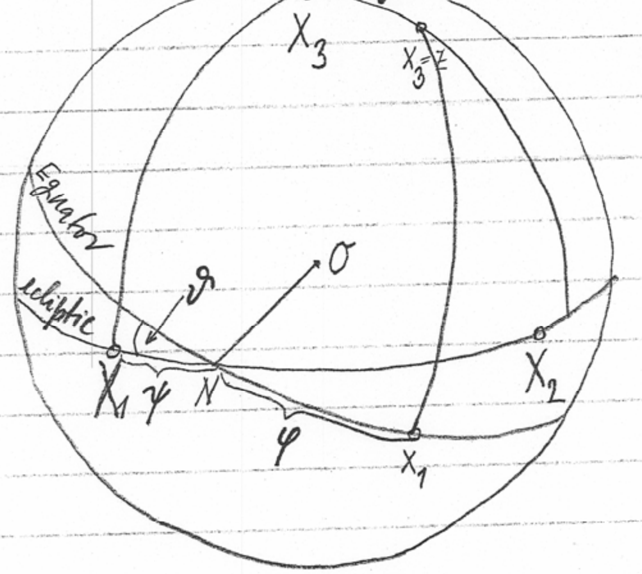


Fig. 2.10: The basic Euler angles on the unit sphere

The angle ψ is the longitude of the node N (jarní bod), ϑ represents the obliquity of the ecliptic ($\doteq 23°27'$), and φ is an angle that measures the rotation of the earth.

Nyní použijeme principu superpozice rotací. Účastní-li se těleso n okamžitých dílčích rotací $\vec{\omega}_i$, $i = 1, \ldots, n$, okolo os, které se protínají v jediném bodě, je výsledný rotační pohyb určen vektorem $\vec{\omega}$ rovným

$$\vec{\omega} = \sum_{i=1}^{n} \vec{\omega}_i$$

Pak:

The angular velocity vector $\vec{\omega}$ may be represented as follows:

$$\vec{\omega} = \dot{\varphi}\, \vec{l_\varphi} + \dot{\vartheta}\, \vec{l_\vartheta} + \dot{\psi}\, \vec{l_\psi} \qquad (2.244)$$

That is, it is decomposed into a rotation with angular velocity $\dot{\varphi} = d\varphi/dt$ around an axis (unit vector $\vec{l_\varphi}$), a rotation with $\dot{\psi}$ around $\vec{l_\psi}$, and a rotation with $\dot{\vartheta}$ around $\vec{e}_\vartheta$.

A rotation in which only φ changes, is a rotation around the $x_3 = z$ axis, so that

$$\vec{e}_\varphi = \vec{e}_z$$

the unit vector of the z-axis. The rotation by ϑ is around the nodal line NO, so that

$$\vec{e}_\vartheta = \overrightarrow{ON}$$

(note that this is indeed a unit vector since the sphere has radius 1), and similarly

$$\vec{e}_\psi = \vec{e}_Z$$

the unit vector of the inertial axis $X_3 \equiv Z$.

Let us now represent the components of $\vec{\omega}, \vec{e}_\psi, \vec{e}_\vartheta, \vec{e}_\varphi$ in the <u>body frame</u> $x_1 x_2 x_3$.

Evidently

$$\vec{e}_\varphi = \vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (2.248)$$

Podle str. 44.6 je rotační matice:

$$R(\psi, \vartheta, \varphi) =$$

$$= \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi, & -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi, & \sin\psi\sin\vartheta \\ \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi, & -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi, & -\cos\psi\sin\vartheta \\ \sin\vartheta\sin\varphi, & \sin\vartheta\cos\varphi, & \cos\vartheta \end{pmatrix}$$

Měli jsme $\vec{e}_i{}' = R_{ji}\,\vec{e}_j$ (viz str. 44.1)

např. složky vektoru $\vec{e}_\psi$ v soustavě $x_1 x_2 x_3$ pevně spojené s tělesem jsou

$$\vec{e}_\psi = R^T \vec{e}_Z = \begin{pmatrix} \cdot, & \cdot, & \sin\vartheta\sin\varphi \\ \cdot, & \cdot, & \sin\vartheta\cos\varphi \\ \cdot, & \cdot, & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$\underline{\vec{e}_\psi = \begin{pmatrix} \sin\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi \\ \cos\vartheta \end{pmatrix}} \qquad (2.2.50)$$

Chybí určit složky $\vec{e}_\vartheta$:

$$\vec{e}_{\vartheta} = R^T(\vartheta,\varphi)\begin{pmatrix}1\\0\\0\end{pmatrix} = \left[R_{x'}(\vartheta)\,R_{z''}(\varphi)\right]^T\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= R^T_{z''}(\varphi)\;R^T_{x'}(\vartheta)\begin{pmatrix}1\\0\\0\end{pmatrix} =$$

$$= \begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\vartheta & \sin\vartheta\\ 0 & -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\varphi, & \sin\varphi\cos\vartheta, & \sin\varphi\sin\vartheta\\ -\sin\varphi, & \cos\varphi\cos\vartheta, & \cos\varphi\sin\vartheta\\ 0, & -\sin\vartheta, & \cos\vartheta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix}$$

$$\underline{\vec{e}_{\vartheta} = \begin{pmatrix}\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix}} \qquad (2.249)$$

Podle rovnice (2.244):

$$\vec{\omega} = \dot{\varphi}\begin{pmatrix}0\\0\\1\end{pmatrix} + \dot{\vartheta}\begin{pmatrix}\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix} + \dot{\psi}\begin{pmatrix}\sin\vartheta\sin\varphi\\ \sin\vartheta\cos\varphi\\ \cos\vartheta\end{pmatrix}$$

The components of the last equation are

$$\boxed{\begin{aligned}\omega_1 &= \dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi\\ \omega_2 &= -\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\varphi\\ \omega_3 &= \dot{\varphi} + \dot{\psi}\cos\vartheta\end{aligned}} \qquad (2.251)$$

* ... spojené s tělesem

The components of the last eq. are:

$$\boxed{\begin{aligned} \omega_1 &= \dot{\vartheta}\sin\varphi - \dot{\psi}\sin\vartheta\cos\varphi \\ \omega_2 &= \dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi \\ \omega_3 &= \dot{\varphi} + \dot{\psi}\cos\vartheta \end{aligned}} \qquad (2.251)$$

These equations provide the fundamental connection between the components of the vector $\vec{\omega}$ and the time derivatives of the Euler angles. They are known as Euler's kinematical equations (Euler's dynamical equations are (2.74))

otoč

Podle Bursy: ψ .... precese
ϑ .... nutace
φ .... vlastní rotace

$\dot{\psi}$ vyjadřuje časovou změnu směru na jarní bod N (ten je definovaný jako průsečnice roviny okamžitého rovníku a roviny „pevné" ekliptiky zvolené epochy)

$\dot{\vartheta}$ časová změna sklonu roviny okamžitého rovníku k ekliptice

$\dot{\varphi}$ okamžitá úhlová rychlost vlastní rotace Země, přičemž referenčním směrem je směr na jarní bod

Let us now express the kinetic energy in terms of Euler angles. For principal axes of inertia and rotational symmetry ($B=A$), eq. (2.62) becomes

$$T = \frac{1}{2} A (\omega_1^2 + \omega_2^2) + \frac{1}{2} C \omega_3^2$$

and on substituting (2.251)

$$T = \frac{1}{2} A (\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2\vartheta) + \frac{1}{2} C (\dot{\varphi} + \dot{\psi}\cos\vartheta)^2$$

