and spheroidal. The toroidal free oscillations are characterized by the radial component of the displacement vector and the volume dilatation being zero. Consequently, these oscillations are not concommitent with changes of density, nor with perturbations of the gravitational potential. The toroidal equations of motion consist of a system of two ordinary differential equations of the 1st order. Whereas the energy of the toroidal oscillation is restricted to the solid elastic regions of the Earth model, spheroidal oscillations may "propagate" even through a liquid. These oscillations are characterized by a zero radial component of the rotation of the displacement vector, but the other quantities are, in general, non-zero. The spheroidal equations of motion consist of a system of six ordinary 1st-order differential equations. Radial oscillations ($n = 0$) are a special case of spheroidal oscillations.

In defining the initial values of numerical integration of the equations of motion and in the matrix solution of free oscillations, the eigenfunctions for the homogeneous model have to be known. We have proved that, for this particular model, the eigenfunctions of the oscillations can be expressed by a combination of spherical Bessel functions. In defining the initial values of numerical integration of systems of equations of motion in the neighbourhood of the model's centre, we used a different method, i.e. the expansion of the eigenfunctions into a power series in r in the neighbourhood of the origin.

Another important problem is determining the roots of the secular function. For the SNREI Earth model, we have derived a relation for computing an improved value of the eigenfrequency, using the variation method with a boundary term, with the aid of the tested frequency and the eigenfunctions computed for this tested frequency. The first three sections of Chapter 8 describe the method of solving the system of ordinary differential equations numerically for the free oscillations of the SNREI Earth model. This then involves the description of program functions, inclusive of instructions for using them, as written for the purpose of solving the problems on hand numerically. In Section 4 of Chapter 8, we present some of the eigenperiods of model 1066A and compare them with observed eigenperiods.

SUPPLEMENT A. TENSOR ANALYSIS

A.1. Introduction

To facilitate the understanding of the principal part of this study, we shall briefly deal with tensor analysis in this supplement. Tensor analysis is a natural expansion of vector analysis. As in the case of vectors, we shall formulate the tensor calculus for an arbitrary coordinate system. We shall introduce tensor with the aid of invariant properties of coordinate transformation. Since physical laws are invariant with respect to a particular coordinate system, the introduction of tensors via their invariant properties will provide a natural and powerful tool for formulating physical laws. There exist a large number of books and monographs of various sophistication on the subject. We recommend [65, 82, 89, 104, 125]. An account particularly suitable to continuum physics can be found in [56—59].

A.2. Curvilinear coordinates

Assume the position of an arbitrary point P in three-dimensional space to be determined by its Cartesian coordinates y^1, y^2, y^3.

Consider the transformation of these coordinates,

$$x^k = x^k(y^1, y^2, y^3), \qquad k = 1, 2, 3, \tag{A.1}$$

under the assumption that functions x^k are defined and continuously differentiable at least up to the first order in a particular region of point $P(y^1, y^2, y^3)$. Also assume that the Jacobian of the transformation,

$$J \equiv \det\left|\frac{\partial y^k}{\partial x^m}\right| = \begin{vmatrix} \partial y^1/\partial x^1 & \partial y^1/\partial x^2 & \partial y^1/\partial x^3 \\ \partial y^2/\partial x^1 & \partial y^2/\partial x^2 & \partial y^2/\partial x^3 \\ \partial y^3/\partial x^1 & \partial y^3/\partial x^2 & \partial y^3/\partial x^3 \end{vmatrix} \tag{A.2}$$

differs from zero in the region being considered. From the implicit function theorem it follows that transformation (A.1) has a uniquely inverse transformation

$$y^k = y^k(x^1, x^2, x^3), \qquad k = 1, 2, 3. \tag{A.3}$$

Under these assumptions the coordinates x^k are uniquely assigned to coordinates y^k and vice versa. Coordinates x^k determine the position of point P in space uniquely and, therefore, they are referred to as the *curvilinear coordinates* of the point (Fig. A1).

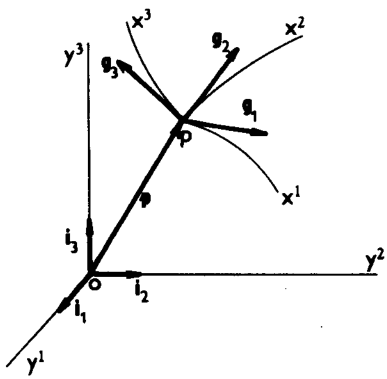

Fig. A1. Curvilinear coordinates.

A set of points in E_3 one of whose curvilinear coordinates is constant, is called a *coordinate surface*. Three different coordinate surfaces may pass through each point in E_3. The line of intersection of two mutually corresponding coordinate surfaces is called the *coordinate line*, i.e. a set of points in E_3 whose two curvilinear coordinates are constant. Once again, three different coordinate lines may pass through each point in E_3.

In Cartesian coordinates, the position vector $\boldsymbol{p}$ of point P is given by the relation

(A.4) $$\boldsymbol{p} = y^k \boldsymbol{i}_k ,$$

where $\boldsymbol{i}_k$ are unit basis vectors in Cartesian coordinates. In Eq. (A.4) and throughout the text as a whole we shall use Einstein's summation rule, i.e. we sum from one to three over all repeated indices which occur in diagonal position. No summation is carried out over the underscore indices.

We shall introduce the *base vectors* $\boldsymbol{g}_k(x^1, x^2, x^3)$ as follows:

(A.5) $$\boldsymbol{g}_k(\boldsymbol{x}) \equiv \frac{\partial \boldsymbol{p}}{\partial x^k} = \frac{\partial y^m}{\partial x^k} \boldsymbol{i}_m .$$

If we multiply (A.5) by $\partial x^k / \partial y^n$, we obtain

(A.6) $$\boldsymbol{i}_n = \frac{\partial x^k}{\partial y^n} \boldsymbol{g}_k .$$

Equation (A.5) implies that the vectors $\boldsymbol{g}_k$ are tangential to coordinate lines x^k like the vectors $\boldsymbol{i}_k$, which are located on the Cartesian axes y^k.

The infinitesimal vector at point P can be expressed as

(A.7) $$\mathrm{d}\boldsymbol{p} = \frac{\partial \boldsymbol{p}}{\partial x^k} \mathrm{d}x^k = \boldsymbol{g}_k \, \mathrm{d}x^k .$$

The square of the distance between two infinitesimally distant points is

(A.8) $$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{p} \, . \, \mathrm{d}\boldsymbol{p} = g_{kl}(\boldsymbol{x}) \, \mathrm{d}x^k \, \mathrm{d}x^l ,$$

where $g_{kl}(\boldsymbol{x})$ is the *covariant metric tensor* defined by the relation

(A.9) $$g_{kl}(\boldsymbol{x}) = \boldsymbol{g}_k \cdot \boldsymbol{g}_l = \frac{\partial y^m}{\partial x^k} \frac{\partial y^n}{\partial x^l} \delta_{mn} ,$$

where δ_{mn} is *Kronecker's delta symbol*, equal to unity if the indices are the same and to zero if the indices are different. If the metric tensor is known, the length of the vector and the angle between two vectors can be determined. Note that in general curvilinear coordinates $g_{kl} \neq 0$ for $k \neq l$. Therefore, vector $\boldsymbol{g}_k$ need not be orthogonal to vector $\boldsymbol{g}_l$. We shall refer to the coordinates as *orthogonal* if $g_{kl} = 0$ everywhere when $k \neq l$. Nor is g_{kk} necessarily equal to unity and, therefore, vectors $\boldsymbol{g}_k$ are *not* necessarily unit vectors. Equation (A.9) further implies that the covariant metric tensor is symmetric, $g_{kl} = g_{lk}$.

The *reciprocal base vectors* $\boldsymbol{g}^k(x)$ is determined by a system of nine equations

$$\boldsymbol{g}^k \cdot \boldsymbol{g}_l = \delta_l^k, \tag{A.10}$$

where δ_l^k is Kronecker's delta symbol. The solution to system (A.10) reads

$$\boldsymbol{g}^k = g^{kl}(\boldsymbol{x})\,\boldsymbol{g}_l, \tag{A.11}$$

where

$$g^{kl}(\boldsymbol{x}) = \frac{\text{alg. cofactor } g_{kl}}{g}, \quad g = \det(g_{kl}). \tag{A.12}$$

From Eqs (A.9), (A.10) and (A.11) it is easy to derive the formulas

$$g^{kl} = \boldsymbol{g}^k \cdot \boldsymbol{g}^l, \; g_l^k = \boldsymbol{g}^k \cdot \boldsymbol{g}_l = g^{km} g_{ml} = \delta_l^k. \tag{A.13}$$

Tensor g^{kl} is called the *contravariant metric tensor.* One can see that it is symmetric, $g^{kl} = g^{lk}$. Tensor g_l^k is a mixed metric tensor with the components $g_l^k = \delta_l^k$, where δ_l^k is Kronecker's delta symbol.

A.3. Tensors

Definition 1: We shall say that *tensor* $\mathbf{A}$ is defined in three-dimensional space if 3^{p+q} numbers $A^{k_1 \dots k_p}$ are assigned to every coordinate system, so that the coordinate transformation $x^{k'} = x^{k'}(x^1, x^2, x^3)$ transforms these numbers according to the relations

$$A^{k'_1 \dots k'_p}{}_{l'_1 \dots l'_q}(\boldsymbol{x}') = G^{k'_1 \dots k'_p l_1 \dots l_q}_{k_1 \dots k_p l'_1 \dots l'_q} A^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\boldsymbol{x}), \tag{A.14}$$

where

$$G^{k'_1 \dots k'_p l_1 \dots l_q}_{k_1 \dots k_p l'_1 \dots l'_q} = \frac{\partial x^{k'_1}}{\partial x^{k_1}} \dots \frac{\partial x^{k'_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial x^{l'_1}} \dots \frac{\partial x^{l_q}}{\partial x^{l'_q}}. \tag{A.15}$$

We shall say that tensor $\mathbf{A}$ is p-times *contravariant* and q-times *covariant*. The total number of indices $p + q$ is the *rank* (*degree*) of the tensor, and the numbers $A^{k_1 \dots k_p}{}_{l_1 \dots l_q}$ are referred to as the *coordinates* of the tensor.

Example 1 (scalar): If we assign the same number A to every coordinate system, the number determines a zero-order tensor ($p = q = 0$), which is called a *scalar*,

$$A'(\boldsymbol{x}') = A(\boldsymbol{x}). \tag{A.16}$$

Example 2 (vector): In changing the coordinates, the contravariant ($p = 1$,

$q = 0$), or covariant ($p = 0$, $q = 1$) coordinates of a *vector* are transformed according to the formulas

$$A^{k'}(\mathbf{x}') = A^k(\mathbf{x})\,\partial x^{k'}/\partial x^k, \tag{A.17}$$

or

$$A_{k'}(\mathbf{x}') = A_k(x)\,\partial x^k/\partial x^{k'}, \tag{A.18}$$

respectively.

An example of a contravariant vector is the differential vector $\mathrm{d}x^k$,

$$\mathrm{d}x^{k'} = (\partial x^{k'}/\partial x^k)\,\mathrm{d}x^k, \tag{A.19}$$

which agrees with (A.17) with $A^k = \mathrm{d}x^k$.

Similarly, the partial derivatives of a scalar is a covariant vector,

$$\partial\Phi/\partial x^{k'} = (\partial\Phi/\partial x^k)\,\partial x^k/\partial x^{k'}, \tag{A.20}$$

which agrees with (A.18) with $A_k = \partial\Phi/\partial x^k$.

Example 3 (2nd-order tensor): In changing the coordinates, the contravariant ($p = 2$, $q = 0$), covariant ($p = 0$, $q = 2$) and mixed ($p = 1$, $q = 1$) coordinates of a *2nd-order tensor* are transformed according to the formulas

$$A^{k'l'}(\mathbf{x}') = A^{kl}(\mathbf{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^{l'}/\partial x^l), \tag{A.21}$$

$$A_{k'l'}(\mathbf{x}') = A_{kl}(\mathbf{x})\,(\partial x^k/\partial x^{k'})\,(\partial x^l/\partial x^{l'}), \tag{A.22}$$

$$A^{k'}{}_{l'}(\mathbf{x}') = A^k{}_l(\mathbf{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^{l'}/\partial x^l). \tag{A.23}$$

An example of a covariant or contravariant 2nd-order tensor is the metric tensor g_{kl} or g^{kl}, respectively, since

$$g_{k'l'}(\mathbf{x}') = \mathbf{g}_{k'}(\mathbf{x}')\,.\,\mathbf{g}_{l'}(\mathbf{x}') = \mathbf{g}_k(\mathbf{x})\,.\,\mathbf{g}_l(\mathbf{x})\,(\partial x^k/\partial x^{k'})\,(\partial x^l/\partial x^{l'}). \tag{A.24}$$

The same applies to quantities g^{kl}. The quantities $g^k{}_l = \delta^k_l$ are the coordinates of a mixed 2nd-order tensor, since

$$\begin{aligned}\delta^k_l(\mathbf{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^l/\partial x^{l'}) &= \\ = (\partial x^{k'}/\partial x^k)\,(\partial x^k/\partial x^{l'}) = \partial x^{k'}/\partial x^{l'} &= \delta^{k'}_{l'}(\mathbf{x}').\end{aligned} \tag{A.25}$$

Lemma 1 (index law): Let $A^{k_1\ldots k_p}{}_{l_1\ldots l_q}$ be any p-times contravariant and q-times covariant tensor and let $s \geq q$, $t \geq p$. If the multiplication

$$A^{k_1\ldots k_p}{}_{l_1\ldots l_q}X^{l_1\ldots l_q\ldots l_s}{}_{k_1\ldots k_p\ldots k_t} = B^{l_{q+1}\ldots l_s}{}_{k_{p+1}\ldots k_t} \tag{A.26}$$

produces an arbitrary (s—q)-times contravariant and (t—p)-times covariant tensor $\mathbf{B}$, the quantity $\mathbf{X}$ is an s-times contravariant and t-times covariant tensor.

