Non rigid Earth

1. Základní kinematické vztahy.

Uvažujme dvě soustavy souřadnic:

- inertial system
- rigid reference frame (ne příliš vhodný název: body frame)

rotuje s úhlovou rychlostí $\vec{\omega}$! kolem osy z.

~~Osa z: The z axis is a principal axis of the inertia tensor~~ $\overleftrightarrow{C}_0$

$$\overleftrightarrow{C}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix}$$

~~Thus $\overleftrightarrow{C}_0$ corresponds to the model of an undeformed earth whose principal axes of inertia coincide with the coordinate axes x, y, z, which has rotational symmetry (B = A) and whose principal moments of inertia are constant in time.~~

Protože se jedná o rotaci rigidní soustavy, úhlová rychlost $\vec{\omega}$ nezávisí na x, y, z, ale pouze na čase t,

$$\vec{\omega} = \vec{\omega}(t)$$

Nechť počátky obou soustav splývají. Potom jsme mohli říci, že časová derivace je

$$\left(\frac{d}{dt}\right)_{\text{inerciální}} = \left(\frac{d}{dt}\right)_{\text{rotující}} + \left(\vec{\omega}\times\ \right)_{\text{rotující}}$$

Např. pro polohový vektor $\vec{r}$ deformujícího se tělesa platí

$$\left(\frac{d\vec{r}}{dt}\right)_{inerc.} = \left(\frac{d\vec{r}}{dt}\right)_{rotující} + (\vec{\omega} \times \vec{r})_{rotující}$$

Pro rotaci tuhého tělesa je

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = 0$$

avšak pro rotaci deformujícího se tělesa je tato veličina nenulová. Proto zavedeme

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = \vec{v}$$

.... deformation velocity
(udává rychlost relativního pohybu vzhledem k rigidní referenční soustavě)

Celkově

$$\boxed{\left(\frac{d\vec{r}}{dt}\right)_{inerc.} = \underbrace{\vec{v} + \vec{\omega} \times \vec{r}}_{\text{vše k rigidní rotující soustavě}}} \qquad (3.28)$$

If the earth is not rigid, then there is no coordinate system in which all particles of the earth are at rest. Thus they move with respect to the rigid reference frame with a velocity $\vec{v}$ which is small becoming zero for a rigid body.

2. Úhlový moment (angular momentum, moment of momentum – moment hybnosti)

$$\vec{H} = \iiint \left(\vec{r} \times \frac{d\vec{r}}{dt} \right)_{\text{v inerciální soust.}} dm$$

$$\overset{(3.28)}{=} \iiint \left(\vec{r} \times (\vec{\omega} \times \vec{r}) + \vec{r} \times \vec{v} \right)_{\text{v rotující rigidní soustavě}} dm$$

První část upravíme stejně jako pro rotaci tuhého tělesa, tj.

$$\iiint \vec{r} \times (\vec{\omega} \times \vec{r})\, dm \overset{\text{str. 8}}{=} \iiint \left[(\vec{r} \cdot \vec{r}) I - \vec{r}\vec{r} \right] \cdot \vec{\omega}\, dm$$

Protože $\vec{\omega}$ udává rotaci tuhé (rigidní) referenční soustavy, je nezávislá na souřadnicích, závisí pouze na čase $\vec{\omega} = \vec{\omega}(t)$. Proto lze vytknout mimo integrál. Jinými slovy: v daném časovém okamžiku je o libov. místě deformujícího se tělesa referenční úhlová rychlost rotace stejná.
Celkově

$$\boxed{\vec{H} = \overleftrightarrow{C} \cdot \vec{\omega} + \vec{h}} \qquad (3.73)$$

$$\overleftrightarrow{C} = \iiint \left[(\vec{r} \cdot \vec{r}) I - \vec{r}\vec{r} \right] dm \quad \ldots \text{tensor setrvačnosti}$$

$$\vec{h} = \iiint (\vec{r} \times \vec{v})\, dm \quad \text{deformační (relativní) úhlový moment}$$

člen $\overleftrightarrow{C}\cdot\vec{\omega}$ je rotační úhlový moment (rotational angular momentum). Má stejný tvar jako pro rotaci tuhého tělesa viz. eq. (2.38). Protože se však těleso deformuje v čase bude tensor setrvačnosti $\overleftrightarrow{C}$ funkcí času $\overleftrightarrow{C} = \overleftrightarrow{C}(t)$.

3. Tisserand axes

Celková kinetická energie

$$T = \frac{1}{2}\iiint \left(\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right)_{\text{inerciální}} dm$$

Dosazení z (3.28):

$$T = \frac{1}{2}\iiint \left(\vec{\omega}\times\vec{r} + \underset{\text{tuhé, rotující}}{\vec{v}}\right)\cdot\left(\vec{\omega}\times\vec{r} + \underset{\text{tuhé, rotující}}{\vec{v}}\right) dm$$

$$T = T_{rot} + T_{def} + T_{rd}$$

kde

$$T_{rot} = \frac{1}{2}\iiint \left(\vec{\omega}\times\vec{r}\right)^2 dm \overset{\text{str. 10}}{=}$$

$$= \cancel{\vec{\omega}\cdot\vec{\omega}} = \frac{1}{2}\,\vec{\omega}\cdot\overleftrightarrow{C}\cdot\vec{\omega} \qquad (3.62)$$

↑ rotational energy.

the deformational energy

$$T_{def} = \frac{1}{2}\iiint (\vec{v}\cdot\vec{v})\, dm$$

The interaction energy T_{rd} between rotation and deformation:

$$T_{rd} = \iiint (\vec{\omega} \times \vec{r}) \cdot \vec{v}\, dm = \iiint \vec{\omega} \cdot (\vec{r} \times \vec{v})\, dm$$

$$\underset{\vec{\omega}\text{ nezávisí na souřadnicích}}{=} \vec{\omega} \cdot \iiint \vec{r} \times \vec{v}\, dm = \vec{\omega} \cdot \vec{h}$$

↑ deformační úhlový moment

$$\underline{T_{rd} = \vec{\omega} \cdot \vec{h}} \qquad (3.66)$$

If the body were rigid and the axes were fixed in it, the deformation velocity $\vec{v}$ would be identically zero, and $T_{def} = T_{rd} = 0$. The rotation vector $\vec{\omega}$ of the rigid reference frame is not uniquely determined by (3.28). We might select a slightly different $\vec{\omega}$ to try to minimize the deformation energy with respect to $\vec{\omega}$

$$T_{def}(\vec{\omega}) = \frac{1}{2} \iiint \vec{v} \cdot \vec{v}\, dm = \text{minimum}$$

$$\delta T = \iiint \vec{v} \cdot \delta \vec{v}\, dm = 0$$

avšak $\quad \vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{inerc} - \vec{\omega} \times \vec{r}$

$$\delta \vec{v} = - \delta \vec{\omega} \times \vec{r}$$

$$\delta T = -\iiint \vec{v}\cdot(\delta\vec{\omega}\times\vec{r})\,dm$$
$$= \iiint \vec{v}\cdot(\vec{r}\times\delta\vec{\omega})\,dm$$
$$= \iiint (\vec{v}\times\vec{r})\cdot\delta\vec{\omega}\,dm$$
$$= \delta\vec{\omega}\cdot\underbrace{\iiint(\vec{r}\times\vec{v})\,dm}_{\vec{h}}$$
$$= -\delta\vec{\omega}\cdot\vec{h}$$