Now we are ready to apply Lagrange's equations. With our present variables (2.243) and with $L = T - U$ they become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = -\frac{\partial U}{\partial \psi}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\vartheta}}\right) - \frac{\partial T}{\partial \vartheta} = -\frac{\partial U}{\partial \vartheta} \qquad (2.254)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = -\frac{\partial U}{\partial \varphi}$$

<u>U does not depend on $\dot{\psi}, \dot{\vartheta}, \dot{\varphi}$!</u> The formation of the partial derivatives is straightforward:

$$\frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \varphi} = 0 \qquad \psi, \varphi \ldots \text{cyklické souřadnice}$$

$$\frac{\partial T}{\partial \vartheta} = A\dot{\psi}^2 \sin\vartheta\cos\vartheta - C(\dot{\varphi} + \dot{\psi}\cos\vartheta)\dot{\psi}\sin\vartheta$$

$$\frac{\partial T}{\partial \dot\psi} = A\dot\psi \sin^2\vartheta + C(\dot\varphi + \dot\psi\cos\vartheta)\cos\vartheta$$

$$\frac{\partial T}{\partial \dot\vartheta} = A\dot\vartheta$$

$$\frac{\partial T}{\partial \dot\varphi} = C(\dot\varphi + \dot\psi\cos\vartheta)$$

Hence (2.254) becomes

$$\frac{d}{dt}\left[A\dot\psi \sin^2\vartheta + C(\dot\varphi + \dot\psi\cos\vartheta)\cos\vartheta\right] = -\frac{\partial U}{\partial \psi}$$

$$A\ddot\vartheta - A\dot\psi^2 \sin\vartheta\cos\vartheta + C(\dot\varphi + \dot\psi\cos\vartheta)\dot\psi\sin\vartheta = -\frac{\partial U}{\partial \vartheta}$$

$$\frac{d}{dt}\left[C(\dot\varphi + \dot\psi\cos\vartheta)\right] = -\frac{\partial U}{\partial \varphi} = 0 \qquad (2.255)$$

In fact, $\partial U/\partial\varphi = 0$ since φ denotes a rotation around the body z-axis, and for a rotationally symmetric body, the potential energy U must remain unchanged by such a rotation so that it cannot depend on φ.

Now the third eq. of the last system immediately gives

$$\dot\varphi + \dot\psi\cos\vartheta = \text{const.} = \Omega \qquad (2.256)$$

The constant has been denoted by Ω since it is essentially the rotational velocity of the earth $\dot\varphi$, the second term $\dot\psi\cos\vartheta$ being much smaller than $\dot\varphi$.

The substitution of (2.256) into (2.255) gives

$$\boxed{\begin{aligned} &A\ddot{\psi}\sin^2\vartheta + 2A\dot{\psi}\dot{\vartheta}\sin\vartheta\cos\vartheta - C\Omega\sin\vartheta\,\dot{\vartheta} = -\frac{\partial U}{\partial\psi} \\ &A\ddot{\vartheta} - A\dot{\psi}^2\sin\vartheta\cos\vartheta + C\Omega\dot{\psi}\sin\vartheta = -\frac{\partial U}{\partial\vartheta} \end{aligned}} \qquad (2.257)$$

Now it is important to consider orders of magnitude. The angle $\dot{\varphi}$ has been seen essentially to be the angular velocity of the earth's rotation. The angles $\dot{\psi}$ and $\dot{\vartheta}$ characterize the change of the earth's axis $x_3 = z$ with respect to the inertial frame. Thus they describe precession and nutation, which are secular (uniformly increasing) and the periodic part of the same phenomenon, namely the change of direction of the earth's axis in inertial space. Since the velocity of precession and nutation is much smaller than the velocity of earth rotation, we have

$$\dot{\psi},\ \dot{\vartheta} \ll \Omega$$

Hence the dominant term in both left-hand sides of (2.257) is the third term. Neglecting the others we obtain

$$\boxed{\begin{aligned} \dot{\vartheta} &= \frac{1}{C\Omega\sin\vartheta}\frac{\partial U}{\partial\psi} \\ \dot{\psi} &= -\frac{1}{C\Omega\sin\vartheta}\frac{\partial U}{\partial\vartheta} \end{aligned}} \qquad (2.259)$$

These are the well-known Poisson equations, which are basic for the classical theory of precession and nutation

An alternative form. The right-hand sides of (2.259) essentially are nothing else than components of the lunisolar torque. This can be seen as follows.

The potential energie $\mathcal{U}$ is the gravitational potential energy of the earth-moon system (plus that of the earth-sun system, which we take for granted without mentioning it explicitly). For the present purpose it is sufficient to regard the moon as a point mass. Then

$$\mathcal{U} = -\mu V, \qquad (2.260)$$

the potential energy being the product of the earth's gravitational potential times the moon's mass μ (potential is potential energy per unit mass), and the minus sign being due to different sign conventions for $\mathcal{U}$ and V!

$$V(r,\vartheta,\varphi) = \frac{GM}{a}\sum_{jm}\left(\frac{a}{r}\right)^{j+1} A_{jm} Y_{jm}(\vartheta,\varphi)$$

avšak do vztahu (2.260) je třeba dosadit potenciál Země v bodě Měsíce. Buď

d .... the distance of the Moon
p ... the polar distance of the Moon
Λ ... the longitude of the Moon

Potom

$$V(d,p,\Lambda) = \frac{GM}{a}\sum_{jm}\left(\frac{a}{d}\right)^{j+1} A_{jm} Y_{jm}(p,\Lambda)$$

p, Λ ... the polar distance and longitude of the Moon in the equatorial system x_1, x_2, x_3

$\downarrow \quad \downarrow$

a, b .... the polar distance and longitude of the Moon in the inertial system $X_1 X_2 X_3$

Eulerovy úhly mezi $X_1 X_2 X_3 \longrightarrow x_1 x_2 x_3$ jsou

$$\psi \leftrightarrow \alpha$$
$$\vartheta \leftrightarrow \beta$$
$$\varphi \leftrightarrow \gamma$$

lze postupovat tak, že

$$Y_{jm'}(p, \Lambda) = \sum_m D^j_{mm'}(\psi, \vartheta, \varphi)\, Y_{jm}(a, b)$$

a nebo podle str. 44.7 – 44.9 platí

$$\cos p = \cos a \cos\vartheta + \sin a \sin\vartheta \sin(\psi - b) \qquad (2.264)$$

$$\sin p \cos(\Lambda + \varphi) = \sin a \cos(b - \psi) \qquad (2.265)$$

$$\sin p \sin(\Lambda + \varphi) = \cos a \sin\vartheta + \sin a \cos\vartheta \sin(b - \psi)$$

$$U(d, p, \Lambda) = -\mu \frac{GM}{a} \sum_{jm} \left(\frac{a}{d}\right)^{j+1} A_{jm} Y_{jm}(p, \Lambda)$$

$$\frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial p}\frac{\partial p}{\partial \psi} + \frac{\partial U}{\partial \Lambda}\frac{\partial \Lambda}{\partial \psi}$$

$$(2.262)$$

$$\frac{\partial U}{\partial \vartheta} = \frac{\partial U}{\partial p}\frac{\partial p}{\partial \vartheta} + \frac{\partial U}{\partial \Lambda}\frac{\partial \Lambda}{\partial \vartheta}$$