Proof: Assume Eq. (A.26) to hold in some coordinate system, i.e.

$$A^{k'_1\ldots k'_p}{}_{l'_1\ldots l'_q}X^{l'_1\ldots l'_q\ldots l'_s}{}_{k'_1\ldots k'_p\ldots k'_t} = B^{l'_{q+1}\ldots l'_s}{}_{k'_{p+1}\ldots k'_t}. \tag{A.27}$$

Since **A** and **B** are tensors, the following transformation relations apply to them,

$$A^{k'_1 \ldots k'_p}{}_{l'_1 \ldots l'_q} = G^{k'_1 \ldots k'_p l_1 \ldots l_q}_{k_1 \ldots k_p l'_1 \ldots l'_q} A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}$$

and the same applies to tensor **B**. By substituting these transformations into (A.27) we obtain

$$\text{(A.28)} \quad G^{k'_1 \ldots k'_p l_1 \ldots l_q}_{k_1 \ldots k_p l'_1 \ldots l'_q} A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q} X^{l'_1 \ldots l'_q \ldots l'_s}{}_{k'_1 \ldots k'_p \ldots k'_t} = G^{l'_q+1 \ldots l'_s k_p+1 \ldots k_t}_{l_q+1 \ldots l_s k'_p+1 \ldots k'_t} B^{l_q+1 \ldots l_s}{}_{k_p+1 \ldots k_t} .$$

If we multiply (A.26) by $G^{l'_q+1 \ldots l'_s k_p+1 \ldots k_t}_{l_q+1 \ldots l_s k'_p+1 \ldots k'_t}$ and subtract the result from (A.28), we arrive at

$$\text{(A.29)} \qquad G^{k'_1 \ldots k'_p l_1 \ldots l_q}_{k_1 \ldots k_p l'_1 \ldots l'_q} A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q} (X^{l'_1 \ldots l'_s}{}_{k'_1 \ldots k'_t} - G^{k_1 \ldots k_t l'_1 \ldots l'_s}_{k'_1 \ldots k'_t l_1 \ldots l_s} X^{l_1 \ldots l_s}{}_{k_1 \ldots k_t}) = 0 ,$$

where we have made use of the following properties of the quantities $G^{k'_1 \ldots k'_p l_1 \ldots l_q}_{k_1 \ldots k_p l'_1 \ldots l'_q}$ defined by Eq. (A.15),

$$\text{(A.30)} \qquad G^{k'_1 \ldots k'_p}_{k_1 \ldots k_p} G^{l_1 \ldots l_q}_{l'_1 \ldots l'_q} = G^{k'_1 \ldots k'_p l_1 \ldots l_q}_{k_1 \ldots k_p l'_1 \ldots l'_q} ,$$

$$G^{k_1 \ldots k_p}_{k'_1 \ldots k'_p} G^{k'_1 \ldots k'_p}_{l_1 \ldots l_p} = \delta^{k_1}_{l_1} \delta^{k_2}_{l_2} \ldots \delta^{k_p}_{l_p} .$$

Since the factor preceding the parentheses in Eq. (A.29) is an arbitrary tensor, the necessary and sufficient condition for (A.29) to be satisfied is that the expression in the parentheses should be zero. It then follows that

$$X^{l'_1 \ldots l'_s}{}_{k'_1 \ldots k'_t} = G^{k_1 \ldots k_t l'_1 \ldots l'_s}_{k'_1 \ldots k'_t l_1 \ldots l_s} X^{l_1 \ldots l_s}{}_{k_1 \ldots k_t} ,$$

Q.E.D.

Definition 2 (transposed tensor): A tensor which is created by the permutation of two superscripts or two subscripts, is referred to as a tensor *transposed* with respect to these indices.

Example 4: The contravariant, covariant and mixed components of a transposed 2nd-order tensor are $(\mathbf{A}^T)^{kl} = A^{lk}$, $(\mathbf{A}^T)_{kl} = A_{lk}$, $(\mathbf{A}^T)^k{}_l = A_l{}^k$, $(\mathbf{A}^T)_l{}^k = A^k{}_l$.

Definition 3 (symmetric tensor): We shall refer to a tensor as *symmetric* with respect to the superscripts or subscripts, provided its coordinates remain unchanged under any permutation of these indices, e.g. tensor $A^n{}_{klm}$ is symmetric with respect to the first two subscripts provided $A^n{}_{klm} = A^n{}_{lkm}$.

Example 5: Metric tensors g_{kl}, g^{kl} and $g^k{}_l$ are symmetric tensors because $g_{kl} = \mathbf{g}_k \cdot \mathbf{g}_l = \mathbf{g}_l \cdot \mathbf{g}_k = g_{lk}$, and similarly for tensors g^{kl} and $g^k{}_l$. This implies the symmetry of Kronecker's delta symbol: $\delta^k{}_l = \delta_l{}^k$.

A.4. Tensor algebra

Definition 4 (equality of tensors): We say two tensors are *equal* if they are p-times contravariant and q-times covariant and if their coordinates are equal at least in one coordinate system. Their coordinates are then equal in any coordinate system. Their coordinates are then equal in any coordinate system.

Definition 5 (addition of tensors): If two tensors are of the same order and type, the *sum* or difference of these tensors is a tensor of the same order and type, e.g.

$$C^{kl}{}_m = A^{kl}{}_m + B^{kl}{}_m . \tag{A.31}$$

Definition 6 (outer product of tensors): The *outer product* of two tensors is obtained by simple multiplication of the tensor components, e.g.

$$C^{kl}{}_m = A^{kl} B_m . \tag{A.32}$$

Lemma 1 implies that this operation yields a tensor whose order is equal to the sum of the orders of the factors.

Example 6 (dyadic product): The outer product of two vectors is called the *dyadic product*,

(A.33)

$$C^{kl} = A^k B^l \quad \text{contravariant component,}$$
$$C_{kl} = A_k B_l \quad \text{covariant component,}$$
$$C^k{}_l = A^k B_l \quad \text{mixed component.}$$

Definition 7 (tensor contraction): The algebraic operation in which we put the covariant and contravariant indices of a tensor equal to each other and add with respect to these identical indices is reffered to as tensor *contraction*, e.g.

$$A^k{}_{kl},\ A^k{}_{lk} . \tag{A.34}$$

Lemma 1 implies that the order of the contracted tensor is lower by two than the order of the original tensor. The type of contracted tensor is determined by the number of free indices. It is easy to prove that no tensor quantity is obtain, if this procedure is applied to two indices of the same type, i.e. either to both covariant or to both contravariant indices.

Definition 8 (raising and lowering the indices): The algebraic operation in which we assign the quantity $A_{k_1 \ldots k_p l_1 \ldots l_q}$ to every p-times contravariant and q-times covariant tensor $A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}$ by the relation

$$A_{k_1 \ldots k_p l_1 \ldots l_q} = g_{m_1 k_1} \cdots g_{m_p k_p} A^{m_1 \ldots m_p}{}_{l_1 \ldots l_q} , \tag{A.35}$$

where g_{kl} are the components of the covariant metric tensor, is referred to as *lowering* the indices of tensor **A**. Lemma 1 implies that the quantity $A_{k_1 \ldots k_p l_1 \ldots l_q}$ is a $(p + q)$-times covariant tensor.

Similarly, the algebraic operation in which we construct to tensor $A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}$ a new $(p + q)$-times contravariant tensor

$$A^{k_1 \ldots k_p l_1 \ldots l_q} = g^{m_1 l_1} \ldots g^{m_q l_q} A^{k_1 \ldots k_p}{}_{m_1 \ldots m_q} , \tag{A.36}$$

where g^{kl} are the components of the contravariant metric tensor, is referred to as *raising* the indices of tensor **A**.

If we raise or lower only some of the indices of a tensor, we again obtain a tensor quantity. With tensors of higher orders than the first we use a gap (sometimes a dot) to indicate the original position of the indices we have lowered or raised. By raising or lowering indicates of a given tensor we obtain so-called *associated tensors.*

Example 7: By lowering the contravariant coordinates v^k of vector $\boldsymbol{v}$, we obtain its covariant coordinates and vice versa,

$$v_k = g_{kl}v^l, \quad v^k = g^{kl}v_l. \tag{A.37}$$

Example 8: Raising the indices of a 2nd-order tensor can be expressed as

$$A^k{}_l = g^{mk}A_{ml}, \quad A_l{}^k = g^{km}A_{lm}. \tag{A.38}$$

In general, tensors $A^k{}_l$ and $A_l{}^k$ are not equal. Only if tensor **A** is symmetric, $A^k{}_l = A_l{}^k$ and the relative positioning of the indices is unimportant.

Similarly, lowering the indicies can be expressed as

$$A^k{}_l = g_{lm}A^{km}, \quad A_l{}^k = g_{lm}A^{mk}. \tag{A.39}$$

The following relations also hold:

$$A^{kl} = g^{km}A_m{}^l = g^{lm}A^k{}_m = g^{km}g^{ln}A_{mn}, \tag{A.40}$$

$$A_{kl} = g_{ml}A_k{}^m = g_{km}A^m{}_l = g_{km}g_{ln}A^{mn}, \tag{A.41}$$

$$A^k{}_l = g^{km}g_{ln}A_m{}^n. \tag{A.42}$$

The associated tensors A^{kl}, $A^k{}_l$, $A_l{}^k$, A_{kl} characterize one and the same 2nd-order tensor **A**.

Definition 9 (inner product of tensors): We define the *inner (scalar) product* of tensors by contraction of the outer product of two tensors. The inner (scalar) product of vectors and tensors will be denoted by a dot.

Example 9 (scalar product of vectors): By contracting the dyadic product of two vectors, we define the *inner (scalar) product of two vectors*:

$$\boldsymbol{u} . \boldsymbol{v} = u_k v^k = u^k v_k. \tag{A.43}$$

Lemma 2: The scalar product of vectors is an *invariant*, i.e. it is independent of the coordinate system.

Proof: With a view to Eqs (A.17) and (A.18)

$$u^{k'} v_{k'}(\boldsymbol{x}') = u^k(\boldsymbol{x})\, v_k(\boldsymbol{x})\,(\partial x^{k'}/\partial x^k)(\partial x^k/\partial x^{k'}) = u^k\, v_k(\boldsymbol{x}). \tag{A.44}$$

Example 10 (scalar product of a vector and 2nd-order tensor): By contracting the outer product of a vector $\boldsymbol{v}$ and 2nd-order tensor **A**, we define the *left-hand* and *right-hand scalar product* of a vector and 2nd-order tensor, (A.45) $$(\boldsymbol{v}.\mathbf{A})^k = A_l^{\ k} v^l = A^{lk} v_l,$$

and

(A.46) $$(\mathbf{A}.\boldsymbol{v})^k = A^k_{\ l} v^l = A^{kl} v_l,$$

respectively. By lowering the indices we obtain the covariant components of these vectors,

(A.47) $$(\boldsymbol{v}.\mathbf{A})_k = A_{lk} v^l = A^l_{\ k} v_l,$$

and

(A.48) $$(\mathbf{A}.\boldsymbol{v})_k = A_{kl} v^l = A_k^{\ l} v_l,$$

respectively.

Lemma 3: Assume φ to be a scalar $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ to be vectors and $\mathbf{A}$, $\mathbf{A}_1$, $\mathbf{A}_2$ to be 2nd-order tensors. It then holds that

(A.49) $$(\mathbf{A}_1 + \mathbf{A}_2).\boldsymbol{v} = \mathbf{A}_1.\boldsymbol{v} + \mathbf{A}_2.\boldsymbol{v},$$

(A.50) $$\mathbf{A}.(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \mathbf{A}.\boldsymbol{v}_1 + \mathbf{A}.\boldsymbol{v}_2,$$

(A.51) $$(\mathbf{A}.\varphi\boldsymbol{v}) = \varphi(\mathbf{A}.\boldsymbol{v}),$$

(A.52) $$\mathbf{A}.\boldsymbol{v} = \boldsymbol{v}.\mathbf{A}^{\mathrm{T}},$$

(A.53) $$(\boldsymbol{v}_1\boldsymbol{v}_2).\boldsymbol{v}_3 = \boldsymbol{v}_1(\boldsymbol{v}_2.\boldsymbol{v}_3),$$

(A.54) $$\boldsymbol{v}_1.(\boldsymbol{v}_2\boldsymbol{v}_3) = (\boldsymbol{v}_1.\boldsymbol{v}_2)\boldsymbol{v}_3,$$

where $\mathbf{A}^{\mathrm{T}}$ is the tensor transposed to tensor $\mathbf{A}$ and $\boldsymbol{v}_1\boldsymbol{v}_2$ is the dyadic product of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Proof: Equations (A.49)—(A.52) follow immediately from the definition of the scalar product of a vector and 2nd-order tensor. Let us prove Eq. (A.53), e.g. for the contravariant component,

$$[(\boldsymbol{v}_1\boldsymbol{v}_2).\boldsymbol{v}_3]^k = (\boldsymbol{v}_1\boldsymbol{v}_2)^k_{\ l}\, v_3^l = v_1^k(\boldsymbol{v}_2)_l v_3^l = [\boldsymbol{v}_1(\boldsymbol{v}_2.\boldsymbol{v}_3)]^k.$$

Equation (A.54) can be proved in very much the same way.

Example 11 (scalar product of 2nd-order tensors): By contracting the outer product of two 2nd-order tensors $\mathbf{A}$ and $\mathbf{B}$, we define the *scalar product* of these tensors:

(A.55) $$(\mathbf{A}.\mathbf{B})^k_{\ l} = A^k_{\ m} B^m_{\ l}.$$

By lowering and raising the indices we obtain the second mixed component of this tensor:

(A.56) $$(\mathbf{A}.\mathbf{B})_l^{\ k} = A_l^{\ m} B_m^{\ k}.$$

Lemma 4: Let φ be a scalar, $\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ vectors, $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{B}, \mathbf{B}_1, \mathbf{B}_2$ 2nd-order tensors. It then holds that

$$(\mathbf{A}_1 + \mathbf{A}_2) . \mathbf{B} = \mathbf{A}_1 . \mathbf{B} + \mathbf{A}_2 . \mathbf{B} , \tag{A.57}$$

$$\mathbf{A} . (\mathbf{B}_1 + \mathbf{B}_2) = \mathbf{A} . \mathbf{B}_1 + \mathbf{A} . \mathbf{B}_2 , \tag{A.58}$$

$$(\varphi \mathbf{A}) . \mathbf{B} = \varphi(\mathbf{A} . \mathbf{B}) , \tag{A.59}$$

$$\mathbf{A} . (\varphi \mathbf{B}) = \varphi(\mathbf{A} . \mathbf{B}) , \tag{A.60}$$

$$(\mathbf{A}_1 . \mathbf{A}_2) . \mathbf{B} = \mathbf{A}_1 . (\mathbf{A}_2 . \mathbf{B}) , \tag{A.61}$$

$$(\mathbf{A} . \mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} . \mathbf{A}^{\mathrm{T}} , \tag{A.62}$$

$$(\mathbf{A} . \mathbf{B}) . \boldsymbol{v} = \mathbf{A} . (\mathbf{B} . \boldsymbol{v}) , \tag{A.63}$$

$$\boldsymbol{v} . (\mathbf{A} . \mathbf{B}) = (\boldsymbol{v} . \mathbf{A}) . \mathbf{B} , \tag{A.64}$$

$$(\boldsymbol{v}_1 \boldsymbol{v}_2) . (\boldsymbol{v}_3 \boldsymbol{v}_4) = (\boldsymbol{v}_2 . \boldsymbol{v}_3) \boldsymbol{v}_1 \boldsymbol{v}_4 . \tag{A.65}$$