$\delta T = 0 \Rightarrow \underline{\vec{h} = 0}$ neboť $\delta\vec{\omega}$ je libovolný vektor

We might use the freedom of changing the three components of $\vec{\omega}$ in such a way that three components of $\vec{h}$ become zero. Geocentric coordinate axes chosen in such a way are called <u>Tisserand axes.</u> Thus, by taking Tisserand axes, the relative angular momentum can be made zero.
Furthermore, the deformation energy is globally minimized in a least-squares sense. The interaction energy $T_{rd} = \vec{\omega}\cdot\vec{h}$ completely vanishes for Tisserand axes.

! It is frequently convenient to take axes other than Tisserand axes for the whole earth. In particular, in the theory of liquid core it is convenient to use axes which satisfy the Tisserand condition $\vec{h} = 0$ only for the mantle (so called <u>Tisserand axes for the mantle</u>), or an auxiliary coordinate system which rotates with constant angular velocity $\vec{\omega}_0$ with respect to the inertial space (the so-called <u>nutation frame</u>). Then of course, rotation and deformation will not be completely separated due to $T_{rd} \neq 0$.

4. <u>Liouville equations</u>

The basic equation

$$\frac{d\vec{H}}{dt} = \vec{L}$$

relating the angular momentum $\vec{H}$ and external torque $\vec{L}$, holds for a deformable as well as for a rigid body.

V rigidní referenční soustavě platí

$$\left(\frac{d}{dt}\right)_{inerc} = \left(\frac{d}{dt}\right)_{rot} + \left(\vec{\omega} \times \quad\right)_{rot}.$$

Tedy pro $\vec{H}$:

$$\left(\frac{dH}{dt}\right)_{rot} + \left(\vec{\omega} \times \vec{H}\right)_{rot} = \vec{L} \qquad (3.72)$$

Označme

$$\dot{H} = \left(\frac{dH}{dt}\right)_{rot} = \frac{\partial H}{\partial t}$$

Dále podle (3.73) je $\vec{H} = \overleftrightarrow{C}\cdot\vec{\omega} + \vec{h}$. Potom

$$\boxed{\left.\frac{d}{dt}\left(\overleftrightarrow{C}\cdot\vec{\omega} + \vec{h}\right)\right|_{rot} + \vec{\omega}\times\left.\left(\overleftrightarrow{C}\cdot\vec{\omega}+\vec{h}\right)\right|_{rot} = \vec{L}} \tag{3.75}$$

Liouville equation

For Tisserand axes we have $\dot{\vec{h}} = 0$ and (3.75) takes the form

$$\left.\frac{d}{dt}\left(\overleftrightarrow{C}\cdot\vec{\omega}\right)\right|_{rot} + \vec{\omega}\times\left.\left(\overleftrightarrow{C}\cdot\vec{\omega}\right)\right|_{rot} = \vec{L}$$

Linearization

The inertia tensor $\overleftrightarrow{C}$:

$$\overleftrightarrow{C} = \overleftrightarrow{C}_0 + \overleftrightarrow{c} \tag{3.77}$$

$$\overleftrightarrow{C}_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{pmatrix} \qquad \overleftrightarrow{c} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \tag{3.78}$$

v tuhé referenční rotující soustavě — v tuhé referen[…] rotující soustavě

Thus, $\overleftrightarrow{C}_0$ corresponds to the model of an undeformed earth whose principal axes of inertia coincide with the coordinate axes, which has rotational symmetry (B=A) and whose principal moments of inertia <u>A and C are constant in time!!</u> The tensor $\overleftrightarrow{c}$ takes into account the deviation of the actual earth from this simplified model.

!! (3.78) $\Rightarrow$ the z-axis is a principal axis of the tensor $\overleftrightarrow{C}_0$. avšak rotuje t[illegible] sebou jinou [illegible]

The rotation vector $\vec{\omega}$ is written as

$$\vec{\omega} = \vec{\omega}_0 + \delta\vec{\omega} \qquad (3.79)$$

where

$$\vec{\omega}_0 = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \qquad (3.80)$$

v tuhé referenční rotující soustavě

corresponds to a rotation with constant angular velocity Ω around the z-axis (of the rigid reference frame), and

$$\delta\vec{\omega} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \Omega \qquad (3.81)$$

v tuhé referenční rotující soustavě.

expresses the deviations of the rotation axis from the z-axis (m_1 and m_2) and variations of the rotational speed (m_3).

Both $\overleftrightarrow{c}$ and $\delta\vec{\omega}$ are considered small quantities whose squares, products and higher powers can be neglected.

We substitute (3.77) – (3.81) into (3.75) and retain linear terms only. Zkusme napsat první složku:

$$(\overleftrightarrow{c} \cdot \vec{\omega})_i = c_{ij}\, \omega_j$$

$$(\vec{c} \cdot \vec{\omega})_1 = c_{1j}\, \omega_j = c_{11}\, \omega_1 + c_{12}\, \omega_2 + c_{13}\, \omega_3 =$$

$$= \left[\left(A + c_{11} \right) m_1 + c_{12} m_2 + c_{13} \left(1 + m_3 \right) \right] \Omega$$

Časová derivace vzhledem k rigidní referenční soustavě

$$\frac{d}{dt} \left(\overleftrightarrow{C} \cdot \vec{\omega} \right)_1 \Big|_{rot} = \Big[\cancel{\dot{c}_{11} m_1} + \cancel{c_{11} \dot{m}_1} + \cancel{\dot{c}_{12} m_2} + \cancel{c_{12} \dot{m}_2} + A \dot{m}_1$$
$$+ \dot{c}_{13} + \cancel{\dot{c}_{13} m_3} + \cancel{c_{13} \dot{m}_3} \Big] \Omega$$

$$\overset{\text{linearizace}}{=} \left(A \dot{m}_1 + \dot{c}_{13} \right) \Omega$$

kde $(\dot{\ })$ značí časovou derivaci vůči rigidní rotující soustavě.