The differention of (2.264) gives

$$-\sin p \frac{\partial p}{\partial \psi} = \sin a \sin \vartheta \cos(\psi - b)$$

$$-\sin p \frac{\partial p}{\partial \vartheta} = -\cos a \sin \vartheta + \sin a \cos \vartheta \sin(\psi - b)$$

On the right-hand sides of these expressions we substitute (2.265), obtaining

$$\begin{aligned} \frac{\partial p}{\partial \psi} &= -\cos(\Lambda + \varphi) \sin \vartheta \\ \frac{\partial p}{\partial \vartheta} &= \sin(\Lambda + \varphi) \end{aligned} \qquad (2.266)$$

Nyní se omezíme na případ, že Zemi je rotačně symetrické těleso

$$U = -\mu \frac{GM}{a} \left[\frac{a}{d} + \left(\frac{a}{d}\right)^3 A_{20} P_{20}(\cos p) \right]$$

$$\Rightarrow \quad \frac{\partial U}{\partial \Lambda} = 0$$

i.e. the angular dependence of U is on p only

$$\frac{\partial U}{\partial p} = -\mu \frac{GM}{a} \left(\frac{a}{d}\right)^3 A_{20} \frac{\partial P_{20}(\cos p)}{\partial p}$$

avšak $\quad P_{20}(\cos p) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2 p - 1)$

$\frac{\partial P_{20}}{\partial p}$ (viz další strana) $\qquad P_{21} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cos p \sin p$

$$\frac{\partial P_{20}}{\partial p} = -\frac{1}{4}\sqrt{\frac{5}{\pi}}\, 6 \cos p \sin p = -\frac{3}{2}\sqrt{\frac{5}{\pi}} \cos p \sin p =$$

$$= 3\sqrt{\frac{5}{\pi}}\sqrt{\frac{2\pi}{15}}\, P_{21}(\cos p) = \sqrt{6}\, P_{21}(\cos p)$$

$$\frac{\partial U}{\partial p} = -GM\mu \frac{a^2}{d^3}\sqrt{6}\, A_{20}\, P_{21}(\cos p)$$

Porovnáme-li poslední výraz ze str 24.3., jsou složky lunisolárního momentu sil dány

$$L_x = -\frac{\partial U}{\partial p} \sin \Lambda$$

$$L_y = \frac{\partial U}{\partial p} \cos \Lambda \qquad (2.269)$$

It is convenient to form the complex combination of these quantities

$$L = L_x + i L_y = \frac{\partial U}{\partial p}(-\sin \Lambda + i \cos \Lambda) = i \frac{\partial U}{\partial p} e^{i\Lambda}$$

viz též 62-1

Dále (2.262) + (2.266):

$$\frac{\partial U}{\partial \psi} = \frac{\partial U}{\partial p}\frac{\partial p}{\partial \psi} = -\frac{\partial U}{\partial p} \sin \vartheta \cos(\Lambda + \varphi)$$

$$\frac{\partial U}{\partial \vartheta} = \frac{\partial U}{\partial p}\frac{\partial p}{\partial \vartheta} = \frac{\partial U}{\partial p} \sin(\Lambda + \varphi) \qquad (2.267)$$

str L9 : alternativní odvození

$$L = L_x + i L_y$$

$$= -\sqrt{6}\, N_2 A_{20}\, i \phi_{21}^{*}$$

$$= -\sqrt{6}\, \cancel{\frac{5}{4\pi}}\, \cancel{M a^2} A_{20} \frac{GM}{\cancel{a^2}} \frac{1}{a^2} \underbrace{i \nu_{21}^{*}(t)}_{a \cancel{\frac{\mu}{M}} \cancel{\frac{4\pi}{5}} \left(\frac{a}{d}\right)^3 Y_{21}(p, \Lambda)}$$

$$= -\sqrt{6}\, A_{20}\, GM \mu\, a^2 \frac{i}{d^3} Y_{21}(p, \Lambda)$$

$$\underline{\underline{L = i \frac{\partial U}{\partial p} e^{i\Lambda}}}$$

Definujme

$$\bar{L}_x = -\frac{\partial U}{\partial p}\sin(\Lambda+\varphi)$$
$$\bar{L}_y = \frac{\partial U}{\partial p}\cos(\Lambda+\varphi) \qquad (2.268)$$

a komplexní veličinu

$$\bar{L} = \bar{L}_x + i\bar{L}_y = \frac{\partial U}{\partial p}\left(-\sin(\Lambda+\varphi) + i\cos(\Lambda+\varphi)\right) =$$

$$= i\frac{\partial U}{\partial p} e^{i(\Lambda+\varphi)} = L e^{i\varphi}$$

tj.

$$\bar{L} = L e^{i\varphi} \qquad (2.273)$$

Potom

$$\frac{\partial U}{\partial \psi} = -\bar{L}_y \sin\vartheta$$
$$\frac{\partial U}{\partial \vartheta} = -\bar{L}_x \qquad (2.270)$$

Dosazení posledních vztahů do Poisson. rovnic (2.259) dá

$$\boxed{\begin{aligned} \dot{\vartheta} &= \frac{1}{C\Omega\sin\vartheta}\frac{\partial U}{\partial \psi} = -\frac{1}{C\Omega}\bar{L}_y \\ \dot{\psi} &= -\frac{1}{C\Omega\sin\vartheta}\frac{\partial U}{\partial \vartheta} = \frac{1}{C\Omega\sin\vartheta}\bar{L}_x \end{aligned}} \qquad (2.271)$$

It will be convenient to form the complex combination of these equations

$$\dot{\vartheta} + i \sin\vartheta \, \dot{\psi} = \frac{i}{C\Omega}\left(\bar{L}_x + i\bar{L}_y\right) = \frac{i}{C\Omega}\bar{L} =$$

$$= \frac{i}{C\Omega} L e^{i\varphi} \qquad (2.272)$$

Now the angle φ measures the rotation of the earth which is very nearly uniform, so that we may put

$$\varphi = \Omega t \qquad (2.274)$$

We finally get

$$\boxed{\dot{\vartheta} + i \sin\vartheta \, \dot{\psi} = \frac{i}{C\Omega} L e^{i\Omega t}} \qquad (2.276)$$

<u>Various axes</u>. The Euler angles ψ, ϑ, φ describe the orientation of the body frame $x_1 x_2 x_3$ with respect to the inertial system $X_1 X_2 X_3$. Thus the changes $\dot{\psi}$ and $\dot{\vartheta}$ of these Euler angles describe the motion of the body frame; in other words, they represent the nutation of the figure axis $x_3 = z$.

The <u>rotation axis</u> obviously is given by its components $\omega_1, \omega_2, \omega_3$ in the body frame $x_1 x_2 x_3$. Thus Euler's kinematical equations (2.251) may be said to connect the motion of the rotation axis (represented by $\omega_1, \omega_2, \omega_3$) with the motion of the figure axis (represented by $\dot{\psi}, \dot{\vartheta}, \dot{\varphi}$).