Proof: Equations (A.57)—(A.60) follow immediately from the definition of the scalar product of two 2nd-order tensors. Therefore, let us only prove the remaining Eqs (A.61)—(A.65):

$$\begin{aligned}
[(\mathbf{A}_1 . \mathbf{A}_2) . \mathbf{B}]^k{}_l &= (\mathbf{A}_1 . \mathbf{A}_2)^k{}_m B^m{}_l = \\
&= (\mathbf{A}_1)^k{}_n (\mathbf{A}_2)^n{}_m \mathbf{B}^m{}_l = (\mathbf{A}_1)^k{}_n (\mathbf{A}_2 . \mathbf{B})^n{}_l = [\mathbf{A}_1 . (\mathbf{A}_2 . \mathbf{B})]^k{}_l ,
\end{aligned}$$

$$[(\mathbf{A} . \mathbf{B})^{\mathrm{T}}]^k{}_l = (\mathbf{A} . \mathbf{B})_l{}^k = A_l{}^m B_m{}^k = (\mathbf{B}^{\mathrm{T}})^k{}_m (\mathbf{A}^{\mathrm{T}})^m{}_l = [(\mathbf{B}^{\mathrm{T}} . \mathbf{A}^{\mathrm{T}})]^k{}_l ,$$

$$[(\mathbf{A} . \mathbf{B}) . \boldsymbol{v}]^k = (\mathbf{A} . \mathbf{B})^k{}_l v^l = A^k{}_m B^m{}_l v^l = A^k{}_m (\mathbf{B} . \boldsymbol{v})^m = [\mathbf{A} . (\mathbf{B} . \boldsymbol{v})]^k ,$$

$$[\boldsymbol{v} . (\mathbf{A} . \mathbf{B})]^k = (\mathbf{A} . \mathbf{B})_l{}^k v^l = A_l{}^m B_m{}^k v^l = (\boldsymbol{v} . \mathbf{A})^m B_m{}^k = [(\boldsymbol{v} . \mathbf{A}) . \mathbf{B}]^k ,$$

$$\begin{aligned}
[(\boldsymbol{v}_1 \boldsymbol{v}_2) . (\boldsymbol{v}_3 \boldsymbol{v}_4)]^k{}_l &= (\boldsymbol{v}_1 \boldsymbol{v}_2)^k{}_m (\boldsymbol{v}_3 \boldsymbol{v}_4)^m{}_l = (\boldsymbol{v}_1)^k (\boldsymbol{v}_2)_m (\boldsymbol{v}_3)^m (\boldsymbol{v}_4)_l = \\
&= [(\boldsymbol{v}_2 . \boldsymbol{v}_3)(\boldsymbol{v}_1 \boldsymbol{v}_4)]^k{}_l .
\end{aligned}$$

The proofs for the other components are similar.

A.5. Physical components

We have so far represented a vector by its contravariant or covariant components

$$\boldsymbol{v} = v^k \boldsymbol{g}_k = v_k \boldsymbol{g}^k , \tag{A.66}$$

where $\boldsymbol{g}_k$ is the vector tangential to the kth coordinate line at point x^k. Since vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are not generally unit vectors, components v^k and v_k do not have the same physical dimension as vector $\boldsymbol{v}$. Let us assign the unit vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}^k$ to vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$. However, the square of the length of a vector is given by the scalar product of this vector with itself; in virtue of Eqs (A.9) and (A.13)

(A.67) $$\boldsymbol{g}_k \cdot \boldsymbol{g}_k = g_{\underline{kk}}, \quad \boldsymbol{g}^k \cdot \boldsymbol{g}^k = g^{\underline{kk}},$$

where no summation is carried out over the underscore indices. The unit vectors to vector $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are then defined by the relations

(A.68) $$\boldsymbol{e}_k = \boldsymbol{g}_k/(g_{\underline{kk}})^{1/2}, \quad \boldsymbol{e}^k = \boldsymbol{g}^k/(g^{\underline{kk}})^{1/2}.$$

With a view to (A.66), vector $\boldsymbol{v}$ can be resolved into these unit vectors as

(A.69) $$\boldsymbol{v} = v^{(k)}\boldsymbol{e}_k = v_{(k)}\boldsymbol{e}^k,$$

where the quantities $v^{(k)}$ and $v_{(k)}$ are the *physical components* of vector $\boldsymbol{v}$. We use the term physical because these components have the same physical dimension as vector $\boldsymbol{v}$. By substituting Eqs (A.66) and (A.68) into (A.69), we obtain the formula for the physical components of vector $\boldsymbol{v}$,

(A.70) $$v^{(k)} = v^k(g_{\underline{kk}})^{1/2}, \quad v_{(k)} = v_k(g^{\underline{kk}})^{1/2}.$$

By substituting Eq. (A.37) into (A.70), we can derive the relation between the two kinds of physical components:

(A.71) $$v_{(k)} = \sum_l g_{kl}(g^{\underline{kk}}/g_{ll})^{1/2}v^{(l)}.$$

If the curvilinear coordinates are orthogonal, $g_{kl} = g^{kl} = 0$ for $k \neq l$,

(A.72) $$v_{(k)} = v^{(k)},$$

i.e. the difference between the two kinds of physical components of the vector vanishes.

This definition of the physical components can also be extended to tensors of higher orders with the aid of their relations with vectors. Let us demonstrate the resolution of the stress tensor $\mathbf{t}$ into physical components. For the time being, let us not assume that the stress tensor $\mathbf{t}$ is a symmetric 2nd-order tensor. The projection of the stress tensor $\mathbf{t}$ onto the unit external normal $\boldsymbol{n}$ defines the stress vector $\boldsymbol{t}$,

(A.73) $$\boldsymbol{t} = \mathbf{t} \,.\, \boldsymbol{n},$$

i.e. the components expressed with the aid of Eqs (A.46) and (A.48) read

(A.74) $$t^k = t^k{}_l n^l = t^{kl}n_l, \quad t_k = t_k{}^l n_l = t_{kl}n^l.$$

If we express vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ in terms of physical components (A.70), we obtain the relations

(A.75) $$t^{(k)} = t^{(k)}{}_{(l)}n^{(l)} = t^{(k)(l)}n_{(l)},$$
$$t_{(k)} = t_{(k)}{}^{(l)}n_{(l)} = t_{(k)(l)}n^{(l)},$$

where the quantities

$$(A.76)\qquad \begin{aligned} t^{(k)}{}_{(l)} &= t^{k}{}_{l}(g_{\underline{kk}}/g_{\underline{ll}})^{1/2},\\ t^{(k)(l)} &= t^{kl}(g_{\underline{kk}}g_{\underline{ll}})^{1/2},\\ t_{(k)}{}^{(l)} &= t_{k}{}^{l}(g_{\underline{ll}}/g_{\underline{kk}})^{1/2},\\ t_{(k)(l)} &= t_{kl}(g_{\underline{kk}}g_{\underline{ll}})^{-1/2}, \end{aligned}$$

are the physical components of the stress tensor $\mathbf{t}$. Let it be emphasized that the quantities $t^{(k)}{}_{(l)}$, $t^{(k)(l)}$, $t_{(k)}{}^{(l)}$ and $t_{(k)(l)}$ are not tensor components. The relation between the right-hand and left-hand physical component can be derived with the aid of (A.42) and (A.76):

$$(A.77)\qquad t_{(k)}{}^{(l)} = \sum_{m,n} (g_{nn}g_{\underline{ll}}/g_{\underline{kk}}g_{mm})^{1/2} g_{km}g^{ln}t^{(m)}{}_{(n)}.$$

If the curvilinear coordinates are orthogonal, $g_{kl} = g^{kl} = 0$,

$$(A.78)\qquad t^{(k)}{}_{(l)} = t^{(k)(l)} = t_{k}{}^{(l)} = t_{(k)(l)},$$

i.e. the physical components of the stress tensor $\mathbf{t}$ in orthogonal curvilinear components are the same for all types of tensor coordinates.

A.6. Covariant derivative

As compared to Cartesian coordinates, the greatest difficulty in the system of curvilinear coordinates is that the basis vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are functions of the curvilinear coordinates x^k, so that in differentiating and integrating these vectors do not behave like constants. Therefore, let us first derive the formulas for differentiating these vectors with respect to the curvilinear coordinates. We shall put

$$(A.79)\qquad \frac{\partial \boldsymbol{g}_k}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\frac{\partial \boldsymbol{p}}{\partial x^k}\right) = \frac{\partial^2 y^m}{\partial x^l \partial x^k}\boldsymbol{i}_m,$$

because the Cartesian unit vectors $\boldsymbol{i}_m$ do not depend on coordinates. After substituting for $\boldsymbol{i}_m$ from (A.6),

$$(A.80)\qquad \frac{\partial \boldsymbol{g}_k}{\partial x^l} = \begin{Bmatrix} m \\ k\ \ l \end{Bmatrix} \boldsymbol{g}_m,$$

where the quantities

$$(A.81)\qquad \begin{Bmatrix} m \\ k\ \ l \end{Bmatrix} = \frac{\partial^2 y^n}{\partial x^k \partial x^l}\frac{\partial x^m}{\partial y^n}$$

are referred to as *Christoffel's symbols of the 2nd kind. Christoffel's symbols of the* 1*st kind* are defined by the relations

(A.82) $$[kl,m] = g_{mn}\begin{Bmatrix} n \\ k\ l \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} m \\ k\ l \end{Bmatrix} = g^{mn}[kl,n].$$

By using (A.9) it is no difficult to prove that

(A.83) $$[kl, m] = \frac{1}{2}\left(\frac{\partial g_{km}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Let it be emphasized that Christoffel's symbols are *not* tensors. However, they are symmetric with respect to indices k, l,

(A.84) $$[kl,m] = [lk,m], \quad \begin{Bmatrix} m \\ k\ l \end{Bmatrix} = \begin{Bmatrix} m \\ l\ k \end{Bmatrix}.$$

By making use of (A.11), we obtain a similar result to (A.79),

(A.85) $$\frac{\partial \boldsymbol{g}^k}{\partial x^l} = -\begin{Bmatrix} k \\ l\ m \end{Bmatrix}\boldsymbol{g}^m.$$

We can now calculate the partial derivatives of vector $\boldsymbol{v}$,

(A.86) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = \frac{\partial}{\partial x^k}(v^m \boldsymbol{g}_m) = \frac{\partial v^m}{\partial x^k}\boldsymbol{g}_m + v^m\frac{\partial \boldsymbol{g}_m}{\partial x^k} = \left(\frac{\partial v^m}{\partial x^k} + \begin{Bmatrix} m \\ k\ l \end{Bmatrix} v^l\right)\boldsymbol{g}_m,$$

which can be abbreviated to

(A.87) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = v^m{}_{;k}\boldsymbol{g}_m,$$

where the expression

(A.88) $$v^m{}_{;k} = \frac{\partial v^m}{\partial x^k} + \begin{Bmatrix} m \\ k\ l \end{Bmatrix} v^l$$

is the *covariant partial derivative* of the contravariant vector v^m.

A covariant partial derivative, or the partial derivative of any tensor is denoted by adding a semi-colon after the last tensor index, or a comma, and a further index appropriate to the coordinate with respect to which the covariant partial derivative, or partial derivative, respectively, is being performed.

By differentiating the expression $\boldsymbol{v} = v_m \boldsymbol{g}^m$, we obtain the covariant partial derivative of the covariant vector v_m,

(A.89) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = v_{m;k}\boldsymbol{g}^m,$$

where

(A.90) $$v_{m;k} = v_{m,k} - \begin{Bmatrix} l \\ m\ k \end{Bmatrix} v_l.$$

The reason for introducing, besides ordinary partial derivatives, also covariant partial derivatives, is that applying the covariant derivative to any tensor increases the order of the tensor by one covariant index, whereas a partial derivative of a tensor is not, in general, a tensor quantity.

Since Christoffel's symbols are identically equal to zero in Cartesian coordinates, covariant partial derivatives in this coordinate system reduce to "ordinary" partial derivatives.

The covariant partial derivative of a scalar is identical with an "ordinary" partial derivative, because a scalar is a covariant tensor of order zero. Covariant partial derivatives of higher-order tensors are defined in a similar fashion as the covariant derivatives of vectors, e.g. the covariant partial derivative of a 2nd-order tensor,

(A.91)
$$A^{kl}{}_{;m} = A^{kl}{}_{,m} + \left\{ {k \atop m\ n} \right\} A^{nl} + \left\{ {l \atop m\ n} \right\} A^{kn},$$

$$A^{k}{}_{l;m} = A^{k}{}_{l,m} - \left\{ {n \atop l\ m} \right\} A^{k}{}_{n} + \left\{ {k \atop m\ n} \right\} A^{n}{}_{l},$$

$$A_{kl;m} = A_{kl,m} - \left\{ {n \atop k\ m} \right\} A_{nl} - \left\{ {n \atop l\ m} \right\} A_{kn},$$

is a tensor of the 3rd order.

Lemma 5 (Ricci): The covariant partial derivatives of any metric tensor are zero,

(A.92)
$$g_{kl;m} = g^{kl}{}_{;m} = g^{k}{}_{l;m} = g_{;k} = 0,$$

where $g = \det(g_{kl})$.

Proof: With a view to $(A.91)_3$,

(A.93)
$$g_{kl;m} = g_{kl,m} - \left\{ {n \atop k\ m} \right\} g_{nl} - \left\{ {n \atop l\ m} \right\} g_{kn}.$$

By using Eqs $(A.82)_2$ and (A.83) we can prove that the r.h.s. of Eq. (A.93) is equal to zero, $g_{kl;m} = 0$. The other relations of (A.92) can be proved in very much the same way.