$$\vec{\omega} \times \left(\overleftrightarrow{C} \cdot \vec{\omega} \right) \Big|_1 = \omega_2 \left(\overleftrightarrow{C} \cdot \vec{\omega} \right)_3 - \omega_3 \left(\overleftrightarrow{C} \cdot \vec{\omega} \right)_2$$

$$= \omega_2 \left(C_{31} \omega_1 + C_{32} \omega_2 + C_{33} \omega_3 \right)$$
$$- \omega_3 \left(C_{21} \omega_1 + C_{22} \omega_2 + C_{23} \omega_3 \right)$$

$$= m_2 \Omega^2 \left(c_{31} m_1 + c_{32} m_2 + \left(C + c_{33} \right) \left(1 + m_3 \right) \right)$$
$$- \left(1 + m_3 \right) \Omega^2 \left(c_{21} m_1 + \left(A + c_{22} \right) m_2 + c_{23} \left(1 + m_3 \right) \right) =$$

$$\overset{\text{linearizace}}{=} m_2 \Omega^2 C - \left(1 + \cancel{m_3} \right) \Omega^2 A m_2 - \Omega^2 c_{23}$$

$$= \left(C - A \right) \Omega^2 m_2 - \Omega^2 c_{23}$$

Dále $\left(\vec{\omega} \times \vec{h}\right)_1 = \omega_2 h_3 - \omega_3 h_2 = m_2 \Omega h_3 - (1+m_3)\Omega h_2$

$$\overset{\text{linearizace}}{=} -\Omega h_2$$

První linearizovaná rovnice:

$$\boxed{A\Omega \dot{m}_1 + (C-A)\Omega^2 m_2 + \Omega \dot{c}_{13} - \Omega^2 c_{23} + \dot{h}_1 - \Omega h_2 = L_1}$$

analogicky se odvodí zbylé dvě:

$$A\Omega \dot{m}_2 - (C-A)\Omega^2 m_1 + \Omega \dot{c}_{23} + \Omega^2 c_{13} + \dot{h}_2 + \Omega h_1 = L_2$$

$$C\Omega \dot{m}_3 + \dot{c}_{33}\Omega + \dot{h}_3 = L_3 \qquad (3.82)$$

Euler-Liouville equations

where the dot !' denotes a time derivative with respect to a rigid rotating system. Equations (3.82) obviously generalize Euler's equations (2.74) or (2.112) and will therefore be called Euler-Liouville equations.

Again it will be convenient to combine the first two equations by using complex notation:

$$m = m_1 + i m_2$$
$$c = c_{13} + i c_{23}$$
$$L = L_1 + i L_2$$
$$h = h_1 + i h_2$$

The first two equations of (3.82) read

$$\boxed{A\Omega \dot{m} - i(C-A)\Omega^2 m + \Omega \dot{c} + i\Omega^2 c + \dot{h} + i\Omega h = L} \qquad (3.84)$$

Eq. (3.84) can be solved for $m = m_1 + i m_2$, obtaining m_1 and m_2 characterizing a deviation of the earth's rotation axis from the z-axis, that is polar motion. Similarly, the third eqs of (3.82) can be solved for m_3 to get variations of the speed of rotation or variations of the length of day.

In other words, m_1 and m_2 describe the variation of the direction of $\vec{\omega}$, and m_3 characterizes a variation of the length of $\vec{\omega}$. It is remarkable that both phenomena, polar motion and variations of the speed of rotation, are separated in the linear approximation.

We shall also linearize eqn. (3.73)

$$\vec{H} = \overleftrightarrow{C} \cdot \vec{\omega} + \vec{h}$$

$$H_1 = C_{1j}\, \omega_j + h_1 = C_{11}\, \omega_1 + C_{12}\, \omega_2 + C_{13}\, \omega_3 + h_1$$

$$= \left[(A + c_{11}) m_1 + c_{12} m_2 + c_{13}(1 + m_3) \right] \Omega + h_1$$

$$\underset{=}{\text{linearizace}} \; (A m_1 + c_{13}) \Omega + h_1$$

$$H_2 = \left[c_{21} m_1 + (A + c_{22}) m_2 + c_{23}(1 + m_3) \right] \Omega + h_2$$

$$\underset{=}{\text{lineariz.}} \; (A m_2 + c_{23}) \Omega + h_2$$

$$H_3 = C_{31}\omega_1 + C_{32}\omega_2 + C_{33}\omega_3 + h_3$$

$$= \left[c_{31} m_1 + c_{32} m_2 + (C + c_{33})(1 + m_3) \right] \Omega + h_3$$

$$\overset{\text{linearizace}}{=} (C + C m_3 + c_{33}) \Omega + h_3$$

$$\boxed{\begin{aligned} H_1 &= (A m_1 + c_{13}) \Omega + h_1 \\ H_2 &= (A m_2 + c_{23}) \Omega + h_2 \\ H_3 &= C\Omega + C\Omega m_3 + c_{33} \Omega + h_3 \end{aligned}}$$

(3.85)

(3.87)

The angular momentum axis (the direction of $\vec{H}$) is defined by the unit vector $\vec{e}^{H}$,

$$\vec{e}^{H} = \frac{\vec{H}}{|\vec{H}|}$$

$$|\vec{H}| = \sqrt{H_1^2 + H_2^2 + H_3^2} \overset{\text{linearizace}}{=} H_3$$

$$\frac{1}{|\vec{H}|} = \frac{1}{C\Omega} \left(1 - m_3 - \frac{c_{33}}{C} - \frac{h_3}{C\Omega} \right)$$

$$\boxed{\begin{aligned} (\vec{e}^{H})_1 &= \frac{1}{C\Omega} \left[(A m_1 + c_{13}) \Omega + h_1 \right] \\ (\vec{e}^{H})_2 &= \frac{1}{C\Omega} \left[(A m_2 + c_{23}) \Omega + h_2 \right] \\ (\vec{e}^{H})_3 &= 1 \end{aligned}}$$

(3.88)

The components $(\vec{e}^{M})_1$ and $(\vec{e}^{M})_2$ define the deviation of the angular momentum axis from the z-axis in the same way as m_1 and m_2 define the deviation of the rotation axis from the z-axis.

Finally, we consider the figure axis which is the axis of maximum inertia for the deformed earth, that is, the maximum principal axis of the tensor $\overleftrightarrow{C}$ (the z-axis is the maximum principal axis of the tensor $\overleftrightarrow{C}_0$). The determination of the principal axes of a symmetric tensor is easy! any of the three principal axes satisfies the eigenvalue equation

$$\overleftrightarrow{C}\cdot\vec{u} = \lambda\vec{u} \qquad \text{or} \qquad (\overleftrightarrow{C}-\lambda I)\cdot\vec{u}=0 \qquad (3.90)$$

denoting such an axis by an eigenvector $\vec{u}$. The eigenvalues are determined by equation

$$\det(\overleftrightarrow{C}-\lambda I) = 0$$

Dosazení z (3.77) dává

$$\begin{vmatrix} A+c_{11}-\lambda & c_{12} & c_{13} \\ c_{12} & A+c_{22}-\lambda & c_{23} \\ c_{13} & c_{23} & C+c_{33}-\lambda \end{vmatrix} = 0$$

$$(A+c_{11}-\lambda)(A+c_{22}-\lambda)(C+c_{33}-\lambda)+2c_{12}c_{23}c_{13}$$
$$-c_{13}^2(A+c_{22}-\lambda)-c_{23}^2(A+c_{11}-\lambda)-c_{12}^2(C+c_{33}-\lambda)=0$$