! The full equations (2.257) clearly describe the nutation of the figure axis. This is also true, to a good approximation, for simplified equations (2.259), Poisson's equations! However, to a still much better approximation, Poisson's equations describe the nutation of the angular momentum axis

In fact, eqs. (2.271) are a direct consequence of the basic equation (2.54),

$$\frac{d\vec{H}}{dt} = \vec{L} \qquad (2.277)$$

relating the time change of the angular momentum $\vec{H}$ to the torque $\vec{L}$, with respect to the inertial space

The angular momentum axis is very close to the figure axis X_3, so that it can analogously be described by angles ϑ_H and φ_H which are very close to ϑ and φ, the Euler angles of the figure axis (Fig. 2.14)

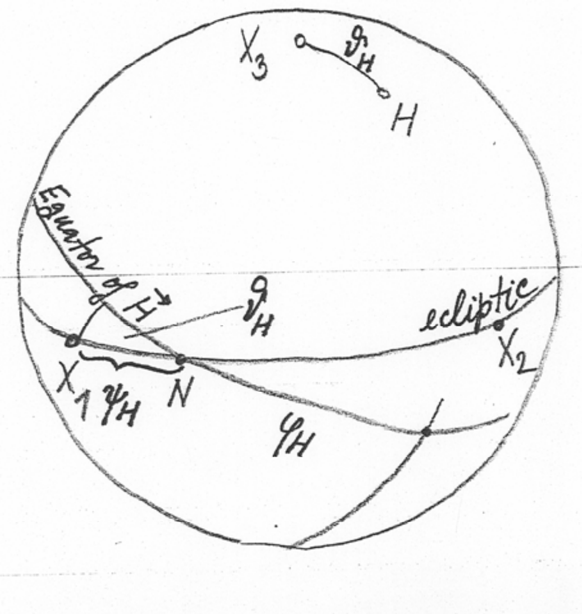


Fig 2.14: The angular momentum axis in space illustrated by means of a unit sphere

In terms of ψ_H and ϑ_H, the components of $\vec{H}$ in the $X_1 X_2 X_3$ system are

$$\vec{H} = |\vec{H}| \begin{pmatrix} \sin\psi_H \sin\vartheta_H \\ -\cos\psi_H \sin\vartheta_H \\ \cos\vartheta_H \end{pmatrix} \qquad (2.278)$$

On substituting (2.278) into (2.277), differentiating and performing some transformations, we get Poisson's eqs (2.271), with ψ_H and ϑ_H in the place of ψ and ϑ.

Odvoďme to. Nejprve dokažme (2.278). Měli jsme

$$\vec{e}_i' = R_{ji} \vec{e}_j$$

a naopak

$$\vec{e}_i = R_{ij} \vec{e}_j'$$

Nyní potřebujeme tento druhý vztah. Nechť $\vec{e}_3'$ leží v úhlovém momentu $\vec{H}$ vzhledem k soustavě souřadnic spjaté s tělesem, tj. jeho kartézské souřadnice konce vektoru jsou

$$x_1' = 0 \qquad x_2' = 0 \qquad x_3' = 1$$

Ptáme se na tyto souřadnice v inerciální soustavě $X_1 X_2 X_3$

$$x_i = R_{ij} x_j'$$

Dosadíme-li za rotační matici $R(\psi_H, \vartheta_H, \varphi_H)$ ze str. 53 a provedeme-li maticové násobení, dostaneme (2.278).

Zcela analogicky pro složky momentu vnější síly

$$\begin{pmatrix} L_{X_1} \\ L_{X_2} \\ L_{X_3} \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi, & -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi, & \sin\psi\sin\vartheta \\ \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi, & -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi, & -\cos\psi\sin\vartheta \\ \sin\vartheta\sin\varphi, & \sin\vartheta\cos\varphi, & \cos\vartheta \end{pmatrix} \begin{pmatrix} L_{x_1} \\ L_{x_2} \\ L_{x_3} \end{pmatrix}$$

(Na chvíli vypouštíme index H u $\psi_H, \vartheta_H, \varphi_H$ a $L_{x_1}, L_{x_2}, L_{x_3}$)

$$L_{X_1} = (\cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi) L_{x_1} + (-\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi) L_{x_2} + \sin\psi\sin\vartheta\, L_{x_3}$$

$$L_{X_2} = (\sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi) L_{x_1} + (-\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi) L_{x_2} - \cos\psi\sin\vartheta\, L_{x_3}$$

$$L_{X_3} = \sin\vartheta\sin\varphi\, L_{x_1} + \sin\vartheta\cos\varphi\, L_{x_2} + \cos\vartheta\, L_{x_3}$$

$$(\cos\psi\,\dot{\psi}\sin\vartheta + \sin\psi\cos\vartheta\,\dot{\vartheta})H + \sin\psi\sin\vartheta\,\dot{H} = L_{X_1} \quad |\cos\psi \,|\sin\psi$$
$$(\sin\psi\,\dot{\psi}\sin\vartheta - \cos\psi\cos\vartheta\,\dot{\vartheta})H - \cos\psi\sin\vartheta\,\dot{H} = L_{X_2} \quad |\sin\psi \,|-\cos\psi$$
$$-\sin\vartheta\,\dot{\vartheta}\,H + \cos\vartheta\,\dot{H} = L_{X_3}$$

kde
$$H = |\vec{H}|$$

$$\dot{\psi}\sin\vartheta\,H = L_{X_1}\cos\psi + L_{X_2}\sin\psi$$
$$\dot{\vartheta}\cos\vartheta\,H + \sin\vartheta\,\dot{H} = L_{X_1}\sin\psi - L_{X_2}\cos\psi \quad |\cos\vartheta$$
$$-\dot{\vartheta}\sin\vartheta\,H + \cos\vartheta\,\dot{H} = L_{X_3} \quad |-\sin\vartheta$$

$$\dot{\vartheta}H = L_{X_1}\sin\psi\cos\vartheta - L_{X_2}\cos\psi\cos\vartheta - L_{X_3}\sin\vartheta$$
$$\dot{\psi}\sin\vartheta\,H = L_{X_1}\cos\psi + L_{X_2}\sin\psi \quad |\cdot i$$

$$H(\dot{\vartheta} + i\sin\vartheta\,\dot{\psi}) = L_{X_1}(\sin\psi\cos\vartheta + i\cos\psi) + L_{X_2}(-\cos\psi\cos\vartheta + i\sin\psi) - L_{X_3}\sin\vartheta$$

Nyní dosaďme za $L_{X_1}, L_{X_2}, L_{X_3}$ pomocí $L_{x_1}, L_{x_2}, L_{x_3}$ a přeznačme $L^H_x := L_{x_1}$, $L^H_y = L_{x_2}$, $L^H_z = L_{x_3}$:

$$L_{X_1}(\sin\psi\cos\vartheta + i\cos\psi) + L_{X_2}(-\cos\psi\cos\vartheta + i\sin\psi) - L_{X_3}\sin\vartheta =$$