Equation $(A.92)_4$ yields the following useful relation:

(A.94)
$$(\log \sqrt{g})_{,k} = \left\{ {m \atop m\ k} \right\}, \qquad g \equiv \det(g_{kl}).$$

Lemma 5 implies that metric tensors under covariant differentiation behave like constants, consequently, whether we raise or lower the index before covariant differentiation or after it is unimportant. It is easy to prove, for example, that

(A.95)
$$A^{k}{}_{;l} = (g^{km} A_m)_{;l} = g^{km} A_{m;l}.$$

It is also easy to prove that the product rule of differentiation holds for the covariant partial differentiation, e.g.,

(A.96) $$(A^k B_{lm})_{;n} = A^k{}_{;n} B_{lm} + A^k B_{lm;n} .$$

Sometimes, by means of the covariant partial derivative, we also introduce the contravariant partial derivative as

(A.97) $$A^k{}_{l;}{}^m = A^k{}_{l;n} g^{nm} .$$

A.7. Invariant differential operators

The invariant differential operators *gradient* (grad) of scalar Φ, *divergence* (div) and *rotation* (rot) of vector $\boldsymbol{v}$, are defined by the relations

(A.98) $$\operatorname{grad} \Phi = \Phi_{,k} \boldsymbol{g}^k ,$$

(A.99) $$\operatorname{div} \boldsymbol{v} = v^k{}_{;k} ,$$

(A.100) $$\operatorname{rot} \boldsymbol{v} = \varepsilon^{klm} v_{m;l} \boldsymbol{g}_k ,$$

(A.101) $$\varepsilon^{klm} = e^{klm}/\sqrt{g} , \qquad \varepsilon_{klm} = e_{klm}\sqrt{g}$$

and e^{klm} and e_{klm} are *Levi-Civita alternating symbols,*

(A.102) $$e^{123} = e^{312} = e^{231} = -e^{213} = -e^{321} = -e^{132} = 1 ,$$

and the other $e^{klm} = 0$. The symbols e_{klm} are defined similarly. Let us remind the reader of some of the important relations:

(A.103) $$\varepsilon_{pkl}\varepsilon^{qmn} = \begin{vmatrix} \delta^q_p & \delta^q_k & \delta^q_l \\ \delta^m_p & \delta^m_k & \delta^m_l \\ \delta^n_p & \delta^n_k & \delta^n_l \end{vmatrix} ,$$

(A.104) $$\varepsilon_{pkl}\varepsilon^{pmn} = \delta^m_k \delta^n_l - \delta^m_l \delta^n_k ,$$

(A.105) $$\varepsilon_{pkl}\varepsilon^{pkn} = 2\delta^n_l , \qquad \varepsilon_{pkl}\varepsilon^{pkl} = 6 .$$

The operators (A.98)—(A.100) are invariant with respect to a general transformation of coordinates.

It is sometimes advantageous to introduce the *nabla operator* ∇,

(A.106) $$\nabla = \boldsymbol{g}^k \frac{\partial}{\partial x^k} .$$

By using this symbol we can express Eqs (A.98)—(A.100) in the following form:

(A.107) $$\operatorname{grad} \Phi = \nabla \Phi = \boldsymbol{g}^k \Phi_{,k} ,$$

(A.108) $$\operatorname{div}\mathbf{v} = \nabla . \mathbf{v} = \mathbf{g}^k . \frac{\partial}{\partial x^k}(v^l \mathbf{g}_l) = \mathbf{g}^k . \mathbf{g}_l v^l{}_{;k} = v^k{}_{;k},$$

(A.109) $$\operatorname{rot}\mathbf{v} = \nabla \times \mathbf{v} = \mathbf{g}^k \times \frac{\partial}{\partial x^k}(v^l g_l) = \mathbf{g}^k \times \mathbf{g}_l v^l{}_{;k} = \varepsilon^{klm} v_{m;l} \mathbf{g}_k .$$

If we use Eq. (A.94), we can express div $\mathbf{v}$ in a more convenient form:

(A.110) $$\operatorname{div}\mathbf{v} = v^k{}_{;k} = v^k{}_{,k} + \begin{Bmatrix} k \\ k \ l \end{Bmatrix} v^l = v^k{}_{,k} + v^k (\log \sqrt{g})_{,k} = \\ = [\sqrt{(g)} v^k]_{,k} / \sqrt{g} .$$

Laplace's operator is

(A.111) $$\nabla^2 \Phi = \operatorname{div}\operatorname{grad}\Phi = (g^{kl}\Phi_{,l})_{;k} = g^{kl}(\Phi_{,l})_{;k} = [\sqrt{(g)} g^{kl} \Phi_{,l}]_{,k} / \sqrt{g} .$$

Let us generalize the above differential operations also for tensors of higher orders. The gradient, divergence and rotation of tensor $\mathbf{A}$ are defined by the relations

(A.112) $$\operatorname{grad}\mathbf{A} \equiv \nabla \mathbf{A},$$

(A.113) $$\operatorname{div}\mathbf{A} \equiv \nabla . \mathbf{A},$$

(A.114) $$\operatorname{rot}\mathbf{A} \equiv \nabla \times \mathbf{A}.$$

If $\mathbf{A}$ is a tensor of order p,

(A.115) $$\mathbf{A} = A^{k_1 \dots k_p} \mathbf{g}_{k_1} \dots \mathbf{g}_{k_p} = A_{k_1 \dots k_p} \mathbf{g}^{k_1} \dots \mathbf{g}^{k_p},$$

and, consequently,

(A.116) $$\operatorname{grad}\mathbf{A} = A_{k_1 \dots k_p ; m} \mathbf{g}^m \mathbf{g}^{k_1} \dots \mathbf{g}^{k_p} = (\operatorname{grad}\mathbf{A})_{m k_1 \dots k_p} \mathbf{g}^m \mathbf{g}^{k_1} \dots \mathbf{g}^{k_p},$$

(A.117) $$\operatorname{div}\mathbf{A} = A^{m k_2 \dots k_p}{}_{;m} \mathbf{g}_{k_2} \dots \mathbf{g}_{k_p} = (\operatorname{div}\mathbf{A})^{k_2 \dots k_p} \mathbf{g}_{k_2} \dots \mathbf{g}_{k_p},$$

(A.118) $$\operatorname{rot}\mathbf{A} = \varepsilon^{nmk_1} A_{k_1 \dots k_p ; m} \mathbf{g}_n \mathbf{g}^{k_2} \dots \mathbf{g}^{k_p} = (\operatorname{rot}\mathbf{A})^n{}_{k_2 \dots k_p} \mathbf{g}_n \mathbf{g}^{k_2} \dots \mathbf{g}^{k_p} .$$

By lowering and raising the indices, we can express the above tensors in terms of associated components.

Example 12: The *gradient of vector* $\mathbf{v} = v_l \mathbf{g}^l$ is defined as

(A.119) $$\operatorname{grad}\mathbf{v} = v_{l;k} \mathbf{g}^k \mathbf{g}^l = (\operatorname{grad}\mathbf{v})_{kl} \mathbf{g}^k \mathbf{g}^l .$$

Example 13: For the 2nd-order tensor $\mathbf{A} = A^{kl} \mathbf{g}_k \mathbf{g}_l = A_{kl} \mathbf{g}^k \mathbf{g}^l$

(A.120) $$\operatorname{div}\mathbf{A} = A^{kl}{}_{;k} \mathbf{g}_l = (\operatorname{div}\mathbf{A})^l \mathbf{g}_l ,$$

(A.121) $$\operatorname{rot}\mathbf{A} = \varepsilon^{klm} A_{mn;l} \mathbf{g}_k \mathbf{g}^n = (\operatorname{rot}\mathbf{A})^k{}_n \mathbf{g}_k \mathbf{g}^n .$$

Lemma 6: Let Φ, Ψ be scalars, $\boldsymbol{u}$, $\boldsymbol{v}$ vectors and $\mathbf{A}$ a 2nd-order tensor. It then holds that

$$\text{grad}(\Phi\Psi) = \Phi \,\text{grad}\, \Psi + \Psi \,\text{grad}\, \Phi, \tag{A.122}$$

$$\text{div}(\Phi \boldsymbol{u}) = \Phi \,\text{div}\, \boldsymbol{u} + \boldsymbol{u}\,.\,\text{grad}\, \Phi, \tag{A.123}$$

$$\text{rot}(\Phi \boldsymbol{u}) = \Phi \,\text{rot}\, \boldsymbol{u} + \text{grad}\, \Phi \times \boldsymbol{u}, \tag{A.124}$$

$$\text{grad}(\boldsymbol{u}\,.\,\boldsymbol{v}) = \text{grad}\, \boldsymbol{u}\,.\,\boldsymbol{v} + \text{grad}\, \boldsymbol{v}\,.\,\boldsymbol{u}, \tag{A.125}$$

$$\text{div}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v}\,.\,\text{rot}\, \boldsymbol{u} - \boldsymbol{u}\,.\,\text{rot}\, \boldsymbol{v}, \tag{A.126}$$

$$\text{rot}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v}\,.\,\text{grad}\, \boldsymbol{u} - \boldsymbol{u}\,.\,\text{grad}\, \boldsymbol{v} + \boldsymbol{u}\,\text{div}\, \boldsymbol{v} - \boldsymbol{v}\,\text{div}\, \boldsymbol{u}, \tag{A.127}$$

$$\boldsymbol{u} \times \text{rot}\, \boldsymbol{v} = \text{grad}\, \boldsymbol{v}\,.\,\boldsymbol{u} - \boldsymbol{u}\,.\,\text{grad}\, \boldsymbol{v}, \tag{A.128}$$

$$\text{rot}\,\text{grad}\, \Phi = 0, \qquad \text{div}\,\text{rot}\, \boldsymbol{u} = 0, \tag{A.129}$$

$$\text{grad}\,\text{div}\, \boldsymbol{u} = \text{div}\,[(\text{grad}\, \boldsymbol{u})^T], \tag{A.130}$$

$$\text{rot}\,\text{rot}\, \boldsymbol{u} = \text{grad}\,\text{div}\, \boldsymbol{u} - \text{div}\,\text{grad}\, \boldsymbol{u}, \tag{A.131}$$

$$\text{grad}(\Phi \boldsymbol{u}) = \Phi \,\text{grad}\, \boldsymbol{u} + (\text{grad}\, \Phi)\boldsymbol{u}, \tag{A.132}$$

$$\text{div}(\Phi \mathbf{A}) = \Phi \,\text{div}\, \mathbf{A} + \text{grad}\, \Phi\,.\,\mathbf{A}, \tag{A.133}$$

$$\text{rot}(\Phi \mathbf{A}) = \Phi \,\text{rot}\, \mathbf{A} + \text{grad}\, \Phi \times \mathbf{A}, \tag{A.134}$$

$$\text{div}(\boldsymbol{u}\boldsymbol{v}) = \boldsymbol{v}\,\text{div}\, \boldsymbol{u} + \boldsymbol{u}\,.\,\text{grad}\, \boldsymbol{v}, \tag{A.135}$$

$$\text{rot}(\boldsymbol{u}\boldsymbol{v}) = (\text{rot}\, \boldsymbol{u})\boldsymbol{v} - \boldsymbol{u} \times \text{grad}\, \boldsymbol{v}. \tag{A.136}$$

Proof:

$$\text{grad}(\Phi\Psi) = (\Phi\Psi)_{,k}\boldsymbol{g}^k = \Phi\Psi_{,k}\boldsymbol{g}^k + \Psi\Phi_{,k}\boldsymbol{g}^k = \Phi\,\text{grad}\,\Psi + \Psi\,\text{grad}\,\Phi,$$

$$\text{div}(\Phi\boldsymbol{u}) = (\Phi\boldsymbol{u})^k{}_{;k} = \Phi u^k{}_{;k} + u^k\Phi_{,k} = \Phi\,\text{div}\,\boldsymbol{u} + \boldsymbol{u}\,.\,\text{grad}\,\Phi,$$

$$\text{rot}(\Phi\boldsymbol{u}) = \varepsilon^{klm}(\Phi\boldsymbol{u})_{m;l}\boldsymbol{g}_k = \varepsilon^{klm}u_m\Phi_{,l}\boldsymbol{g}_k + \Phi\varepsilon^{klm}u_{m;l}\boldsymbol{g}_k = \Phi\,\text{rot}\,\boldsymbol{u} + \text{grad}\,\Phi \times \boldsymbol{u},$$

$$\begin{aligned}\text{grad}(\boldsymbol{u}\,.\,\boldsymbol{v}) &= (\boldsymbol{u}\,.\,\boldsymbol{v})_{;k}\boldsymbol{g}^k = (u^l v_l)_{;k}\boldsymbol{g}^k = u^l{}_{;k}v_l\boldsymbol{g}^k + u^l v_{l;k}\boldsymbol{g}^k = \\ &= (\text{grad}\,\boldsymbol{u})^l{}_k v_l\boldsymbol{g}^k + (\text{grad}\,\boldsymbol{v})_{kl}u^l\boldsymbol{g}^k = \text{grad}\,\boldsymbol{u}\,.\,\boldsymbol{v} + \text{grad}\,\boldsymbol{v}\,.\,\boldsymbol{u},\end{aligned}$$

$$\begin{aligned}\text{div}(\boldsymbol{u} \times \boldsymbol{v}) &= (\boldsymbol{u} \times \boldsymbol{v})^k{}_{;k} = (\varepsilon^{klm}u_l v_m)_{;k} = \varepsilon^{klm}(u_l v_m)_{;k} = \varepsilon^{klm}(u_{l;k}v_m + u_l v_{m;k}) = \\ &= \varepsilon^{klm}u_{l;k}v_m - \varepsilon^{lkm}u_l v_{m;k} = \boldsymbol{v}\,.\,\text{rot}\,\boldsymbol{u} - \boldsymbol{u}\,.\,\text{rot}\,\boldsymbol{v},\end{aligned}$$

$$\begin{aligned}\text{rot}(\boldsymbol{u} \times \boldsymbol{v}) &= \varepsilon^{klm}(\boldsymbol{u} \times \boldsymbol{v})_{m;l}\boldsymbol{g}_k = \varepsilon^{klm}(\varepsilon_{mpq}u^p v^q)_{;l}\boldsymbol{g}_k = \\ &= (\delta_p{}^k\delta_q{}^l - \delta_q{}^k\delta_p{}^l)(u^p{}_{;l}v^q + u^p v^q{}_{;l})\boldsymbol{g}_k = \\ &= (u^k{}_{;l}v^l - u^l v^k{}_{;l} + u^k v^l{}_{;l} - u^l{}_{;l}v^k)\boldsymbol{g}_k = \\ &= (v^l(\text{grad}\,\boldsymbol{u})_l{}^k - u^l(\text{grad}\,\boldsymbol{v})_l{}^k + u^k\,\text{div}\,\boldsymbol{v} - v^k\,\text{div}\,\boldsymbol{u})\boldsymbol{g}_k = \\ &= \boldsymbol{v}\,.\,\text{grad}\,\boldsymbol{u} - \boldsymbol{u}\,.\,\text{grad}\,\boldsymbol{v} + \boldsymbol{u}\,\text{div}\,\boldsymbol{v} - \boldsymbol{v}\,\text{div}\,\boldsymbol{u},\end{aligned}$$

$$\boldsymbol{u} \times \operatorname{rot} \boldsymbol{v} = \varepsilon_{klm} u^l (\operatorname{rot} \boldsymbol{v})^m \boldsymbol{g}^k = \varepsilon_{mkl} \varepsilon^{mpq} u^l v_{q;p} \boldsymbol{g}^k = (\delta_k^p \delta_l^q - \delta_l^p \delta_k^q) u^l v_{q;p} \boldsymbol{g}^k =$$
$$= (u^l v_{l;k} - u^l v_{k;l}) \boldsymbol{g}^k = (u^l (\operatorname{grad} \boldsymbol{v})_{kl} - u^l (\operatorname{grad} \boldsymbol{v})_{lk}) \boldsymbol{g}^k = \operatorname{grad} \boldsymbol{v} . \boldsymbol{u} - \boldsymbol{u} . \operatorname{grad} \boldsymbol{v},$$