Linearizujme poslední rovnici tak, že vynecháme kvadratické a kubické členy v c_{ij}:

$$\left(A+c_{11}-\lambda\right)\left(A+c_{22}-\lambda\right)\left(C+c_{33}-\lambda\right)=0$$

$$\lambda_{1,2,3} = \begin{matrix} A+c_{11} \\ A+c_{22} \\ C+c_{33} \end{matrix}$$

Nyní předpokládejme, že $C-A \gg c_{ij}$. Potom největší vlastní číslo (v linearizovaném případě) je

$$\lambda_{max} = C+c_{33}$$

The axis of maximum inertia is very close to the z-axis, its unit vector $\vec{u}$ has the form

$$\vec{u} = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix} \qquad (3.91)$$

with b_1 and b_2 small of first order, so that $|\vec{u}|=1$ apart from second-order terms. Poznamenejme, že

$$\vec{u} \neq \begin{pmatrix} b_1 \\ b_2 \\ 1+b_3 \end{pmatrix}$$

neboť v tomto případě $|\vec{u}| = 1+b_3 \neq 1$ (pro linearizovaný případ)! We substitute (3.91) into (3.90):

$$\begin{aligned} (A-\lambda+c_{11})b_1 + c_{12}b_2 + c_{13} &= 0 \\ c_{12}b_1 + (A-\lambda+c_{22})b_2 + c_{23} &= 0 \\ c_{13}b_1 + c_{23}b_2 + (C-\lambda+c_{33}) &= 0 \end{aligned} \qquad (3.92)$$

Substituting the eigenvalue $\lambda_{max} = C + c_{33}$ with the maximum moment of inertia & neglecting second-order terms such as $c_{11} f_1$, $c_{12} f_2$, etc the third equation in (3.92) is identically satisfied, again up to second-order terms, and the first two equations reduce to

$$(A-C) f_1 + c_{13} = 0$$
$$(A-C) f_2 + c_{23} = 0$$

whence

$$\boxed{\begin{aligned} f_1 &= \frac{c_{13}}{C-A} \\ f_2 &= \frac{c_{23}}{C-A} \end{aligned}} \qquad (3.94)$$

Mac Cullagh's formula

Gravitační potenciál vně Země:

$$V(r,\vartheta,\varphi) = G \iiint\limits_V \frac{\varrho(r',\vartheta',\varphi')}{\ell}\, dV$$

Pro pozorovatele, který se nachází vně koule obklopující všechny hmoty, platí

$$\frac{1}{\ell} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\psi}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\psi)$$

Použijeme-li adiční teorém pro $P_n(\cos\psi)$

$$\frac{1}{\ell} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_{nm}(\vartheta,\varphi)\, Y^*_{nm}(\vartheta',\varphi')$$

Dosazení do vzorce pro potenciál a záměna integrace a sumace dává

$$\boxed{V(r,\vartheta,\varphi) = G\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n} V_{nm}\, Y_{nm}(\vartheta,\varphi)} \qquad (3.95)$$

kde

$$\boxed{V_{nm} = \frac{4\pi}{2n+1}\iiint\limits_V (r')^n \varrho(r',\vartheta',\varphi')\, Y^*_{nm}(\vartheta',\varphi')\, dV}$$

Vyjádřeme speciální koeficient V_{21}.

$$Y_{21}(\vartheta,\varphi) = -\frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin 2\vartheta\, e^{i\varphi}$$

$$V_{21} = \frac{4\pi}{5}\iiint_V (r')^2 \varrho(P')\left(-\frac{1}{4}\right)\sqrt{\frac{15}{2\pi}}\,\sin 2\vartheta\, e^{-i\varphi}\,dV$$

$$\underline{V_{21} = -\sqrt{\frac{3\pi}{10}}\iiint_V (r')^2 \varrho(P')\,\sin 2\vartheta\, e^{-i\varphi}\,dV} \qquad (*)$$

Deviační momenty:

$$I_{xz} = -\iiint_V x'z'\varrho(P')\,dV = -\iiint_V \varrho(P')(r')^2 \sin\vartheta'\cos\vartheta'\cos\varphi'\,dV$$

$$I_{yz} = -\iiint_V y'z'\varrho(P')\,dV = -\iiint_V \varrho(P')(r')^2 \sin\vartheta'\cos\vartheta'\sin\varphi'\,dV$$

$$I_{xz} - i\,I_{yz} = -\iiint_V \varrho(P')(r')^2\,\frac{1}{2}\sin 2\vartheta'\, e^{-i\varphi'}\,dV \qquad (**)$$

Porovnání (*) a (**):

$$\boxed{V_{21} = \sqrt{\frac{6\pi}{5}}\,(I_{xz} - i\,I_{yz})} \qquad \underline{\text{MacCullagh's formula}} \qquad (3.96)$$

i.e. the relation between the products of inertia I_{xz} and I_{yz} and the potential coefficient V_{21}.

<u>Tidal deformation</u>. The tesseral tidal potential v_{21} corresponds to $n=2$ and $m=1$ and thus affects c_{13} and c_{23}.

In fact, the lunisolar attraction slightly deforms the elastic earth. For a deformed earth, the gravitational potential will be slightly different from its gravitational potential of the undeformed state, because the masses that produce the gravitational attraction will, after deformation, have a slightly different location. Denote $\delta V'$ the change of gravitational potential due to deformation.

As we shall see later, for an external perturbing potential v (of spherical harmonic degree $n=2$), the change δV of the gravitational potential of the earth will be proportional to the perturbing potential:

$$\delta V = k v \qquad (3.97)$$

The constant k is called a Love number. The linear relation (3.97) may be considered a consequence of Hooke's law basic for elasticity: elastic deformation is proportional to the force applied to the elastic body; and a small deformation causes a small change of gravitational potential V which is proportional to the deformation.

This rather intuitive way looking at the problem must be made more rigorous. First, even for a purely elastic body, the Love number k in (3.97) depends on degree n of the spherical harmonic. Thus, if the lunisolar potential is expanded into a series of spherical harmonics, then each term corresponding to a certain n is multiplied by a different coefficient k. "The" Love number k corresponds to $n=2$ since this is the dominant term in spherical harmonic series for tidal potential. Second, in the presence of a fluid core, the Love number k will, in addition, slightly depend on the frequency w_k of the tidal wave under consideration.

We only note that, for the earth, we approximately have

$$k \doteq 0.3 \qquad (3.98)$$

The case of a completely rigid earth also fits into this scheme. For a rigid body, there are no elastic deformations, and the shape of such a body and its gravitational potential V do not change. Hence, $\delta V = 0$, and (3.97) implies

$$k = 0 \qquad \text{for a rigid body.}$$

Therefore, formulas for a rigid earth can be obtained from those for an elastic earth by simply putting $k = 0$.