$$= L_x^H\Big(\overset{1.}{\cancel{\sin\psi\cos\vartheta\cos\psi\cos\varphi}} - \sin^2\psi\cos^2\vartheta\sin\varphi + i\cos^2\psi\cos\varphi$$
$$- \overset{2.}{\cancel{i\sin\psi\cos\psi\cos\vartheta\sin\varphi}} - \overset{1.}{\cancel{\sin\psi\cos\psi\cos\vartheta\cos\varphi}} - \cos^2\psi\cos^2\vartheta\sin\varphi$$
$$+ i\sin^2\psi\cos\varphi + \overset{2.}{\cancel{i\sin\psi\cos\psi\cos\vartheta\sin\varphi}} - \sin^2\vartheta\sin\varphi\Big)$$
$$+ L_y^H\Big(-\overset{3.}{\cancel{\sin\psi\cos\psi\cos\vartheta\sin\varphi}} - \sin^2\psi\cos^2\vartheta\cos\varphi - i\cos^2\psi\sin\varphi$$
$$- \overset{4.}{\cancel{i\sin\psi\cos\psi\cos\vartheta\cos\varphi}} + \overset{3.}{\cancel{\sin\psi\cos\psi\cos\vartheta\sin\varphi}} - \cos^2\psi\cos^2\vartheta\cos\varphi$$
$$- i\sin^2\psi\sin\varphi + \overset{4.}{\cancel{i\sin\psi\cos\psi\cos\vartheta\cos\varphi}} - \sin^2\vartheta\cos\varphi\Big)$$
$$+ L_z^H\Big(\sin^2\psi\sin\vartheta\cos\vartheta + \overset{5.}{\cancel{i\sin\psi\cos\psi\sin\vartheta}} + \cos^2\psi\sin\vartheta\cos\vartheta - \overset{5.}{\cancel{i\sin\psi\cos\psi\sin\vartheta}}$$
$$- \sin\vartheta\cos\vartheta\Big) =$$

$$= L_x^H(-\cos^2\vartheta\sin\varphi + i\cos\varphi - \sin^2\vartheta\sin\varphi) +$$
$$+ L_y^H(-\cos^2\vartheta\cos\varphi - i\sin\varphi - \sin^2\vartheta\cos\varphi)$$
$$+ L_z^H(\cancel{\sin\vartheta\cos\vartheta - \sin\vartheta\cos\vartheta}) =$$

$$= L_x^H(-\sin\varphi + i\cos\varphi) + L_y^H(-\cos\varphi - i\sin\varphi)$$
$$= i L_x^H(\cos\varphi + i\sin\varphi) - L_y^H(\cos\varphi + i\sin\varphi) =$$
$$= e^{i\varphi}(iL_x^H - L_y^H) = i e^{i\varphi}(L_x^H + iL_y^H) =$$
$$= i e^{i\varphi} L^H$$

kde
$$L^H = L_x^H + iL_y^H$$

Nakonec dostáváme

$$\boxed{\dot{\vartheta}_H + i \sin\vartheta\, \dot{\psi}_H = \frac{i e^{i\psi_H} L^H}{H}}$$

We got Poisson's equations (2.271) with ϑ_H and ψ_H in place of ϑ and ψ. Since the rotation axis is even much closer to the angular momentum axis than the figure axis (see Fig. 2.7), we may say that Poisson's equations best describe the motion of the angular momentum axis and very well also the motion of the rotation axis. The description of the motion of figure axis by Poisson's eqns is somewhat less accurate. At the first glance this is surprising since we have derived Poisson's equations in order to describe the motion of the figure axis (i.e. of the body frame $x_1 x_2 x_3$), but this paradox is easily resolved by noting that the terms neglected in going from (2.257) to (2.259) precisely correspond to the transition from the figure axis to the angular momentum axis.

Zbývá vyjádřit velikost H vektoru úhlového momentu $\vec{H}$,

$$H = |\vec{H}|$$

Platí

$$\vec{H} = \overleftrightarrow{C} \cdot \vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ H \end{pmatrix}_H$$

v soustavě spojené s $\vec{H}$

kde

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \Omega \end{pmatrix}_H$$

Tedy

$$C_{11}^H \omega_1 + C_{12}^H \omega_2 + C_{13}^H \Omega = 0 \quad \Big| -C_{22}^H \Big| -C_{12}^H$$

$$C_{12}^H \omega_1 + C_{22}^H \omega_2 + C_{23}^H \Omega = 0 \quad \Big| C_{12}^H \Big| C_{11}^H$$

$$C_{13}^H \omega_1 + C_{23}^H \omega_2 + C_{33}^H \Omega = H$$

Z prvních dvou rovnic vyjádříme ω_1 a ω_2 a dosadíme do třetí rovnice

$$\omega_1 \left(-C_{11}^H C_{22}^H + (C_{12}^H)^2\right) = \left(C_{13}^H C_{22}^H - C_{23}^H C_{12}^H\right)\Omega$$

$$\omega_2 \left(C_{11}^H C_{22}^H - (C_{12}^H)^2\right) = \left(C_{13}^H C_{12}^H - C_{23}^H C_{11}^H\right)\Omega$$

$$\omega_1 = \frac{-C_{13}^H C_{22}^H + C_{23}^H C_{12}^H}{C_{11}^H C_{22}^H - (C_{12}^H)^2}\,\Omega$$

$$\omega_2 = \frac{C_{13}^H C_{12}^H - C_{23}^H C_{11}^H}{C_{11}^H C_{22}^H - (C_{12}^H)^2}\,\Omega$$

Potom

$$H = C^H_{13}\,\omega_1 + C^H_{23}\,\omega_2 + C^H_{33}\,\Omega$$

$$H = \frac{-(C^H_{13})^2 C^H_{22} - (C^H_{23})^2 C^H_{11} + 2\,C^H_{12} C^H_{13} C^H_{23}}{C^H_{11} C^H_{22} - (C^H_{12})^2}\,\Omega + C^H_{33}\,\Omega$$

Na druhé straně, moment setrvačnosti v soustavě pevně spojené s tělesem má tvar

$$\overleftrightarrow{C}\Big|_X = \begin{pmatrix} A & & 0 \\ & A & \\ 0 & & C \end{pmatrix}$$

Otočme soustavu $x_1 x_2 x_3$ pevně spojenou s tělesem do soustavy pevně spojené s úhlovým momentem $\vec{H}$.

$$S_X \xleftarrow{R(\alpha,\beta,\gamma)} S_H$$

Moment setrvačnosti se transformuje ve tvaru

$$\overleftrightarrow{C}\Big|_H = R(\alpha,\beta,\gamma)\cdot \overleftrightarrow{C}\Big|_X \cdot R^T(\alpha,\beta,\gamma)$$

kde rotační matice je dána na str. 53. Složkově

$$C^H_{ij} = \sum_{kl} R_{ik}\,\underbrace{C^X_{kl}}_{\delta_{kl} C^X_k}\,(R^T)_{lj} = \sum_k R_{ik} R_{jk}\, C^X_k$$

$$= A(R_{i1}R_{j1} + R_{i2}R_{j2}) + C\,R_{i3}R_{j3}$$

$$R(\alpha,\beta,\gamma)=\left|\begin{matrix} \cos\alpha\cos\gamma-\sin\alpha\cos\beta\sin\gamma, & -\cos\alpha\sin\gamma-\sin\alpha\cos\beta\cos\gamma, & \sin\alpha\sin\beta \\ \sin\alpha\cos\gamma+\cos\alpha\cos\beta\sin\gamma, & -\sin\alpha\sin\gamma+\cos\alpha\cos\beta\cos\gamma, & -\cos\alpha\sin\beta \\ \sin\beta\sin\gamma, & \sin\beta\cos\gamma, & \cos\beta \end{matrix}\right.$$

$$C_{11}^{H}=R_{1k}^{2}C_{k}^{X}$$

$$=A\big(\cos^2\alpha\cos^2\gamma-\cancel{2\sin\alpha\cos\alpha\cos\beta\sin\gamma\cos\gamma}+\sin^2\alpha\cos^2\beta\sin^2\gamma$$
$$+\cos^2\alpha\sin^2\gamma+\cancel{2\sin\alpha\cos\alpha\cos\beta\sin\gamma\cos\gamma}+\sin^2\alpha\cos^2\beta\cos^2\gamma\big)$$
$$+C\left(\sin^2\alpha\sin^2\beta\right)=$$