$$\operatorname{rot} \operatorname{grad} \Phi = \varepsilon^{klm} (\operatorname{grad} \Phi)_{m;l} \boldsymbol{g}_k = \varepsilon^{klm} \Phi_{,ml} \boldsymbol{g}_k = 0,$$

$$\operatorname{div} \operatorname{rot} \boldsymbol{u} = (\operatorname{rot} \boldsymbol{u})^k{}_{;k} = \varepsilon^{klm} u_{m;lk} = 0,$$

$$\operatorname{grad} \operatorname{div} \boldsymbol{u} = (\operatorname{div} \boldsymbol{u})_{;k} \boldsymbol{g}^k = u^l{}_{;lk} \boldsymbol{g}^k = u^l{}_{;kl} \boldsymbol{g}^k = [(\operatorname{grad} \boldsymbol{u})^{\mathrm{T}}]^l{}_{k;l} \boldsymbol{g}^k = \operatorname{div} [(\operatorname{grad} \boldsymbol{u})^{\mathrm{T}}],$$

$$\operatorname{rot} \operatorname{rot} \boldsymbol{u} = \varepsilon_{klm} (\operatorname{rot} \boldsymbol{u})^m{}_{;}{}^l \boldsymbol{g}^k = \varepsilon_{mkl} \varepsilon^{mpq} u_{q;p}{}^l \boldsymbol{g}^k = (\delta_k{}^p \delta_l{}^q - \delta_l{}^p \delta_k{}^q) u_{q;p}{}^l \boldsymbol{g}^k =$$
$$= u_{l;k}{}^l \boldsymbol{g}^k - u_{k;l}{}^l \boldsymbol{g}^k = u^l{}_{;lk} \boldsymbol{g}^k - u_{k;l}{}^l \boldsymbol{g}^k = (\operatorname{div} \boldsymbol{u})_{;k} \boldsymbol{g}^k - (\operatorname{grad} \boldsymbol{u})^l{}_{k;l} \boldsymbol{g}^k =$$
$$= \operatorname{grad} \operatorname{div} \boldsymbol{u} - \operatorname{div} \operatorname{grad} \boldsymbol{u},$$

$$\operatorname{grad} (\Phi \boldsymbol{u}) = (\Phi \boldsymbol{u})_{l;k} \boldsymbol{g}^k \boldsymbol{g}^l = (\Phi u_{l;k} + \Phi_{,k} u_l) \boldsymbol{g}^k \boldsymbol{g}^l =$$
$$= [\Phi (\operatorname{grad} \boldsymbol{u})_{kl} + (\operatorname{grad} \Phi)_k u_l] \boldsymbol{g}^k \boldsymbol{g}^l = \Phi \operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \Phi) \boldsymbol{u},$$

$$\operatorname{div} (\Phi \mathbf{A}) = (\Phi \mathbf{A})^{kl}{}_{;k} \boldsymbol{g}_l = \Phi A^{kl}{}_{;k} \boldsymbol{g}_l + A^{kl} \Phi_{,k} \boldsymbol{g}_l = \Phi \operatorname{div} \mathbf{A} + \operatorname{grad} \Phi . \mathbf{A},$$

$$\operatorname{rot} (\Phi \mathbf{A}) = \varepsilon^{klm} (\Phi \mathbf{A})_{mn;l} \boldsymbol{g}_k \boldsymbol{g}^n = \Phi \varepsilon^{klm} A_{mn;l} \boldsymbol{g}_k \boldsymbol{g}^n + \varepsilon^{klm} \Phi_{,l} A_{mn} \boldsymbol{g}_k \boldsymbol{g}^n =$$
$$= \Phi \operatorname{rot} \mathbf{A} + \operatorname{grad} \Phi \times \mathbf{A},$$

$$\operatorname{div} (\boldsymbol{u}\boldsymbol{v}) = (\boldsymbol{u}\boldsymbol{v})^k{}_{l;k} \boldsymbol{g}^l = (u^k v_l)_{;k} \boldsymbol{g}^l = (u^k{}_{;k} v_l + u^k v_{l;k}) \boldsymbol{g}^l =$$
$$= [v_l \operatorname{div} \boldsymbol{u} + u^k (\operatorname{grad} \boldsymbol{v})_{kl}] \boldsymbol{g}^l = \boldsymbol{v} \operatorname{div} \boldsymbol{u} + \boldsymbol{u} . \operatorname{grad} \boldsymbol{v},$$

$$\operatorname{rot} (\boldsymbol{u}\boldsymbol{v}) = \varepsilon^{klm} (u_{m;l} v_n + u_m v_{n;l}) \boldsymbol{g}_k \boldsymbol{g}^n = (\operatorname{rot} \boldsymbol{u}) \boldsymbol{v} - \boldsymbol{u} \times \operatorname{grad} \boldsymbol{v}$$

A.8. Orthogonal curvilinear coordinates

Let us first express the differential operators given above in general orthogonal curvilinear coordinates, and then in spherical coordinates. Since the physical components of tensors, and vectors, are the same in orthogonal curvilinear coordinates for all kinds of tensor components, we shall represent the tensors, and vectors, by physical components.

In *orthogonal curvilinear coordinates holds*:

(A.137) $$g_{kl} = g^{kl} = 0 \quad \text{for } k \neq l, \qquad g^{\underline{kk}} = 1/g_{\underline{kk}},$$
$$\boldsymbol{g}^k = g^{\underline{kk}} \boldsymbol{g}_k, \qquad g = g_{11} g_{22} g_{33},$$

(A.138) $$\mathrm{d}s^2 = g_{11} (\mathrm{d}x^1)^2 + g_{22} (\mathrm{d}x^2)^2 + g_{33} (\mathrm{d}x^3)^2,$$
$$\begin{Bmatrix} k \\ l\ m \end{Bmatrix} = \frac{1}{g_{\underline{kk}}} [lm,k] = \frac{1}{2 g_{\underline{kk}}} \left[\frac{\partial g_{kk}}{\partial x^m} \delta_{kl} + \frac{\partial g_{mm}}{\partial x^l} \delta_{km} - \frac{\partial g_{ll}}{\partial x^k} \delta_{lm} \right],$$

(A.139) $$\operatorname{grad} \Phi = \frac{1}{(g_{11})^{1/2}} \frac{\partial \Phi}{\partial x^1} \boldsymbol{e}_1 + \frac{1}{(g_{22})^{1/2}} \frac{\partial \Phi}{\partial x^2} \boldsymbol{e}_2 + \frac{1}{(g_{33})^{1/2}} \frac{\partial \Phi}{\partial x^3} \boldsymbol{e}_3,$$

(A.140)
$$\operatorname{div}\boldsymbol{u}=\frac{1}{(g_{11}g_{22}g_{33})^{1/2}}\left\{\frac{\partial}{\partial x^1}\left[(g_{22}g_{33})^{1/2}u^{(1)}\right]+\right.$$
$$\left.+\frac{\partial}{\partial x^2}\left[(g_{33}g_{11})^{1/2}u^{(2)}\right]+\frac{\partial}{\partial x^3}\left[(g_{11}g_{22})^{1/2}u^{(3)}\right]\right\},$$

(A.141)
$$\operatorname{rot}\boldsymbol{u}=\frac{1}{(g_{22}g_{33})^{1/2}}\left\{\frac{\partial}{\partial x^2}\left[(g_{33})^{1/2}u^{(3)}\right]-\right.$$
$$\left.-\frac{\partial}{\partial x^3}\left[(g_{22})^{1/2}u^{(2)}\right]\right\}\boldsymbol{e}_1+\frac{1}{(g_{33}g_{11})^{1/2}}\left\{\frac{\partial}{\partial x^3}\left[(g_{11})^{1/2}u^{(1)}\right]-\right.$$
$$\left.-\frac{\partial}{\partial x^1}\left[(g_{33})^{1/2}u^{(3)}\right]\right\}\boldsymbol{e}_2+\frac{1}{(g_{11}g_{22})^{1/2}}\left\{\frac{\partial}{\partial x^1}\left[(g_{22})^{1/2}u^{(2)}\right]-\right.$$
$$\left.-\frac{\partial}{\partial x^2}\left[(g_{11})^{1/2}u^{(1)}\right]\right\}\boldsymbol{e}_3,$$

(A.142)
$$\nabla^2\Phi=\frac{1}{(g_{11}g_{22}g_{33})^{1/2}}\left\{\frac{\partial}{\partial x^1}\left[\frac{(g_{22}g_{33})^{1/2}}{(g_{11})^{1/2}}\frac{\partial\Phi}{\partial x^1}\right]+\right.$$
$$\left.+\frac{\partial}{\partial x^2}\left[\frac{(g_{33}g_{11})^{1/2}}{(g_{22})^{1/2}}\frac{\partial\Phi}{\partial x^2}\right]+\frac{\partial}{\partial x^3}\left[\frac{(g_{11}g_{22})^{1/2}}{(g_{33})^{1/2}}\frac{\partial\Phi}{\partial x^3}\right]\right\},$$

(A.143)
$$\operatorname{grad}\boldsymbol{u}=u^l{}_{;k}\boldsymbol{g}^k\boldsymbol{g}_l=(\operatorname{grad}\boldsymbol{u})_k{}^l\boldsymbol{g}^k\boldsymbol{g}_l=(\operatorname{grad}\boldsymbol{u})_{(k)}{}^{(l)}\boldsymbol{e}_k\boldsymbol{e}_l,$$

(A.144)
$$(\operatorname{grad}\boldsymbol{u})_{(k)}{}^{(l)}=\frac{1}{(g_{kk})^{1/2}}\frac{\partial u^{(l)}}{\partial x^k}-$$
$$-\frac{u^{(k)}}{(g_{kk}g_{ll})^{1/2}}\frac{\partial(g_{kk})^{1/2}}{\partial x^l}+\frac{\delta_{kl}}{(g_{kk})^{1/2}}\sum_{m=1}^{3}\frac{u^{(m)}}{(g_{mm})^{1/2}}\frac{\partial(g_{ll})^{1/2}}{\partial x^m},$$

(A.145)
$$\operatorname{div}\mathbf{A}=A^k{}_{l;k}\boldsymbol{g}^l=(\operatorname{div}\mathbf{A})_l\boldsymbol{g}^l=(\operatorname{div}\mathbf{A})_{(l)}\boldsymbol{l}_l,$$

(A.146)
$$(\operatorname{div}\mathbf{A})_{(l)}=\sum_{k=1}^{3}\left\{\frac{1}{(g)^{1/2}}\frac{\partial}{\partial x^k}\left[A^{(k)}{}_{(l)}\frac{(g)^{1/2}}{(g_{kk})^{1/2}}\right]+\right.$$
$$\left.+\frac{1}{(g_{kk}g_{ll})^{1/2}}\frac{\partial(g_{ll})^{1/2}}{\partial x^k}A^{(k)}{}_{(l)}-\frac{1}{(g_{kk}g_{ll})^{1/2}}\frac{\partial(g_{kk})^{1/2}}{\partial x^l}A^{(k)}{}_{(k)}\right\}.$$

Note: Sometimes it is advantageous in orthogonal curvilinear coordinates to introduce *Lame's coefficients* H_k by

(A.147)
$$H_k=(g_{kk})^{1/2}.$$

Example 14: Express above relation in spherical coordinates r, ϑ, φ.

The definition relation between Cartesian and *spherical coordinates* is

(A.148)
$$y^1=r\sin\vartheta\cos\varphi,$$
$$y^2=r\sin\vartheta\sin\varphi,$$
$$y^3=r\cos\vartheta.$$

Lamé's coefficients read

(A.149) $$H_r = 1,\ H_\vartheta = r,\ H_\varphi = r\sin\vartheta.$$

Christoffel's symbols of the 2nd kind,

(A.150) $$\begin{Bmatrix} r \\ \vartheta\ \vartheta \end{Bmatrix} = -r,\ \begin{Bmatrix} r \\ \varphi\ \varphi \end{Bmatrix} = -r\sin^2\vartheta,\ \begin{Bmatrix} \vartheta \\ \mathrm{r}\ \vartheta \end{Bmatrix} = 1/\mathrm{r},$$
$$\begin{Bmatrix} \vartheta \\ \varphi\ \varphi \end{Bmatrix} = -\sin\vartheta\cos\vartheta,\ \begin{Bmatrix} \varphi \\ r\ \varphi \end{Bmatrix} = 1/r,$$
$$\begin{Bmatrix} \varphi \\ \vartheta\ \varphi \end{Bmatrix} = \cot\vartheta,\ \text{others}\begin{Bmatrix} k \\ l\ m \end{Bmatrix} = 0.$$

(A.151) $$\operatorname{grad}\boldsymbol{\Phi} = \frac{\partial\Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\varphi}\boldsymbol{e}_\varphi,$$

(A.152) $$\operatorname{div}\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(u_\vartheta\sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial u_\varphi}{\partial\varphi},$$

(A.153) $$\operatorname{rot}\boldsymbol{u} = \frac{1}{r\sin\vartheta}\left[\frac{\partial(u_\varphi\sin\vartheta)}{\partial\vartheta} - \frac{\partial u_\vartheta}{\partial\varphi}\right]\boldsymbol{e}_r +$$
$$+\left[\frac{1}{r\sin\vartheta}\frac{\partial \mathrm{u_r}}{\partial\varphi} - \frac{1}{r}\frac{\partial(ru_\varphi)}{\partial r}\right]\boldsymbol{e}_\vartheta + \frac{1}{r}\left[\frac{\partial(ru_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right]\boldsymbol{e}_\varphi,$$

(A.154) $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}.$$

A.9. Tensor of small deformations and equations of motion of the continuum in curvilinear orthogonal coordinates

We shall introduce the *tensor of small deformations* $\boldsymbol{\mathsf{e}}$ as (see Supplement B)

(A.155) $$\boldsymbol{\mathsf{e}} = \tfrac{1}{2}[\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}],$$

where $\boldsymbol{u}$ is the *displacement vector*. Using Eq. (A.144), we can express this tensor in terms of the physical components,

(A.156) $$\boldsymbol{\mathsf{e}} = e^{(k)}{}_{(l)}\boldsymbol{e}_k\boldsymbol{e}_l,$$

(A.157) $$e^{(k)}{}_{(l)} = e^k{}_l(g_{\underline{kk}}/g_{\underline{ll}})^{1/2} =$$
$$= \frac{1}{2}\left\{\frac{(g_{\underline{kk}})^{1/2}}{(g_{\underline{ll}})^{1/2}}\frac{\partial}{\partial x^l}\left[\frac{u^{(k)}}{(g_{\underline{kk}})^{1/2}}\right] + \frac{g_{\underline{ll}})^{1/2}}{(g_{\underline{kk}})^{1/2}}\frac{\partial}{\partial x^k}\left[\frac{u^{(l)}}{(g_{\underline{ll}})^{1/2}}\right]\right\} +$$
$$+ \frac{\delta_{kl}}{(g_{\underline{ll}})^{1/2}}\sum_{m=1}^{3}\frac{u^{(m)}}{(g_{\underline{mm}})^{1/2}}\frac{\partial(g_{\underline{kk}})^{1/2}}{\partial x^m}.$$

Example 16: Express the components of the tensor of small deformations in spherical coordinates r, ϑ, φ.