The tesseral tidal potential ($n=2, m=1$) has the form (viz. str 19.B)

$$v_{21}(r,\vartheta,\varphi) = \frac{GM}{a^2}\left(\frac{r}{a}\right)^2\left[v_{21}(t)\,Y_{21}(\vartheta,\varphi) + v_{2-1}(t)\,Y_{2-1}(\vartheta,\varphi)\right] \qquad (3.100)$$

where

$$v_{21}(t) = a\,\frac{M}{M}\,\frac{4\pi}{5}\left(\frac{a}{d}\right)^3 Y^*_{21}(p,\Lambda)$$

at the earth's surface $r=a$:

$$v_{21}(a,\vartheta,\varphi) = \frac{GM}{a^2}\left[v_{21}(t)\,Y_{21}(\vartheta,\varphi) + v_{2-1}(t)\,Y_{2-1}(\vartheta,\varphi)\right]$$

By (3.97), this tidal potential produces a change of the gravitational potential of the earth given by

$$(3.103) \quad \delta V = k\,v_{21}(a,\vartheta,\varphi) = k\,\frac{GM}{a^2}\left[v_{21}(t)\,Y_{21}(\vartheta,\varphi) + v_{2-1}(t)\,Y_{2-1}(\vartheta,\varphi)\right]$$

at the earth's surface $r=a$. Outside this surface it has the form

(3.104) $$\delta V(r,\vartheta,\varphi) = k\frac{GM}{a^2}\left(\frac{a}{r}\right)^3\left[v_{21}(t)Y_{21}(\vartheta,\varphi) + v_{2-1}(t)Y_{2-1}(\vartheta,\varphi)\right]$$

because the dependence of the external gravitational potential of the earth is proportional $1/r^{n+1} = 1/r^3$, and for $r=a$ (3.104) must reduce to (3.103).

The comparison of (3.104) with (3.95) for $n=2$ and $m=\pm 1$ gives

$$\delta V_{21} = k\, M a\, v_{21}(t) \qquad (3.105)$$

and analogously for δV_{2-1}. MacCullagh's formula (3.96) now yields

$$c_{13} - i\,c_{23} = \sqrt{\frac{5}{6\pi}}\, k\, M a\, v_{21}(t) \qquad (3.105')$$

$$= \sqrt{\frac{5}{6\pi}}\, k\, \cancel{M} a \cdot a \frac{\mu}{\cancel{M}} \frac{4\pi}{5}\left(\frac{a}{d}\right)^3 Y^*_{21}(\varphi,\Lambda)$$

$$\boxed{c_{13} + i\,c_{23} = 2\sqrt{\frac{2\pi}{15}}\, k\, a^5 \frac{G\mu}{d^3}\, Y_{21}(\varphi,\Lambda)}$$

The function $\varkappa\, Y_{21}(\varphi,\Lambda)$, $\varkappa = -\sqrt{\frac{24\pi}{5}}\frac{G\mu}{d^3}$, is expanded into (theoretically infinite) trigonometric series (viz. str. 24.4)

$$\varkappa\, Y_{21}(\varphi,\Lambda) = \sum_k C_k\, e^{-i(\omega_k t + \beta_k)}$$

Then

$$\boxed{c_{13} + i\,c_{23} = -\frac{1}{3}\frac{k a^5}{G}\sum_k C_k\, e^{-i(\omega_k t + \beta_k)}} \qquad (3.106)$$

which determines c_{13} and c_{23} for the tidal deformatio[n]

<u>Rotational deformation</u>. If the earth rotates about an axis which deviates from the axis of symmetry, there occur centrifugal forces which tend to distort it, and an elastic earth yields to this distortion; this is similar to distorting forces acting on an unbalanced wheel.

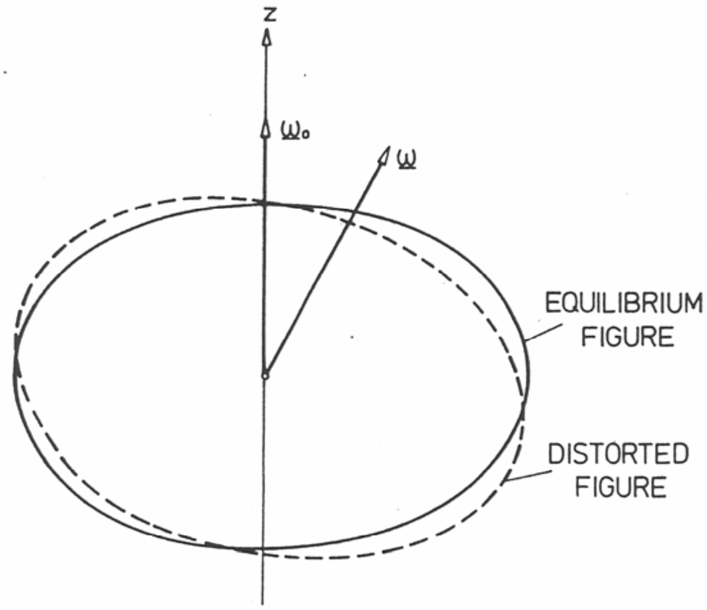

Fig. 3.3. A deviation of the rotation axis (ω) from the equilibrium symmetry axis (ω_0, corresponding to the z-axis), produces a distortion

The centrifugal force on a unit mass is well known to be given by

$$\vec{f_c} = -\vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad (3.107)$$

where $\vec{r}$ is the position vector and $\vec{\omega}$ the angular velocity vector as usual ($\vec{r}$ je posiční vektor v tuhé referentní rotující soustavě). We are only interested in the differential force produced by a deviation of $\vec{\omega}$ from the "normal" angular velocity vector (3.80) which coincides with the equilibrium figure axis (Fig. 3.3). This differential force $\delta\vec{f_c}$ thus is the difference between (3.107) and the equilibrium centrifugal force

$$\vec{f}_{c,\phi} = -\vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r})$$
$$= -\Omega \vec{e}_3 \times (\Omega \vec{e}_3 \times \vec{r})$$
$$= -\Omega^2 \vec{e}_3 \times (x \vec{e}_2 - y \vec{e}_1)$$
$$= \Omega^2 (x \vec{e}_1 + y \vec{e}_2) \qquad (3.108)$$