$$\underline{C_{11}^{H}=A\left(\cos^2\alpha+\sin^2\alpha\cos^2\beta\right)+C\sin^2\alpha\sin^2\beta}$$
$$=\underline{A+(C-A)\sin^2\alpha\sin^2\beta}$$

$$C_{22}^{H}=R_{2k}^{2}C_{k}^{X}$$

$$\underline{C_{22}^{H}=A\left(\sin^2\alpha+\cos^2\alpha\cos^2\beta\right)+C\cos^2\alpha\sin^2\beta}$$
$$=\underline{A+(C-A)\cos^2\alpha\sin^2\beta}$$

$$\underline{C_{33}^{H}=A\sin^2\beta+C\cos^2\beta}$$
$$=\underline{C+(A-C)\sin^2\beta}$$

$$C_{12}^{H}=R_{1k}R_{2k}C_{k}^{X}$$

$$=A\big(\sin\alpha\cos\alpha\cos^2\gamma-\sin^2\alpha\,\cancel{\cos\beta\sin\gamma\cos\gamma}^{1.}+\cos^2\alpha\,\cancel{\cos\beta\sin\gamma\cos\gamma}^{2.}$$
$$-\sin\alpha\cos\alpha\cos^2\beta\sin^2\gamma+\sin\alpha\cos\alpha\sin^2\gamma+\sin^2\alpha\,\cancel{\cos\beta\sin\gamma\cos\gamma}$$
$$-\cos^2\alpha\,\cancel{\cos\beta\sin\gamma\cos\gamma}-\sin\alpha\cos\alpha\cos^2\beta\cos^2\gamma\big)-$$
$$-C\sin\alpha\cos\alpha\sin^2\beta$$

$$=A\left(\sin\alpha\cos\alpha-\sin\alpha\cos\alpha\cos^2\beta\right)-C\sin\alpha\cos\alpha\sin^2\beta$$

$$\underline{C_{12}^{H} = (A-C)\sin\alpha\cos\alpha\sin^2\beta} = C_{21}^{H}$$

$$C_{13}^{H} = R_{1k}R_{3k}C_{k}^{x}$$

$$= A\left(\cos\alpha\,\cancel{\sin\beta\sin\gamma\cos\gamma} - \sin\alpha\sin\beta\cos\beta\sin^2\gamma\right.$$
$$\left. - \cos\alpha\,\cancel{\sin\beta\sin\gamma\cos\gamma} - \sin\alpha\sin\beta\cos\beta\cos^2\gamma\right)$$
$$+ C\sin\alpha\sin\beta\cos\beta$$

$$\underline{C_{13}^{H} = (C-A)\sin\alpha\sin\beta\cos\beta} = C_{31}^{H}$$

$$C_{23}^{H} = R_{2k}R_{3k}C_{k}^{x}$$

$$= A\left(\sin\alpha\,\cancel{\sin\beta\sin\gamma\cos\gamma} + \cos\alpha\sin\beta\cos\beta\sin^2\gamma\right.$$
$$\left. - \sin\alpha\,\cancel{\sin\beta\sin\gamma\cos\gamma} + \cos\alpha\sin\beta\cos\beta\cos^2\gamma\right)$$
$$+ C\left(-\cos\alpha\cos\beta\sin\beta\right)$$

$$\underline{C_{23}^{H} = (A-C)\cos\alpha\sin\beta\cos\beta}$$

Nyní konečně dosadíme C_{ij}^{H} do výrazu pro H (na str. 69.3)

$$\cancel{C_{11}^{H}C_{22}^{H} - (C_{12}^{H})^2 = \left[A(\cos^2\alpha + \sin^2\alpha\cos^2\beta) + C\sin^2\alpha\sin^2\beta\right]\cdot}$$
$$\cancel{\cdot\left[A(\sin^2\alpha + \cos^2\alpha\cos^2\beta) + C\cos^2\alpha\sin^2\beta\right] -}$$
$$\cancel{-(A-C)^2\sin^2\alpha\cos^2\alpha\sin^4\beta} =$$

lépe

$$= \left[A + (C-A)\sin^2\alpha\sin^2\beta\right]\left[A + (C-A)\cos^2\alpha\sin^2\beta\right] - (A-C)^2\sin^2\alpha\cos^2\alpha\sin^4\beta$$

$$= A^2 + A(C-A)\left(\sin^2\alpha\sin^2\beta + \cos^2\alpha\sin^2\beta\right) + \cancel{(C-A)^2\sin^2\alpha\cos^2\alpha\sin^4\beta}$$
$$\cancel{-(A-C)^2\sin^2\alpha\cos^2\alpha\sin^4\beta}$$

$$C_{11}^H C_{22}^H - (C_{12}^H)^2 = A\left[A+(C-A)\sin^2\beta\right] = A\left(A\cos^2\beta + C\sin^2\beta\right)$$

$$\text{čitatel} = -(C_{13}^H)^2 C_{22}^H - (C_{23}^H)^2 C_{11}^H + 2C_{12}^H C_{13}^H C_{23}^H$$
$$+ C_{33}^H\left[C_{11}^H C_{22}^H - (C_{12}^H)^2\right] =$$

$$= -(C-A)^2 \sin^2\alpha \sin^2\beta \cos^2\beta\left[A+(C-A)\cos^2\alpha\sin^2\beta\right]$$
$$-(A-C)^2\cos^2\alpha\sin^2\beta\cos^2\beta\left[A+(C-A)\sin^2\alpha\sin^2\beta\right]$$
$$+2(C-A)^3\sin^2\alpha\cos^2\alpha\sin^4\beta\cos^2\beta$$
$$+\left[C+(A-C)\sin^2\beta\right]A\left(A\cos^2\beta + C\sin^2\beta\right)$$

$$= -A(C-A)^2\sin^2\beta\cos^2\beta - \cancel{(C-A)^3\sin^2\alpha\cos^2\alpha}\sin^4\beta\cos^2\beta$$
$$-(C-A)^3\sin^2\alpha\,\cancel{\cos^2\alpha\sin^4\beta\cos^2\beta} + 2(C-A)^3\cancel{\sin^2\alpha\cos^2\alpha}\sin^4\beta\cos^2\beta$$
$$+\left[C+(A-C)\sin^2\beta\right]A\left(A\cos^2\beta + C\sin^2\beta\right)$$

$$= A\Big\{-(C-A)^2\sin^2\beta\cos^2\beta + C\left(A\cos^2\beta + C\sin^2\beta\right)$$
$$+(A-C)A\sin^2\beta\cos^2\beta + (A-C)C\sin^4\beta\Big\}$$

$$= A\Big\{\underbrace{\left(-C^2+2AC-\cancel{A^2}+\cancel{A^2}-AC\right)}_{C(A-C)}\sin^2\beta\cos^2\beta + C\left(A\cos^2\beta + C\sin^2\beta\right)$$
$$+(A-C)C\sin^4\beta\Big\}$$

$$= AC\Big\{(A-C)\sin^2\beta\cos^2\beta + A\cos^2\beta + C\sin^2\beta + (A-C)\sin^4\beta\Big\}$$

$$= AC\{(A-C)\sin^2\beta(\sin^2\beta+\cos^2\beta)+A\cos^2\beta+C\sin^2\beta\}$$

$$= AC\{A\sin^2\beta-C\sin^2\beta+A\cos^2\beta+C\sin^2\beta\}=A^2C$$

Konečně

$$\boxed{H=\frac{AC}{A\cos^2\beta+C\sin^2\beta}\Omega}$$

Úhel β je malý úhel, neboť vzdálenost osy úhlového momentu a „figure axis x_3" (= třetí hlavní osa momentu setrvačnosti) je malá. (Neplatí to však pro zbývající Euler. úhly Ω a μ, které mohou být konečné velikosti i pro infinitesimální rotaci).