Let us denote the physical components of the displacement vector $\boldsymbol{u}$ by the symbols (u, v, w). Then

$$(\text{A.158}) \qquad \begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, \quad e_{\vartheta\vartheta} = \frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{u}{r}, \\ e_{\varphi\varphi} &= \frac{1}{r\sin\vartheta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v}{r}\cot\vartheta, \\ 2e_{r\vartheta} &= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \vartheta} - \frac{v}{r}, \\ 2e_{r\varphi} &= \frac{\partial w}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial \varphi} - \frac{w}{r}, \\ 2e_{\vartheta\varphi} &= \frac{1}{r}\left(\frac{\partial w}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial v}{\partial \varphi} - w\cot\vartheta\right), \end{aligned}$$

where e_{rr}, $e_{r\vartheta}$, ..., $e_{\vartheta\varphi}$ are the physical components of the tensor of small deformations $\mathbf{e}$.

We shall express the *equations of motion* of the continuum in vectorial form (see Supplement B),

$$(\text{A.159}) \qquad \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} = \varrho\boldsymbol{f} + \operatorname{div}\mathbf{t},$$

where ϱ is the *density*, $\boldsymbol{f}$ the body force per unit mass, $\boldsymbol{u}$ the *displacement vector* and $\mathbf{t}$ *Cauchy's stress tensor*. If we use Eq. (A.146), the 1th equation of motion, expressed in terms of physical components will read

$$(\text{A.160}) \qquad \sum_{k=1}^{3}\left\{\frac{1}{(g)^{1/2}}\frac{\partial}{\partial x^k}\left[t^{(k)}{}_{(l)}\frac{(g)^{1/2}}{(g_{\underline{kk}})^{1/2}}\right] + \frac{1}{(g_{\underline{kk}}g_{\underline{ll}})^{1/2}}\frac{\partial(g_{\underline{ll}})^{1/2}}{\partial x^k}t^{(l)}{}_{(k)} - \right. \\ \left. - \frac{1}{(g_{\underline{kk}}g_{\underline{ll}})^{1/2}}\frac{\partial(g_{\underline{kk}})^{1/2}}{\partial x^l}t^{(k)}{}_{(k)}\right\} + \varrho f_{(l)} = \varrho\frac{\mathrm{d}^2u_{(l)}}{\mathrm{d}t^2}.$$

Example 17: Express the equations of motion of the continuum in spherical coordinates r, ϑ, φ.

Let us denote the physical components of Cauchy's stress tensor, of the body force and displacement vector in spherical coordinates by the symbols $(t_{rr}, t_{r\vartheta}, \ldots t_{\vartheta\varphi})$, $(f_r, f_\vartheta, f_\varphi)$ and (u, v, w). In these coordinates the equations of motion of the continuum can be expressed as follows:

$$(\text{A.161}) \qquad \varrho\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \varrho f_r + \frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\vartheta}}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial t_{r\varphi}}{\partial \varphi} + \\ + \frac{1}{r}(2t_{rr} - t_{\vartheta\vartheta} - t_{\varphi\varphi} + t_{r\vartheta}\cot\vartheta),$$

$$\varrho \frac{d^2 v}{dt^2} = \varrho f_\vartheta + \frac{\partial t_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\vartheta\vartheta}}{\partial \vartheta} + \frac{1}{r \sin\vartheta}\frac{\partial t_{\vartheta\varphi}}{\partial \varphi} + $$
$$+ \frac{1}{r}[3t_{r\vartheta} + (t_{\vartheta\vartheta} - t_{\varphi\varphi})\cot\vartheta],$$
$$\varrho \frac{d^2 w}{dt^2} = \varrho f_\varphi + \frac{\partial t_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial t_{\vartheta\varphi}}{\partial \vartheta} + \frac{1}{r \sin\vartheta}\frac{\partial t_{\varphi\varphi}}{\partial \varphi} + $$
$$+ \frac{1}{r}(3t_{r\varphi} + 2t_{\vartheta\varphi}\cot\vartheta).$$

A.10. Two-point tensor field

Definition 10: The quantities $A^k{}_K(\boldsymbol{x}, \boldsymbol{X})$ that transform like tensors with respect to the indices k and K under transformation of the coordinate systems x^k and X^K, are referred to as *two-point tensors.*

Therefore, if

(A.162) $$x^{k'} = x^{k'}(\boldsymbol{x}), \quad X^{K'} = X^{K'}(\boldsymbol{X})$$

are differentiable transformations of coordinates, and if

(A.163) $$A^{k'}{}_{K'}(\boldsymbol{x}', \boldsymbol{X}') = A^m{}_M(\boldsymbol{x}, \boldsymbol{X}) \frac{\partial x^{k'}}{\partial x^m} \frac{\partial X^M}{\partial X^{K'}},$$

then $A^k{}_K$ is a two point tensor. If $\boldsymbol{g}_k$ and $\boldsymbol{G}_K$ are base vectors and $\boldsymbol{g}^k$ and $\boldsymbol{G}^K$ vectors reciprocal to the former in coordinate systems x^k and X^K, then $A^k{}_K$ are components of the tensor

(A.164) $$\mathbf{A}(\boldsymbol{x}, \boldsymbol{X}) = A^k{}_K(\boldsymbol{x}, \boldsymbol{X})\, \boldsymbol{g}_k(\boldsymbol{x})\, \boldsymbol{G}^K(\boldsymbol{X}).$$

An example of a two-point tensor are *shifters* defined by

(A.165) $$g^k{}_K(\boldsymbol{x}, \boldsymbol{X}) = \boldsymbol{g}^k(\boldsymbol{x}) . \boldsymbol{G}_K(\boldsymbol{X}),$$
$$g^K{}_k(\boldsymbol{x}, \boldsymbol{X}) = \boldsymbol{G}^K(\boldsymbol{X}) . \boldsymbol{g}_k(\boldsymbol{x}).$$

Another example are *deformation gradients*

(A.166) $$x^k{}_{,K} = \frac{\partial x^k}{\partial X^K}, \quad X^K{}_{,k} = \frac{\partial X^K}{\partial x^k}.$$

The two-point tensor character of these quantities is implied by the relation

(A.167) $$\frac{\partial x^{k'}}{\partial X^{K'}} = \frac{\partial x^{k'}}{\partial x^m} \frac{\partial x^m}{\partial X^M} \frac{\partial X^M}{\partial X^{K'}},$$

where we have made use of rule of chaine of differentiation. Equation (A.167) has the form of (A.163). Multiple-point tensors of higher orders are similarly defined.

Definition 11: The *total covariant derivative* of the two-point tensor $A^k{}_K(\boldsymbol{x}, \boldsymbol{X})$, when $\boldsymbol{x}$ is related to $\boldsymbol{X}$ by the a mapping $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$, is defined by

(A.168) $$A^k{}_{K:L} = A^k{}_{K;L} + A^k{}_{K;l}x^l{}_{,L},$$

where $A^k{}_{K;L}$ is the covariant partial derivative of $A^k{}_K$ with respect to metric G_{KL} at the fixed point $\boldsymbol{x}$, and $A^k{}_{K;l}$ is the covariant partial derivative with respect to metric g_{kl} at the fixed point $\boldsymbol{X}$, i.e.,

(A.169) $$A^k{}_{K;L} = \frac{\partial A^k{}_K}{\partial X^L} - \begin{Bmatrix} M \\ L\ K \end{Bmatrix} A^k{}_M,$$
$$A^k{}_{K;l} = \frac{\partial A^k{}_K}{\partial x^l} + \begin{Bmatrix} k \\ l\ m \end{Bmatrix} A^m{}_K.$$

Therefore,

(A.170) $$A^k{}_{K:L} = \frac{\partial A^k{}_K}{\partial X^L} - \begin{Bmatrix} M \\ L\ K \end{Bmatrix} A^k{}_M + \left[\frac{\partial A^k{}_K}{\partial x^l} + \begin{Bmatrix} k \\ l\ m \end{Bmatrix} A^m{}_K\right]\frac{\partial x^l}{\partial X^L}.$$

Note that this result is produced by differentiating Eq. (164) with respect to X^K and by using Eqs (A.80) and (A.85) to express the derivatives of vectors $\boldsymbol{g}_k$ and $\boldsymbol{G}^K$. Therefore,

(A.171) $$\partial \mathbf{A}/\partial X^K = A^k{}_{L:K}\boldsymbol{g}_k\boldsymbol{G}^L.$$

By using Eq. (A.170) for $x^k{}_{,K}(\boldsymbol{X})$, where the vector $\boldsymbol{x}$ is missing in the argument $x^k{}_{,K}$, we arrive at

(A.172) $$(x^k{}_{,K})_{:L} = \frac{\partial^2 x^k}{\partial X^L \partial X^K} - \begin{Bmatrix} M \\ L\ K \end{Bmatrix}\frac{\partial x^k}{\partial X^M} + \begin{Bmatrix} k \\ l\ m \end{Bmatrix}\frac{\partial x^m}{\partial X^K}\frac{\partial x^l}{\partial X^L}.$$

Note that (A.168) is a generalization of the total derivative of the scalar function of two variables, $\Phi(x, X)$ with $x = x(X)$, i.e.,

(A.173) $$\frac{\mathrm{d}\Phi}{\mathrm{d}X} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \Phi}{\partial X}.$$

The same formal rules apply to the total covariant derivative as to the covariant partial derivative, e.g.,

(A.174) $$g^k{}_{K:M} = G_{KL:m} = g_{kl:M} = 0,$$
$$(A^k{}_K B^L{}_l)_{:M} = A^k{}_{K:M}B^L{}_l + A^k{}_K B^L{}_{l:M},$$
$$(A^k{}_K + B^k{}_K)_{:M} = A^k{}_{K:M} + B^k{}_{K:M}.$$

For other accounts, see [58].

A.11. Projection of tensors onto a surface

Let S be an oriented surface in three-dimensional space represented in *Gaussian form*,

(A.175) $$\boldsymbol{x} = \boldsymbol{x}(p^a) \qquad a = 1, 2, \quad \text{or}$$
$$x^k = x^k(p^1, p^2) \qquad k = 1, 2, 3,$$

where p^1, p^2 are curvilinear coordinates on surface S and x^k are space curvilinear coordinates of point $\boldsymbol{x}$ on surface S. Assume $\boldsymbol{n}(\boldsymbol{x})$ to be the *unit normal* external to surface S at point $\boldsymbol{x}$ on S.

We shall refer to vector $\boldsymbol{v}$ on S as a *vector tangential* to S, if $\boldsymbol{n}\,.\,\boldsymbol{v} = 0$ at every point $\boldsymbol{x}$ on S. We shall refer to the 2nd-order tensor $\mathbf{A}$ on S as a *tensor tangential* to S, if $\boldsymbol{n}\,.\,\mathbf{A} = \mathbf{A}\,.\,\boldsymbol{n} = 0$ at every point $\boldsymbol{x}$ on S. If $\mathbf{I}$ is a *three-dimensional identical tensor*, $I^k{}_l = \delta^k{}_l$, i.e. a 2nd-order tensor such that $\boldsymbol{v}\,.\,\mathbf{I} = \mathbf{I}\,.\,\boldsymbol{v} = \boldsymbol{v}$ and $\mathbf{A}\,.\,\mathbf{I} = \mathbf{I}\,.\,\mathbf{A} = \mathbf{A}$ for any vector $\boldsymbol{v}$ and 2nd-order tensor $\mathbf{A}$, then the equation

(A.176) $$\mathbf{I}_s = \mathbf{I} - \boldsymbol{n}\boldsymbol{n}$$

defines the tangential 2nd-order tensor on S which we refer to as the *surface identical tensor* since, if $\boldsymbol{v}$ is the vector tangential to S and $\mathbf{A}$ the 2nd-order tensor tangential to S, then $\boldsymbol{v}\,.\,\mathbf{I}_s = \mathbf{I}_s\,.\,\boldsymbol{v} = \boldsymbol{v}$, $\mathbf{A}\,.\,\mathbf{I}_s = \mathbf{I}_s\,.\,\mathbf{A} = \mathbf{A}$.

The projection of vector $\boldsymbol{v}$ on surface S is the tangential vector $\boldsymbol{v}_s$,

(A.177) $$\boldsymbol{v}_s = \boldsymbol{v}\,.\,\mathbf{I}_s = \mathbf{I}_s\,.\,\boldsymbol{v}.$$

If $\boldsymbol{v}$ is the vector tangential to S, then $\boldsymbol{v}_s = \boldsymbol{v}$. Assume grad to be the gradient operator in three-dimensional space. The *surface gradient*, grad_s, at point $\boldsymbol{x}$ on surface S is defined as the projection of operator grad onto surface S,

(A.178) $$\text{grad}_s = \mathbf{I}_s\,.\,\text{grad} = \text{grad} - \boldsymbol{n}(\boldsymbol{n}\,.\,\text{grad}).$$

For example, if φ is a scalar field on S,

(A.179) $$\text{grad}_s\,\varphi = \text{grad}\,\varphi - \boldsymbol{n}(\boldsymbol{n}\,.\,\text{grad}\,\varphi).$$

Since grad_s only contains derivatives in the direction tangential to surface S, the operator grad_s may be applied to any field, defined on surface S, regardless of whether this field is defined elsewhere in space or not.