Hence

$$\delta f_c = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) \qquad (3.109)$$

ale

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega_i \vec{e}_i \times (\omega_j \vec{e}_j \times x_k \vec{e}_k)$$
$$= \omega_i \omega_j x_k \, \vec{e}_i \times (\vec{e}_j \times \vec{e}_k) = \omega_i \omega_j x_k \, \vec{e}_i \times \varepsilon_{jkm} \vec{e}_m$$
$$= \omega_i \omega_j x_k \, \varepsilon_{jkm} \varepsilon_{imn} \vec{e}_n = -\omega_i \omega_j x_k \, \varepsilon_{jkm} \varepsilon_{inm} \vec{e}_n$$
$$= -\omega_i \omega_j x_k (\delta_{ji} \delta_{kn} - \delta_{jn} \delta_{ki}) \vec{e}_n = -\omega_i^2 x_k \vec{e}_k + \omega_i \omega_j x_i \vec{e}_j$$
$$= (-\omega_i^2 x_j + \omega_i \omega_j x_i) \vec{e}_j$$

Now, we substitute from (3.79)-(3.80):

$\underline{j=1}$

$$-\omega_i^2 x_1 + \omega_i x_i \omega_1 = -x_1 \left(\cancel{m_1^2} + m_2^2 + (1+m_3)^2\right)\Omega^2 +$$
$$+ m_1 \Omega^2 \left(\cancel{m_1 x_1} + m_2 x_2 + (1+m_3) x_3\right)$$
$$\overset{\text{linearizace}}{=} \left[-x_1 (1+2m_3) + m_1 x_3\right] \Omega^2$$

$$\underline{j=2} \qquad -\omega_i^2 x_2 + \omega_i \cdot x_i \cdot \omega_2 = -x_2\left(m_1^2 + \cancel{m_2^2} + (1+m_3)^2\right)\Omega^2$$
$$+ m_2 \Omega^2 \left(m_1 x_1 + \cancel{m_2 x_2} + (1+m_3) x_3\right)$$

$$\overset{\text{linear'z.}}{=} \left[-x_2(1+2m_3) + m_2 x_3\right]\Omega^2$$

$$\underline{j=3} \qquad -\omega_i^2 x_3 + \omega_i \cdot x_i \cdot \omega_3 = -x_3\left(m_1^2 + m_2^2 + (1+m_3)^2\right)\Omega^2$$
$$+ (1+m_3)\Omega^2\left(m_1 x_1 + m_2 x_2 + (1+m_3) x_3\right)$$

$$\overset{\text{linear.}}{=} \left[-x_3(1+\cancel{2m_3}) + m_1 x_1 + m_2 x_2 + (1+\cancel{2m_3}) x_3\right]\Omega^2$$

$$= \left[m_1 x_1 + m_2 x_2\right]\Omega^2$$

The differential force (3.109) now reads

$$\delta f_c = \Omega^2 \left\{ \left[2m_3 x_1 - m_1 x_3\right]\vec{e}_1 + \left[2m_3 x_2 - m_2 x_3\right]\vec{e}_2 - \left[m_1 x_1 + m_2 x_2\right]\vec{e}_3 \right\}$$

This can be expressed in the form

$$\delta f_c = \operatorname{grad} \phi_c \tag{3.112}$$

where

$$\phi_c = \Omega^2\left[m_3(x_1^2 + x_2^2) - m_1 x_1 x_3 - m_2 x_2 x_3\right] \tag{3.113}$$

Thus δf_c is the gradient of a certain function ϕ_c which may be considered an <u>incremental centrifugal potential</u>.

Since we are dealing with polar motion only, disregarding variations in the speed of rotation, we put

$$m_3 = 0$$

Then

$$\phi_c = -\Omega^2 \left(m_1 x_1 x_3 + m_2 x_2 x_3 \right)$$

Vyjádříme nyní ϕ_c pomocí sférických harmonik

$$x_1 = r \sin\vartheta \cos\varphi$$
$$x_2 = r \sin\vartheta \sin\varphi$$
$$x_3 = r \cos\vartheta$$

$\rightarrow$

$$x_1 x_3 = r^2 \sin\vartheta \cos\vartheta \cos\varphi$$
$$x_2 x_3 = r^2 \sin\vartheta \cos\vartheta \sin\varphi$$

$$Y_{21} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\vartheta \cos\vartheta \, e^{i\varphi}$$ $\cos\varphi + i\sin\varphi$

$$Y_{2-1} = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\vartheta \cos\vartheta \, e^{-i\varphi}$$ $\cos\varphi - i\sin\varphi$

$$-Y_{21} + Y_{2-1} = \sqrt{\frac{15}{2\pi}} \sin\vartheta \cos\vartheta \cos\varphi$$
$$Y_{21} + Y_{2-1} = -i\sqrt{\frac{15}{2\pi}} \sin\vartheta \cos\vartheta \sin\varphi$$

$$x_1 x_3 = r^2 \sqrt{\frac{2\pi}{15}} \left(Y_{2-1} - Y_{21} \right)$$
$$x_2 x_3 = r^2 \sqrt{\frac{2\pi}{15}} \, i \left(Y_{2-1} + Y_{21} \right)$$

Nakonec

$$\phi_c = -\Omega^2 r^2 \sqrt{\frac{2\pi}{15}} \left[m_1 \left(Y_{2-1} - Y_{21} \right) + m_2 i \left(Y_{2-1} + Y_{21} \right) \right]$$
$$= -\Omega^2 r^2 \sqrt{\frac{2\pi}{15}} \left[\left(m_1 + i m_2 \right) Y_{2-1} + \left(-m_1 + i m_2 \right) Y_{21} \right] \quad (3.113)$$

The comparison of (3.113') with (3.100) shows that Φ_c acts like some kind of fictitious external potential: this is not surprising since inertial forces such as the centrifugal force behave, in many respects, like genuine forces:

$$\frac{GM}{a^2} v_{21}^c(t) = -\Omega^2 a^2 \sqrt{\frac{2\pi}{15}} \left(-m_1 + i m_2\right) \qquad (3.114)$$

Thus

$$v_{21}^c(t) = \frac{\Omega^2 a^4}{GM} \sqrt{\frac{2\pi}{15}} \left(m_1 - i m_2\right)$$

Using the correspondence (3.114) we obtain the products of inertia for rotational deformation from (3.105')

$$c_{13} - i c_{23} = \sqrt{\frac{5}{6\pi}}\, k M a\, v_{21}^c(t)$$

$$= \sqrt{\frac{5}{6\pi}}\, k \cancel{M} a \frac{\Omega^2 a^4}{G\cancel{M}} \sqrt{\frac{2\pi}{15}} \left(m_1 - i m_2\right)$$

$$c_{13} - i c_{23} = \frac{1}{3} k \frac{\Omega^2 a^5}{G} \left(m_1 - i m_2\right) \qquad (3.115)$$

Introducing

$$k_s := \frac{3G(C-A)}{\Omega^2 a^5} \qquad (3.117)$$

we have

$$\boxed{c_{13} - i c_{23} = \frac{k}{k_s} (C-A)(m_1 - i m_2)} \qquad (3.116)$$

k_s bears the somewhat unfortunate name of "secular Love number". It is a dimensionless quantity

Its value

$$k_s = 0.94 \qquad (3.118)$$

can be computed from (3.117). Despite its name, the secular Love number k_s has nothing to do with the elastic Love number k; this is also visible from the numerical values (3.98) and (3.118)

If the earth were not elastic, but completely fluid, then, according to Munk & MacDonald (1960, p. 26, eq. (5.4.3)) the 'elastic' Love number

$$k = 0.96 \quad \text{for a fluid body} \qquad (3.119)$$

Thus the elastic Love number k may lie between the extreme values $k=0$ (rigid body) and $k=0.96$ (fluid body); the value $k=0.3$ for the elastic earth expresses the fact that the surface of the earth yields to the tidal potential to about one third of the maximum possible value.