Tedy $|\beta| \ll 1$ $\Rightarrow$

$$\underline{H} = \frac{AC}{A+C\beta}\Omega \approx \underline{C\Omega}$$

Solution of Poisson's equations. We shall use their complex combination (2.276):

$$\dot{\vartheta}+i\sin\vartheta\,\dot{\psi}=\frac{i}{C\Omega}Le^{i\Omega t}$$

The complex torque L is taken from (2.108)

$$L=(C-A)\Omega^2\sum_k B_k e^{-i(w_k t+\beta_k)}$$

The substitution of this expression into (2.276) yield

$$\dot{\vartheta}_H + i\,\dot{\psi}_H \sin\vartheta_H = i\frac{C-A}{C}\,\Omega \sum_k B_k\, e^{-i(\Delta\omega_k t + \beta_k)} \qquad (2.280)$$

with

$$\Delta\omega_k = \omega_k - \Omega\;;$$

we have written ϑ_H and ψ_H since Poisson's equations describe the angular momentum axis.

Before integrating these equations we must distinguish the cases $\Delta\omega_k = 0$ and $\Delta\omega_k \neq 0$. The first case also occurs since Ω is a frequency that appears in the development of the tidal potential (sec. 1.3.2 M+M). Let us number the frequencies ω in such a way that $k=0$ corresponds to the frequency Ω:

$$\omega_0 = \Omega\,,\quad \Delta\omega_0 = \Omega - \Omega = 0$$

Then (2.280) can be split up by distinguishing the cases $k=0$ and $k\neq 0$ (it is not difficult to see that $\beta_0 = 0$; why?)

$$\dot{\vartheta}_H + i\sin\vartheta_H\,\dot{\psi}_H = i\frac{C-A}{C}\,\Omega B_0 + i\frac{C-A}{C}\,\Omega \sum_{k\neq 0} B_k\, e^{-i(\Delta\omega_k t + \beta_k)}$$

This equation can be integrated immediately since we may assume that $\sin\vartheta_H$ on the left side to be constant without losing accuracy. This gives

$$\vartheta_H + i\sin\vartheta_H\,\psi_H = i\frac{C-A}{C}\,B_0(t-t_0) - \frac{C-A}{C}\sum_{k\neq 0}\frac{\Omega}{\Delta\omega_k}B_k\, e^{-i(\Delta\omega_k t+\beta_k)} + C_1 + iC_2$$

where $C_1 + iC_2$ is a complex constant of integration. On putting

$$\vartheta_H - C_1 = \Delta\vartheta_H \quad , \quad \psi_H \sin\vartheta_H - C_2 = \Delta\psi_H \sin\vartheta_H$$

this takes the final form

$$\boxed{\Delta\vartheta_H + i\sin\vartheta_H\,\Delta\psi_H = i\frac{C-A}{C}B_0(t-t_0) - i\frac{C-A}{C}\sum_{k\neq 0}\frac{\Omega}{\Delta\omega_k}B_k\,e^{-i(\Delta\omega_k t+\beta_k)}} \qquad (2.284)$$

The first term on the right-hand side increases linearly with time; it is a secular term expressing precession. The second term is periodic (or rather almost-periodic, see sec. 1.3.2.) and expresses nutation. Real expressions for $\Delta\psi_H$ and $\Delta\vartheta_H$ can be obtained by separating this equation into real and imaginary part. This immediately shows that there is no secular part in $\Delta\vartheta_H$.

Precession affects only $\Delta\psi_H$. On the other hand nutation affects both $\Delta\psi_H$ (nutation in longitude) and $\Delta\vartheta_H$ (in obliquity).

Again we see that the nutational frequency $\Delta\omega_k$ is obtained from the tidal frequency ω_i by subtracting the siderial frequency (rotational speed) Ω. This fact, of course, is intuitively evident since the tidal potential refers to an earth-fixed coordinate system $x_1 x_2 x_3$, and precession and nutation refer to an inertial system $X_1 X_2 X_3$; both system are rotating relatively to each other with angular velocity Ω.

Since

$$\omega_k \doteq \Omega \quad \Rightarrow \quad \Delta\omega_k \ll \Omega$$

(pro <u>celodenní slapové vlny</u>) the small $\Delta\omega_k$ in the denominator is the mathematical reason, why forced nutation (maximum amplitude on the order of 17" ~ 520 m) is so much larger than forced polar motion (maximum amplitude on the order of 0.02" corresponding to 60 cm

In Table are listed the nutation waves. The main nutations come from tidal waves which have very small amplitudes (see Melchior str. 50-51). The principal nutation originates from two tidal waves (with Doodson argument numbers 165.545 and 165.565). They are due to the motion of the node of the moon's orbit which has a period 18.6 years.

Tidal wave K_1, consisting of the lunar part $^mK_1$ and the solar part $^sK_1$, produces precession since it corresponds to the sidereal frequency Ω. Lunisolar precession amounts $\dot{\psi} = -50.''39$/year.
Perioda precesního pohybu je

$$T_\psi = \frac{2\pi}{\dot{\psi}} \approx 25\,770 \text{ tropických let}$$
(Platonian year)

za 72 let činí precesní pohyb jarního bodu asi 1°.

asi za 1200 let bude zemská osa (rotační) mířit přibližně k α Lyrae (Vega). V důsledku precese nesouhlasí již znamení zvěrokruhu s příslušnými souhvězdími; posun v ekliptikální délce činí již téměř 30°.

We also note from (2.254) that the precession (as well as nutation) is proportional to the constant

$$H = \frac{C-A}{C}.$$

This fact makes it possible to determine the constant H, which is called dynamical flattening (ellipticity) and is of basic importance for physical geodesy.

$$H = \frac{1}{305.459 \pm 0.002}$$

Tidal Argument	Symbol	Nutation Argument	Period in days	Amplitude $\Delta\psi$	Amplitude $\Delta\vartheta$	Type Origin
$\tau-s=(\tau+s)-2s$ $\tau+3s=(\tau+s)+2s$	O_1 OO_1	$2s$	13.7	-0.″204	0.″088	semi-monthly moon
$t-h=(t+h)-2h$ $t+3h=(t+h)+2h$	P_1 ϕ_1	$2h$	183	-1.″273	0.″552	semi-annual sun
$(\tau+s)+N$ $(\tau+s)-N$	- -	$N=-N'$	6798 =18.6 yrs	-17.″233	9.″210	principal nutation moon's node
$t+h=\tau+s$	K_1	0	∞			precession