Assume $\mathbf{Q}$ to be a scalar, vector or tensor field defined on surface S. If we move this field from point $\boldsymbol{x}$ on S to a point infinitesimally close, $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$, also lying on S, the field $\mathbf{Q}$ will change by the value $\mathrm{d}\mathbf{Q}$:

(A.180) $$\mathrm{d}\mathbf{Q} = \mathrm{d}\boldsymbol{x}\,.\,\text{grad}_s\,\mathbf{Q}.$$

The projection of the 2nd-order tensor $\mathbf{A}$ on surface S is tensor

(A.181) $$\mathbf{A}_s = \mathbf{I}_s\,.\,\mathbf{A} = \mathbf{A} - \boldsymbol{n}\boldsymbol{n}\,.\,\mathbf{A} = \boldsymbol{A} - \boldsymbol{n}(\boldsymbol{n}\,.\,\mathbf{A}).$$

Tensor $\mathbf{A}_s$ is, in general, not tangential to surface S. If $\mathbf{A} = \text{grad}\, \boldsymbol{v}$, the surface gradient of vector $\boldsymbol{v}$ is defined bys the relation

(A.182) $$\text{grad}_s \boldsymbol{v} \equiv \text{grad}\, \boldsymbol{v} - \boldsymbol{n}(\boldsymbol{n} . \text{grad}\, \boldsymbol{v}) .$$

The *surface divergence* of vector $\boldsymbol{v}$ is defined as

(A.183) $$\text{div}_s \boldsymbol{v} \equiv \text{tr}(\text{grad}_s \boldsymbol{v}) = \text{div}\, \boldsymbol{v} - \boldsymbol{n} . \text{grad}\, \boldsymbol{v} . \boldsymbol{n} .$$

Using (A.132), (A.123) and (A.125), it is easy to prove the identities

(A.184) $$\begin{aligned} \text{grad}_s(\varphi \boldsymbol{v}) &= \varphi\, \text{grad}_s \boldsymbol{v} + (\text{grad}_s \varphi) \boldsymbol{v} , \\ \text{div}_s(\varphi \boldsymbol{v}) &= \varphi\, \text{div}_s \boldsymbol{v} + \boldsymbol{v} . \text{grad}_s \varphi , \\ \text{grad}_s(\boldsymbol{u} . \boldsymbol{v}) &= \text{grad}_s \boldsymbol{u} . \boldsymbol{v} + \text{grad}_s \boldsymbol{v} . \boldsymbol{u} . \end{aligned}$$

SUPPLEMENT B. FUNDAMENTAL RELATIONS OF THE THEORY OF ELASTICITY

This supplement is devoted to a brief recapitulation of the fundamental relations of the theory of elastic bodies. Strain geometry is described with the aid of the theory of differential geometry, and laws of conservation are described in natural (deformed) and reference (undeformed) systems of coordinates.

A detailed discussion of theory of elasticity and continuum physics is given in [28, 37, 64, 73, 74, 76, 91, 95, 96, 99, 109, 129, 130]. Our brief description follows books of Eringen [56—60].

B.1. Strain tensor

B.1.1. Coordinates, deformation, motion

Consider an continuum body at two different states of time. In the first, assume the body to be unstrained, in pre-strain state, or the initial undeformed state. In the second, assume the body to be strained, in the post-strain state, or deformed state. Assume the undeformed body B to have volume V and surface S. Assume the deformed body b to have volume v and surface s. The position of material point P in body B will be described by the curvilinear coordinates X^1, X^2, X^3, or by the position vector $\boldsymbol{P}$ (also $\boldsymbol{X}$) which extends from the origin O of the coordinates to point P. In the deformed state, assume the material point p to be represented by a new set of curvilinear coordinates x^1, x^2, x^3, or by a position vector $\boldsymbol{p}$ (also $\boldsymbol{x}$) that extends from the origin o of the new coordinates to point p. Often it is advantageous to select these two systems of coordinates. The coordinates X^K are called the *Lagrangian* or *material* coordinates and x^k the *Eulerian* or *spatial* coordinates.

The motion of the body carries various material points through various spatial positions. This is expressed by

$$x^k = x^k(X^K, t), \quad X^K = X^K(x^k, t) \tag{B.1}$$

for $k = 1, 2, 3$ and $K = 1, 2, 3$. (B.1) can be abbreviated to read

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \quad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t). \tag{B.2}$$

Equation $(B.1)_1$ states that at time t a material point X^K of B occupies the spatial position x^k in b. Equation $(B.1)_2$ describes the opposite.

We shall assume that functions x^k and X^K are continuously differentiable at least up to the first order in the neighbourhood of point $P(\boldsymbol{X})$, or $p(\boldsymbol{x})$, and that the Jacobian of transformation is not identically zero, i.e.

$$j \equiv \det\left(\frac{\partial x^k}{\partial X^K}\right) \neq 0 \tag{B.3}$$

describes unique inverse transformations.

The assumptions mentioned express the *axiom of continuity*, the consequence of which is, on the one hand, the *axiom of indestructability* of matter, i.e. no region of a finite positive volume can be deformed into a region of zero volume, and, on the other, the *axiom of impenetrability* of matter, i.e. under motion every volume is again transformed into a volume, every surface into a surface, and every curve into a curve. However, in some cases it must be assumed that, within a particular interval of time, there may exist singular surfaces, curves and points in which the axiom of continuity is not satisfied.

We shall denote the quantities relating to the undeformed body B by capital letters, to the deformed body b by lower-case letters. The components of vectors and tensors relative to coordinates X^K will have capital Roman letter indices, those relative to coordinates x^k lower-case Roman letters. For example, $G_{KL}(\boldsymbol{X})$ and $g_{kl}(\boldsymbol{x})$ are the covariant metric tensors in B and b, respectively.

B.1.2. Base vectors, metric tensors, shifters

The position vector $\boldsymbol{P}$ of point P in B and the position vector $\boldsymbol{p}$ of point p in b are expressed in Cartesian coordinates Y^K and y^k as

$$\boldsymbol{P} = Y^K \boldsymbol{I}_K, \quad \boldsymbol{p} = y^k \boldsymbol{i}_k, \tag{B.4}$$

where $\boldsymbol{I}_K$ and $\boldsymbol{i}_k$ are unit base vectors in Cartesian coordinates Y^K and y^k. We are again going to use Einstein's summation rule, i.e. we summate from one to three over every diagonally repeated index (Fig. B1).

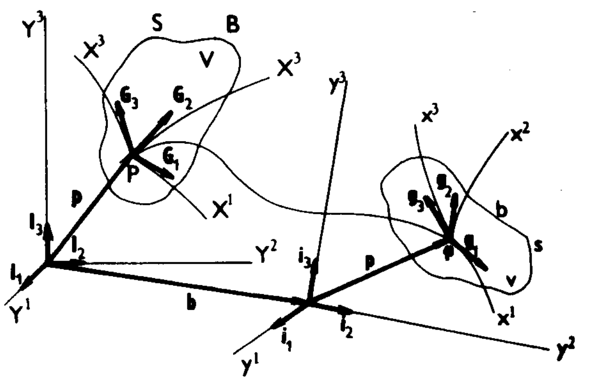

Fig. B1. Coordinate systems for an undeformed body B and a deformed body b.

We shall introduce the *base vectors* $\boldsymbol{G}_K(\boldsymbol{X})$ and $\boldsymbol{g}_k(\boldsymbol{x})$ at X^K and x^k, respectively,

(B.5) $$\boldsymbol{G}_K(\boldsymbol{X}) = \frac{\partial \boldsymbol{P}}{\partial X^K} = \frac{\partial Y^M}{\partial X^K}\boldsymbol{I}_M, \quad \boldsymbol{g}_k(\boldsymbol{x}) = \frac{\partial \boldsymbol{p}}{\partial x^k} = \frac{\partial y^m}{\partial x^k}\boldsymbol{i}_m .$$

The infinitesimal differential vectors $\mathrm{d}\boldsymbol{P}$ at point P and $\mathrm{d}\boldsymbol{p}$ at point p are

(B.6) $$\mathrm{d}\boldsymbol{P} = \frac{\partial \boldsymbol{P}}{\partial X^K}\mathrm{d}X^K = \boldsymbol{G}_K \mathrm{d}X^K, \quad \mathrm{d}\boldsymbol{p} = \frac{\partial \boldsymbol{p}}{\partial x^k}\mathrm{d}x^k = \boldsymbol{g}_k \mathrm{d}x^k .$$

The base vectors $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ are tangential to the coordinate lines X^K and x^k.

The squares of the lenghts in B and b are

(B.7) $$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{P}.\mathrm{d}\boldsymbol{P} = G_{KL}\mathrm{d}X^K\mathrm{d}X^L, \quad \mathrm{d}s^2 = \mathrm{d}\boldsymbol{p}.\mathrm{d}\boldsymbol{p} = g_{kl}\mathrm{d}x^k\mathrm{d}x^l,$$

respectively, where

(B.8) $$G_{KL}(\boldsymbol{X}) = \boldsymbol{G}_K.\boldsymbol{G}_L = \frac{\partial Y^M}{\partial X^K}\frac{\partial Y^N}{\partial X^L}\delta_{MN}, \quad g_{kl}(\boldsymbol{x}) = \boldsymbol{g}_k.\boldsymbol{g}_l = \frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^l}\delta_{mn}$$

are metric tensors in B and b, respectively. Kronecker's delta symbols δ_{MN}, δ_{mn}, δ^M_N and δ^m_n are equal to unity if their indices are the same and to zero if they are different.

The *reciprocal base vectors* $\boldsymbol{G}^K(\boldsymbol{X})$ and $\boldsymbol{g}^k(\boldsymbol{x})$ are defined by the equations

(B.9) $$\boldsymbol{G}^K.\boldsymbol{G}_L = \delta^K_L, \quad \boldsymbol{g}^k.\boldsymbol{g}_l = \delta^k_l .$$

The solution to these equations reads

(B.10) $$\boldsymbol{G}^K = G^{KL}\boldsymbol{G}_L, \quad \boldsymbol{g}^k = g^{kl}\boldsymbol{g}_l,$$

where

(B.11) $$G^{KL} = \frac{\text{alg. cofactor } G_{KL}}{\det(G_{KL})}, \quad g^{kl} = \frac{\text{alg. cofactor } g_{kl}}{\det(g_{kl})}.$$

The scalar product of (B.10) with vectors $\boldsymbol{G}^L$ and $\boldsymbol{g}^l$ yields

(B.12) $$G^{KL} = \boldsymbol{G}^K . \boldsymbol{G}^L, \quad g^{kl} = \boldsymbol{g}^k . \boldsymbol{g}^l .$$

The representation of vectors and tensors with respect to coordinates X^K or x^k is separated, e.g. the components of the position vectors $\boldsymbol{P}$ and $\boldsymbol{p}$ in coordinates X^K and x^k are

(B.13) $$P^K = \boldsymbol{P} . \boldsymbol{G}^K, \quad p^k = \boldsymbol{p} . \boldsymbol{g}^k .$$

We would like to express the vectors and tensors from one coordinate system in terms of their projection into the other coordinate system and vice versa. For this purpose, let us shift vector $\boldsymbol{p}$ parallelly to point $P(\boldsymbol{X})$. If p^K are the components of vector $\boldsymbol{p}$ in X^K,

(B.14) $$\boldsymbol{p} = p^K \boldsymbol{G}_K(\boldsymbol{X}) = p^k \boldsymbol{g}_k(\boldsymbol{x}) .$$

The scalar product of (B.14) with the vectors $\boldsymbol{G}^L$ and $\boldsymbol{g}^l$ yields

(B.15) $$p^K = g^K{}_k p^k, \quad p^k = g^k{}_K p^K ,$$

where

(B.16) $$g^K{}_k(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{G}^K(\boldsymbol{X}) . \boldsymbol{g}_k(\boldsymbol{x}), \quad g^k{}_K(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}^k(\boldsymbol{x}) . \boldsymbol{G}_K(\boldsymbol{X})$$

are so-called *shifters*. These are two-point tensor (see A.10); i.e., they transform as tensors with respect to indices K and k under transformation of coordinates X^K and x^k. With the aid of shifters it is possible to express vectors and tensors from one coordinate system with the aid of their projection into another coordinate system.

In very much the same way we now define

(B.17) $$\begin{aligned} g_{Kk}(\boldsymbol{X}, \boldsymbol{x}) &= g_{kK}(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}_k(\boldsymbol{x}) . \boldsymbol{G}_K(\boldsymbol{X}) , \\ g^{Kk}(\boldsymbol{X}, \boldsymbol{x}) &= g^{kK}(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}^k(\boldsymbol{x}) . \boldsymbol{G}^K(\boldsymbol{X}) . \end{aligned}$$

By raising and lowering the capital-letter indices with the aid of tensors G^{KL} and G_{KL}, and by raising and lowering the lower-case indices with the aid of metric tensors g^{kl} and g_{kl}, we arrive at

(B.18) $$\begin{aligned} g_{Kk} &= g_{kl} g^l{}_K = G_{KL} g^L{}_k = g_{kl} G_{KL} g^{lL} , \\ g^{Kk} &= g^{kl} g^K{}_l = G^{KL} g^k{}_L = g^{kl} G^{KL} g_{lL} , \\ g^K{}_k &= g_{kl} g^{lK} = G^{KL} g_{kL} = g_{kl} G^{KL} g^l{}_L , \\ g^K{}_k g^l{}_K &= \delta^l{}_k , \quad g^K{}_k g^k{}_L = \delta^K{}_L . \end{aligned}$$

By substituting (B.5) into (B.17) we obtain

(B.19) $$g_{Kk} = \delta_{Ll}\frac{\partial Y^L}{\partial X^K}\frac{\partial y^l}{\partial x^k}, \quad \delta_{Ll} = \boldsymbol{I}_L \cdot \boldsymbol{I}_l .$$

This equation implies not only the two-point character of tensor g_{Kk}, but also the relation

(B.20) $$g_{Kk} = \delta_{Kk}, \quad g^K{}_k = \delta^K{}_k ,$$

provided both coordinates X^K and x^k are Cartesian.

B.1.3. Deformation gradients, deformation tensors

Equation (B.1) for a fixed time yields

(B.21) $$\mathrm{d}x^k = x^k{}_{,K}\mathrm{d}X^K, \quad \mathrm{d}X^K = X^K{}_{,k}\mathrm{d}x^k ,$$

where the indices following the commas represent partial derivatives with respect to X^K, if the index is a capital letter, and with respect to x^k, if the index is a lower-case letter, i.e.,

(B.22) $$x^k{}_{,K} = \frac{\partial x^k}{\partial X^K}, \quad X^K{}_{,k} = \frac{\partial X^K}{\partial x^k} .$$

The quantities defined by Eq. (B.22) are referred to as *deformation gradients.* According to the chain rule of partial differentiation,

(B.23) $$x^k{}_{,K}X^K{}_{,l} = \delta^k{}_l, \quad X^K{}_{,k}x^k{}_{,L} = \delta^K{}_L .$$

Each of these systems represents nine linear algebraic equations for nine unknowns $x^k{}_{,K}$ or $X^K{}_{,k}$. Since the Jacobian of transformation is non-zero by assumption, there exists a unique solution to these equations. According to Cramer's determinant rule,

(B.24) $$X^K{}_{,k} = \frac{\text{alg. cofactor } x^k{}_{,K}}{j} = \frac{1}{2j}e^{KLM}e_{klm}x^l{}_{,L}x^m{}_{,M} ,$$

where e^{KLM} and e_{klm} are Levi-Civita's permutation symbols and

(B.25) $$j \equiv \det(x^k{}_{,K}) = \frac{1}{3!}e^{KLM}e_{klm}x^k{}_{,K}x^l{}_{,L}x^m{}_{,M} .$$

By differentiating (B.24) and (B.25) we obtain two important *Jacobi's identities*:

(B.26) $$(jX^K{}_{,k})_{,K} = 0 \text{ and } (j^{-1}x^k{}_{,K})_{,k} = 0 ,$$
$$\frac{\partial j}{\partial x^k{}_{,K}} = \text{alg. cofactor } x^k{}_{,K} = jX^K{}_{,k} .$$

By substituting (B.21) into (B.7) we obtain

(B.27) $$\mathrm{d}S^2 = c_{kl}(\boldsymbol{x}, t)\,\mathrm{d}x^k\,\mathrm{d}x^l, \quad \mathrm{d}s^2 = C_{KL}(\boldsymbol{X}, t)\,\mathrm{d}X^K\,\mathrm{d}X^L,$$

where

(B.28) $$\begin{aligned} c_{kl}(\boldsymbol{x}, t) &= G_{KL}(\boldsymbol{X})\,X^K{}_{,k}X^L{}_{,l}, \\ C_{KL}(\boldsymbol{X}, t) &= g_{kl}(\boldsymbol{x})\,x^k{}_{,K}x^l{}_{,L} \end{aligned}$$

are *Cauchy's and Green's deformation tensors.* Both tensors are symmetric, $c_{kl} = c_{lk}$, $C_{KL} = C_{LK}$, and both are positive definite. Equations (B.28) indicate that the metric tensor $G_{KL}(\boldsymbol{X})$ transforms to tensor $c_{kl}(\boldsymbol{x}, t)$ through the motion. Tensor C_{KL} can be said to do the same in inverse motion.