Now k_s as given by (3.118) is very close to this maximum value, 0.96, for a fluid earth. We may think that for perturbing forces which are constant for a very long time (say, over a billion years) the earth behaves like a fluid body: thus we understand the reasonably good success of explaining the earth's flattening by the theory of hydrostatic equilibrium of a rotating fluid body. Now, "secular" is used in astronomy in the sense of "constant or uniformly acting for a long time", whence the term "secular Love number".

Polar motion

Now we are in a position to transform and solve the Euler-Liouville equation (3.84) for m, defining the direction of the rotation axis. Then (3.88) gives the angular momentum axis. Finally, the figure axis is obtained from (3.94). Let us now carry out this task:

Transformation of the Euler-Liouville equation. We choose Tisserand axes (sec. 3.1). Then the relative angular momentum vanishes

$$\vec{h} = 0$$

and hence also

$$h = h_1 + i h_2 = 0$$

Then the Euler-Liouville equation (3.84) reduces to

$$A\Omega \dot{m} - i(C-A)\Omega^2 m + \Omega \dot{c} + i\Omega^2 c = L \qquad (3.122)$$

We recall the notations:

$$m = m_1 + i m_2$$
$$c = c_{13} + i c_{23}$$
$$L = L_1 + i L_2$$

are complex numbers characterizing the rotation axis (m), off-diagonal elements of the inertia tensor (c), and the torque (L). Ω is an average value for the earth's rotational velocity, and A and C are average values for the principal moments of inertia.

The part of c due to tidal deformation is given by eqn. (3.106):

$$c_{tidal} = -\frac{1}{3}\frac{k a^5}{G}\sum_k C_k \, e^{-i(\omega_k t+\beta_k)} \tag{3.124}$$

whereas the part due to rotational deformation is given by eqn. (3.116):

$$c_{rot} = \frac{k}{k_s}(C-A)\,m \,. \tag{3.125}$$

The secular Love number k_s, defined by eqn. (3.117), may also be introduced in (3.124), which gives

$$c_{tid} = -\frac{k}{k_s}\frac{C-A}{\Omega^2}\sum_k C_k \, e^{-i(\omega_k t+\beta_k)} \,. \tag{3.126}$$

The total c is the sum of both contributions (3.125) and (3.126):

$$\underline{c = \frac{k}{k_s}(C-A)\left(m - \frac{1}{\Omega^2}\sum_k C_k \, e^{-i(\omega_k t+\beta_k)}\right)} \tag{3.127}$$

Finally, the torque L is given on page 24.5:

$$L_{tid} = -i(C-A)\sum_k C_k \, e^{-i(\omega_k t+\beta_k)} \tag{3.132}$$

Now we substitute (3.127) and (3.132) into the Euler-Liouville equation (3.122):

$$A\Omega \dot{m} - i(C-A)\Omega^2 m + \Omega\frac{k}{k_s}(C-A)\left[\dot{m} - \frac{1}{\Omega^2}\sum_k(-i\omega_k)C_k \, e^{-i(\omega_k t+\beta_k)}\right]$$

$$+ i\Omega^2\frac{k}{k_s}(C-A)\left[m - \frac{1}{\Omega^2}\sum_k C_k \, e^{-i(\omega_k t+\beta_k)}\right] =$$

$$= -i(C-A)\sum_k C_k \, e^{-i(\omega_k t+\beta_k)}$$

$$\Omega\left[A+\frac{k}{k_s}(C-A)\right]\dot{m}-i\Omega^2(C-A)\left(1-\frac{k}{k_s}\right)m=$$

$$=-i(C-A)\left[1-\frac{k}{k_s}\,\frac{\omega_h-\Omega}{\Omega}\right]\sum_h C_h\,e^{-i(\omega_h t+\beta_h)} \tag{3.134}$$

This is an inhomogeneous differential equation of first order for m; of course, $\dot{m}$ is the time derivative of m. As is well known, the general solution of such an equation is obtained as the sum of the general solution of the corresponding homogeneous equation plus a particular solution of the given inhomogeneous equation. We thus shall first consider the homogeneous equation.

Free polar motion. This homogeneous equation is obtained from (3.134) by replacing the right-hand side by zero:

$$\Omega\left[A+\frac{k}{k_s}(C-A)\right]\dot{m}-i\Omega^2(C-A)\left(1-\frac{k}{k_s}\right)m=0$$

It corresponds to free polar motion in the absence of external forces ($C_h=0$). On putting

$$\sigma_c=\frac{(C-A)\left(1-\frac{k}{k_s}\right)}{A+\frac{k}{k_s}(C-A)}\,\Omega \tag{3.136}$$

this equation reduces to

$$\dot{m}-i\sigma_c m=0. \tag{3.137}$$

It has the same form as Euler's equation in complex form (see eq. (2.85)). The solution of (3.137) is

$$m = m_0 \, e^{i \sigma_c t} \qquad (3.139)$$

where m_0 is a complex constant.

So far, everything has been analogous to sec. 2.2. The big difference, however, consists in the fact that the Euler frequency σ_E

$$\sigma_E = \frac{C-A}{A} \Omega$$

completely differs from the Chandler frequency σ_c. The Chandler frequency (3.136) may be expressed in terms of the Euler frequency as

$$\boxed{\sigma_c = \sigma_E \frac{1 - \frac{k}{k_s}}{1 + \frac{k}{k_s} \frac{\sigma_E}{\Omega}}}$$

With the numerical values

$$k \doteq 0.3 \qquad k_s \doteq 1 \qquad \frac{\sigma_E}{\Omega} \doteq \frac{1}{300}$$

this gives, approximately

$$\underline{\sigma_c \doteq 0{,}7 \, \sigma_E}$$

which means that the Euler period of about 304 days (for a rigid earth) is lengthened by a factor $1/0.7 \doteq 1{,}4$ to give the observed Chandler period of about 430 days. This a strong indication that the earth is, in fact, behaving like an elastic rather than a rigid body!