TABLE 2.1. Correspondence between tides and nutations

TABLE 2.1. Correspondence between Tides and Nutations

Diurnal tides					
Argument	Origin Symbol	Doodson argument number	Frequencies f_1 in degrees per hour	Period	Amplitude of North-South component at the equator in microgals
$\tau-2s+p=(\tau+s)-(3s-p)$	L Q_1	135.655	13°398 660 9	26h52m48s	+ 5.942 8
$(\tau+s)+(3s-p)$	L -	195.455	16°683 476 3	21h34m42s	- 0.256 1
$\tau-s=(\tau+s)-2s$	L O_1	145.555	13°943 035 6	25h49m 8s	+31.039 1
$\tau+3s=(\tau+s)+2s$	L OO_1	185.555	16°139 101 7	22h18m22s	- 1.336 6
$(\tau+s)-(s-p)$	eL NO_1	155.655	14°496 694 0	24h49m59s	- 2.441 0
$(\tau+s)+(s-p)$	eL J_1	175.455	15°585 443 3	23h 5m53s	- 2.441 0
$t-2h+p_s=(t+h)-(3h-p_s)$	S π_1	162.556	14°917 864 7	24h 7m55s	+ 0.847 4
$(t+h)+(3h-p_s)$	S -	168.554	15°164 272 4	23h44m24s	- 0.036 2
$t-h=(t+h)-2h$	S P_1	163.555	14°958 931 4	24h 3m54s	+14.481 5
$t+3h=(t+h)+2h$	S ϕ_1	167.555	15°123 205 9	23h48m14s	- 0.622 6
$(t+h)-(h-p_s)$	eS S_1	164.556	15°000 002 0	23h59m56s	- 0.348 4
$(t+h)+(h-p_s)$	eS ψ_1	166.554	15°082 135 3	23h52m 8s	- 0.348 4
$t+h-2N$	L -	165.575	15°045 481 4	23h56m39s	+ 0.126 8
$(t-h)+N$	L -	165.545	15°038 862 2	23h56m16s	- 0.864 7
$(t+h)-N$	L -	165.565	15°043 275 1	23h55m51s	+ 5.914 8
$(t+h)$	LS K_1	165.555	15°041 068 6	23h56m 3s	-43.689 8

L: Lunar tides eL: elliptic lunar tides
S: Solar tides eS: elliptic solar tides

Z.M.

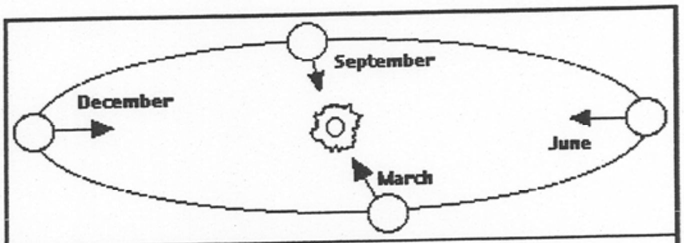
TABLE 2.1. Correspondence between Tides and Nutations

Nutations				
Frequency in degrees per hour $\frac{1}{2}(f_2-f_1)$	Period in days	Amplitude in the longitude $\Delta\psi$	Amplitude in the obliquity $\Delta\vartheta$	Name
$3\dot{s}-\dot{p}=\ell+2F+2\Omega=1°642\ 408$	9.1	- 0″026 1	+0″011 3	
$2\dot{s}=2F+2\Omega=1°098\ 033\ 1$	13.7	- 0″203 7	+0″088 4	fortnightly
$\dot{s}-\dot{p}=\ell=0°544\ 374\ 594$	27.6	+ 0″067 5	0	monthly
$3\dot{h}-\dot{p}_s=\ell'+2F-2D+2\Omega=0°123\ 203\ 9$	122	- 0″049 7	-0″021 6	
$2\dot{h}=2F-2D+2\Omega=0°082\ 137\ 3$	183	- 1″272 9	+0″552 2	semi-annual
$\dot{h}-\dot{p}_s=\ell'=0°041\ 066\ 678$	365	+ 0″126 1	0	annual
$2\dot{N}=2\dot{\Omega}=0°004\ 412\ 8$	3 399	+ 0″208 8	-0″090 4	
$\dot{N}=\dot{\Omega}=0°002\ 206\ 4$	6 798	-17″232 7	+9″210 0	principal nutation of 18.years 66
0	secular		0	precession

$\Delta\psi$ $\Delta\vartheta$

The Ecliptic

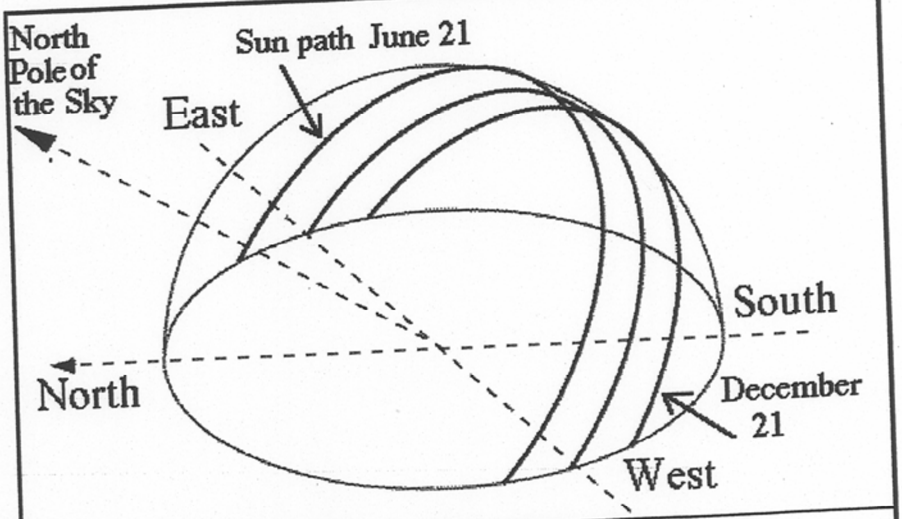
The path of the Sun across the celestial sphere is very close to that of the planets and the moon. After clocks became available, it was a relatively straightforward job for astronomers to relate the path of the Sun in the daytime to the one of stars at night, and to draw it on their star charts. Because of its relation to eclipses, that path is known as the **ecliptic**.



The orbit of the Earth around the Sun. This is a perspective view, the shape of the actualorbit is very close to a circle.

The significance of the ecliptic is evident if we examine the Earth's orbit around the Sun. That orbit lies in a **plane**, flat like a tabletop, called the **plane of the ecliptic** (or sometimes just "the ecliptic"). In one year, as the Earth completes a full circuit around the Sun (drawing above), the Earth-Sun line and its continuation past Earth sweep the entire plane. The far end of that line then traces the ecliptic on the celestial sphere; if you have a star chart handy (it is often included in an atlas), you will find the ecliptic traced there, too.

(2) The Path of the Sun, the Ecliptic



The apparent path of the Sun across the sky.
In summer, the Sun's path is longest, and so are the days.
In winter, the Sun's path is shortest, and so are the days.

The Columbia Encyclopedia, Sixth Edition. 2001.

equinox

(ē´kwĭnŏks) (KEY) , either of two points on the celestial sphere where the ecliptic and the celestial equator intersect. The vernal equinox, also known as "the first point of Aries," is the point at which the sun appears to cross the celestial equator from south to north. This occurs about Mar. 21, marking the beginning of spring in the Northern Hemisphere. At the autumnal equinox, about Sept. 23, the sun again appears to cross the celestial equator, this time from north to south; this marks the beginning of autumn in the Northern Hemisphere. On the date of either equinox, night and day are of equal length (12 hr each) in all parts of the world; the word *equinox* is often used to refer to either of these dates. The equinoxes are not fixed points on the celestial sphere but move westward along the ecliptic, passing through all the constellations of the zodiac in 26,000 years. This motion is called the precession of the equinoxes. The vernal equinox is a reference point in the equatorial coordinate system.