Forced polar motion. Let us now turn to the inhomogeneous eq. (3.134), incorporating the lunisolar forces through it's right-hand term. Using (3.136) this equation may be written in the simpler form

$$\dot{m} - i\sigma_c m = -i \frac{\sigma_E}{\Omega^2} \frac{1 - \frac{k}{k_s}\frac{\omega_k - \Omega}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} \sum_k C_k e^{-i(\omega_k t + \beta_k)}$$

a particular solution is

$$m_p = \frac{\sigma_E}{\sigma_c + \omega_k} \frac{1 - \frac{k}{k_s}\frac{\omega_k - \Omega}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} \frac{1}{\Omega^2} \sum_k C_k e^{-i(\omega_k t + \beta_k)}$$

as we immediately verify by substitution.

a different form is obtained by using (3.132)

$$m_p = \frac{i}{\sigma_c + \omega_k} \frac{1 - \frac{k}{k_s}\frac{\omega_k - \Omega}{\Omega}}{1 + \frac{k}{k_s}\frac{\sigma_E}{\Omega}} \frac{1}{A\Omega} L_{tid} \qquad (3.146)$$

This equation shows that the forced polar motion m_p is directly proportional to the lunisolar torque L_{tid}.

Modification of Liouville's equations

Let us consider the Liouville equations (3.82). We divide them by $(C-A)\Omega^2$ and $C\Omega$, respectively, and rearrange, taking

$$\sigma_E = \frac{C-A}{A}\,\Omega$$

for the Eulerian frequency. The result is

$$\boxed{\begin{aligned} \frac{1}{\sigma_E}\dot{m}_1 + m_2 &= \psi_2 \\ \frac{1}{\sigma_E}\dot{m}_2 - m_1 &= -\psi_1 \\ \dot{m}_3 &= \dot{\psi}_3 \end{aligned}} \qquad (5.87)$$

where we have put <u>(excitation functions)</u>:

$$\psi_1 = \frac{1}{(C-A)\Omega^2}\left(\Omega^2 c_{13} + \Omega \dot{c}_{23} + \Omega h_1 + \dot{h}_2 - L_2\right)$$

$$\psi_2 = \frac{1}{(C-A)\Omega^2}\left(\Omega^2 c_{23} - \Omega \dot{c}_{13} + \Omega h_2 - \dot{h}_1 + L_1\right)$$

$$\dot{\psi}_3 = \left(-\Omega \dot{c}_{33} - \dot{h}_3 + L_3\right)\frac{1}{C\Omega}$$

so that

$$\psi_3 = \frac{1}{C\Omega}\left(-\Omega c_{33} - h_3 + \int_0^t L_3\, dt\right)$$

The first two equations can be combined by employing complex number

$$\frac{i}{\sigma_E}\dot{m} + m = \psi \qquad (5.84)$$

with $\quad m = m_1 + i m_2 \quad , \quad \psi = \psi_1 + i\psi_2$

The differential equations (5.81) can be solved explicitly. The solution of the third equation is,

$$m_3 = \psi_3 + \text{konst.}$$

whereas the solution of (5.84) is

$$m(t) = e^{i\sigma_E t}\left[m(0) - i\sigma_E \int_0^t \psi(\tau) e^{-i\sigma_E \tau} d\tau\right]$$

This is immediately verified by forming $m(t)e^{-i\sigma_E t}$ and differentiating. The result is

$$\dot{m}(t) = i\sigma_E m(t) + e^{i\sigma_E t}\left[-i\sigma_E \psi(t) e^{-i\sigma_E t}\right]$$

or

$$(\dot{m} - i\sigma_E m) = -i\sigma_E \psi(t)$$

which is clearly equivalent to (5.84).

A certain disadvantage of the excitation functions ψ_i is the fact that ψ_1 and ψ_2 contain time derivatives of h_1 and h_2. Since h_i are usually given empirically, the time differentiation of an empirical function increases its inaccuracies. That is why we overcome this disadvantage by introducing, instead of ψ_1 and ψ_2, the so-called <u>angular momentum functions</u>

$$\chi_1 = \frac{1}{(C-A)\Omega}(\Omega c_{13} + h_1)$$

$$\chi_2 = \frac{1}{(C-A)\Omega}(\Omega c_{23} + h_2)$$

or, as a complex combination

$$\chi = \frac{1}{(C-A)\Omega}(\Omega c + h) \qquad (5.89)$$

with $\chi = \chi_1 + i\chi_2$

$c = c_{13} + i c_{23}$

$h = h_1 + i h_2$

Then the first two equations of (5-79) take the form

$$\frac{1}{\sigma_E} \dot{m}_1 + m_2 = \chi_2 - \frac{1}{\Omega} \dot{\chi}_1$$

$$\frac{1}{\sigma_E} \dot{m}_2 - m_1 = -\chi_1 - \frac{1}{\Omega} \dot{\chi}_2$$

or, in complex notation

$$\underline{m + \frac{i}{\sigma_E} \dot{m} = \chi - \frac{i}{\Omega} \dot{\chi}} \qquad (5.92)$$

Again, there is an explicit solution:

$$\boxed{\begin{aligned} m(t) = e^{i\sigma_E t} \Big[m(0) - i\sigma_E \Big(1 + \frac{\sigma_E}{\Omega}\Big) \int_0^t \chi(\tau) e^{-i\sigma_E \tau} d\tau \Big] \\ - \frac{\sigma_E}{\Omega} \Big[\chi(t) - e^{i\sigma_E t} \chi(0) \Big] \end{aligned}}$$

which again can easily be verified by differentiation.

Effects of elasticity

As we have seen, the rotational deformation of an elastic earth introduces a change in its inertia tensor; by eq. (3.116) we have

$$c^R = c_{13}^R + i\, c_{23}^R = \frac{k}{k_s}(C-A)m$$

where the tidal Love number $k \doteq 0{,}3$ an the "secular Love number" $k_s \doteq 0{,}94$. The label 'R' indicates rotational deformation.

This rotational change c^R produces now a corresponding χ^R by (5.89):

$$\chi^R = \frac{1}{(C-A)\Omega}\left(\Omega c^R + \underbrace{h^R}_{=0}\right) = \frac{k}{k_s} m \qquad (5.122)$$

which must be added to the original χ to take elasticity into account. Thus, in (5.92) χ is to be replaced by $\chi + \chi^R$, and using we then have

$$m + \frac{i}{\sigma_E}\dot{m} = \frac{k}{k_s} m - \frac{i}{\Omega}\frac{k}{k_s}\dot{m} + \chi - \frac{i}{\Omega}\dot{\chi}$$

This is readily given the form

$$m + \frac{i}{\sigma_c}\dot{m} = \frac{k_s}{k_s - k}\left(\chi - \frac{i}{\Omega}\dot{\chi}\right) \qquad (5.123)$$

where σ_c is the Chandler frequency (3.141).

The comparison of (5.123) and (5.92) shows that the effect of elasticity consists in replacing the Euler frequency σ_E by the Chandler frequency σ_C and the angular momentum function by the "effective angular momentum function"

$$\frac{k_s}{k_s - k}\chi \doteq 1{,}43\,\chi .$$