New base vectors, so-called *Cauchy's and Green's base vectors* $\boldsymbol{c}_k(\boldsymbol{x}, t)$ and $\boldsymbol{C}_K(\boldsymbol{X}, t)$, can be defined with respect to these two new tensors:

(B.29) $$\begin{aligned} \boldsymbol{c}_k(\boldsymbol{x}, t) &= \frac{\partial \boldsymbol{P}}{\partial x^k} = \frac{\partial \boldsymbol{P}}{\partial X^K}\frac{\partial X^K}{\partial x^k} = \boldsymbol{G}_K(\boldsymbol{X})\,X^K{}_{,k}, \\ \boldsymbol{C}_K(\boldsymbol{X}, t) &= \frac{\partial \boldsymbol{p}}{\partial X^K} = \frac{\partial \boldsymbol{p}}{\partial x^k}\frac{\partial x^k}{\partial X^K} = \boldsymbol{g}_k(\boldsymbol{x})\,x^k{}_{,K}. \end{aligned}$$

This immediately yields

(B.30) $$c_{kl} = c_{lk} = \boldsymbol{c}_k \cdot \boldsymbol{c}_l, \quad C_{KL} = C_{LK} = \boldsymbol{C}_K \cdot \boldsymbol{C}_L.$$

Equations (B.29) indicate that the base vectors $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ deform to vectors $\boldsymbol{c}_k$ and $\boldsymbol{C}_K$ through the motion.

We now have two different representations for the differential vectors $\mathrm{d}\boldsymbol{P}$ and $\mathrm{d}\boldsymbol{p}$. One in coordinate system X^K and the other in x^k, i.e.,

(B.31) $$\begin{aligned} \mathrm{d}\boldsymbol{P} &= \boldsymbol{G}_K(\boldsymbol{X})\,\mathrm{d}X^K = \boldsymbol{c}_k(\boldsymbol{x}, t)\,\mathrm{d}x^k, \\ \mathrm{d}\boldsymbol{p} &= \boldsymbol{C}_K(\boldsymbol{X}, t)\,\mathrm{d}X^K = \boldsymbol{g}_k(\boldsymbol{x})\,\mathrm{d}x^k. \end{aligned}$$

Similarly, the square of length elements are

(B.32) $$\begin{aligned} \mathrm{d}S^2 &= G_{KL}(\boldsymbol{X})\,\mathrm{d}X^K\,\mathrm{d}X^L = c_{kl}(\boldsymbol{x}, t)\,\mathrm{d}x^k\,\mathrm{d}x^l, \\ \mathrm{d}s^2 &= C_{KL}(\boldsymbol{X}, t)\,\mathrm{d}X^K\,\mathrm{d}X^L = g_{kl}(\boldsymbol{x})\,\mathrm{d}x^k\,\mathrm{d}x^l. \end{aligned}$$

B.1.4. Strain tensors, displacement vectors

Lagrange's and Euler's strain tensors are defined as

(B.33) $$\begin{aligned} E_{KL} &= \tfrac{1}{2}[C_{KL}(\boldsymbol{X}, t) - G_{KL}(\boldsymbol{X})], \\ e_{kl} &= \tfrac{1}{2}[g_{kl}(\boldsymbol{x}) - c_{kl}(\boldsymbol{x}, t)]. \end{aligned}$$

(B.32) and (B.33) then yield the following important relation

(B.34) $$ds^2 - dS^2 = 2E_{KL}(\boldsymbol{X}, t)\, dX^K dX^L = 2e_{kl}(\boldsymbol{x}, t)\, dx^k dx^l .$$

When the body undergoes only a *rigid displacement* there will be no change in the differential length in which case the difference $ds^2 - dS^2$ given by (B.34) vanishes. If this is true for all directions dX^K and dx^k, then E_{KL} and e_{kl} vanish. Therefore, these tensors represent a measure of deformation of the body.

Equation (B.34) immediately yields

(B.35) $$E_{KL} = e_{kl} x^k{}_{,K} x^l{}_{,L}, \quad e_{kl} = E_{KL} X^K{}_{,k} X^L{}_{,l} .$$

These relations indicate that E_{KL} and e_{kl} are 2nd-order tensors.

Strain tensors can also be expressed in terms of the *displacement vector* $\boldsymbol{u}$, defined as the vector extending from point P of the undeformed body B to its spatial point p of the deformed body b (see Fig. B2):

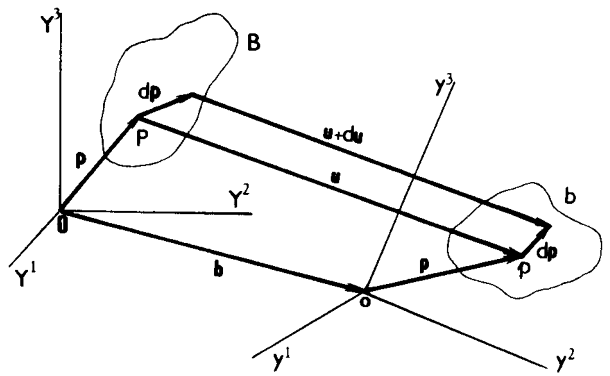

Fig. B2. Displacement vector.

(B.36) $$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{P} + \boldsymbol{b} .$$

The displacement vector can be represented by Lagrange's or Euler's components U^K and u^k,

(B.37) $$\boldsymbol{u} = U^K \boldsymbol{G}_K = u^k \boldsymbol{g}_k .$$

The scalar product of both sides of Eq. (B.36) with vectors $\boldsymbol{G}^K$ and $\boldsymbol{g}^k$ yields

(B.38) $$U^K = p^K - P^K + B^K, \quad u^k = p^k - P^k + b^k ,$$

where p^K, P^K, B^K and p^k, P^k, b^k are the components of vectors $\boldsymbol{p}$, $\boldsymbol{P}$ and $\boldsymbol{b}$ in X^K and x^k, respectively.

Let us express the strain tensors in terms of the displacement vector. By substituting $(B.28)_2$ and $(B.29)_2$ into $(B_{33})_1$ we can express Lagrange's strain tensor as

(B.39) $$E_{KL} = \tfrac{1}{2}(\boldsymbol{g}_k \cdot \boldsymbol{g}_l x^k{}_{,K} x^l{}_{,L} - G_{KL}).$$

Substituting from $(B.5)_2$ into the last equation yields

(B.40) $$E_{KL} = \frac{1}{2}\left(\frac{\partial \boldsymbol{p}}{\partial X^K} \cdot \frac{\partial \boldsymbol{p}}{\partial X^L} - G_{KL}\right).$$

If we also make use of (B.36) we obtain

(B.41) $$E_{KL} = \tfrac{1}{2}[(U_{M;K}\boldsymbol{G}^M + \boldsymbol{G}_K) \cdot (U_{M;L}\boldsymbol{G}^M + \boldsymbol{G}_L) - G_{KL}],$$

in which the semi-colon indicates covariant partial differentiation, $\partial \boldsymbol{u}/\partial X^K =$ $= U_{M;K}\boldsymbol{G}^M$. After some algebra, (B.41) yields

(B.42) $$E_{KL} = \tfrac{1}{2}(U_{K;L} + U_{L;K} + U_{M;K}U^M{}_{;L}).$$

Euler's strain tensor can be expressed in very much the same way:

(B.43) $$e_{kl} = \tfrac{1}{2}(u_{k;l} + u_{l;k} - u_{m;k}u^m{}_{;l}).$$

B.1.5. Changes of lengths and angles

Let us demonstrate the geometric significance of the components of the strain tensor. According to (B.31), the parallelepiped with sides $\boldsymbol{G}_1\,dX^1$, $\boldsymbol{G}_2\,dX^2$, $\boldsymbol{G}_3\,dX^3$, located at point $P(\boldsymbol{X})$ deforms into a parallelepiped with sides $\boldsymbol{C}_1\,dX^1$, $\boldsymbol{C}_2\,dX^2$, $\boldsymbol{C}_3\,dX^3$, located at point $p(\boldsymbol{x})$. It holds that

(B.44) $$d\boldsymbol{X} = \boldsymbol{G}_K dX^K, \quad d\boldsymbol{x} = \boldsymbol{C}_K dX^K, \quad \boldsymbol{C}_K = \boldsymbol{g}_k x^k_{,K}.$$

The unit vectors $\boldsymbol{N}$ and $\boldsymbol{n}$ along $d\boldsymbol{X}$ and $d\boldsymbol{x}$ are defined as

(B.45) $$N^K = \frac{dX^K}{|d\boldsymbol{X}|} = \frac{dX^K}{dS}, \quad n^k = \frac{dx^k}{|d\boldsymbol{x}|} = \frac{dx^k}{ds},$$

where dS and ds are the lengths of vectors $d\boldsymbol{X}$ and $d\boldsymbol{x}$. The *relative change of the length* of vector $\boldsymbol{N}$ is defined by

(B.46) $$E_{(\boldsymbol{N})} = e_{(\boldsymbol{n})} = \frac{ds - dS}{dS}.$$

Let us express Langrange's strain tensor in terms of the quantity $E_{(\boldsymbol{N})}$. Equations (B.34) and (B.45) yield

(B.47) $$2E_{KL}N^K N^L = \frac{ds^2 - dS^2}{dS^2} = E_{(\boldsymbol{N})}(E_{(\boldsymbol{N})} + 2).$$

If $\boldsymbol{N}$ is a vector tangential to coordinate line X^1, $N^1 = dX^1/dS = 1/(G_{11})^{1/2}$, $N^2 = N^3 = 0$, then

(B.48) $$2E_{11}/G_{11} = E_{(1)}(E_{(1)} + 2).$$

The last equation can also be expressed as

(B.49) $$E_{(1)} = -1 + (1 + 2E_{11}/G_{11})^{1/2}.$$

If the strains are small, $E_{(1)} \ll 1$, the following approximate relation applies:

(B.50) $$E_{11}/G_{11} \doteq E_{(1)}.$$

Analogous relations also hold for components E_{22} and E_{33}.

Now, assume $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ to be unit vectors along $\mathrm{d}\boldsymbol{X}_1$, $\mathrm{d}\boldsymbol{X}_2$ at point $\boldsymbol{X}$, and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ unit vectors along $\mathrm{d}\boldsymbol{x}_1$, $\mathrm{d}\boldsymbol{x}_2$ at point $\boldsymbol{x}$. The angle $\Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)}$ between $\mathrm{d}\boldsymbol{X}_1$ and $\mathrm{d}\boldsymbol{X}_2$ deforms into angle $\vartheta_{(\boldsymbol{n}_1, \boldsymbol{n}_2)}$ between $\mathrm{d}\boldsymbol{x}_1$ and $\mathrm{d}\boldsymbol{x}_2$ (see Fig. B3). We also have

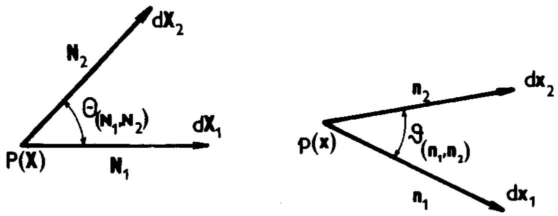

Fig. B3. Angle change.

(B.51) $$\boldsymbol{N}_a = \frac{\mathrm{d}\boldsymbol{X}_a}{|\mathrm{d}\boldsymbol{X}_a|}, \quad \boldsymbol{n}_a = \frac{\mathrm{d}\boldsymbol{x}_a}{|\mathrm{d}\boldsymbol{x}_a|}, \quad a = 1, 2.$$

Let us now calculate the angles $\Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)}$ and $\vartheta_{(\boldsymbol{n}_1, \boldsymbol{n}_2)}$ from

(B.52) $$\cos \Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = \boldsymbol{N}_1 . \boldsymbol{N}_2 = \frac{\mathrm{d}\boldsymbol{X}_1}{|\mathrm{d}\boldsymbol{X}_1|} \cdot \frac{\mathrm{d}\boldsymbol{X}_2}{|\mathrm{d}\boldsymbol{X}_2|} = \\ = \frac{G_{KL}\,\mathrm{d}X^K{}_1\,\mathrm{d}X^L{}_2}{|\mathrm{d}\boldsymbol{X}_1|\,|\mathrm{d}\boldsymbol{X}_2|} = G_{KL}N^K{}_1 N^{L2}.$$

Similarly,

(B.53) $$\cos \vartheta_{(\boldsymbol{n}_1, \boldsymbol{n}_2)} = \boldsymbol{n}_1 . \boldsymbol{n}_2 = \frac{\mathrm{d}\boldsymbol{x}_1}{|\mathrm{d}\boldsymbol{x}_1|} \cdot \frac{\mathrm{d}\boldsymbol{x}_2}{|\mathrm{d}\boldsymbol{x}_2|} = \\ = \frac{C_{KL}\,\mathrm{d}X^K{}_1\,\mathrm{d}X^L{}_2}{(C_{MN}\,\mathrm{d}X^M{}_1\,\mathrm{d}X^N{}_2)^{1/2}(C_{PQ}\,\mathrm{d}X^P{}_1\,\mathrm{d}X^Q{}_2)^{1/2}} = \frac{C_{KL}N^K{}_1 N^L{}_2}{(E_{(\boldsymbol{N}_1)} + 1)(E_{(\boldsymbol{N}_2)} + 1)}.$$

The difference $\Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} - \vartheta_{(\boldsymbol{n}_1, \boldsymbol{n}_2)}$ determines the *change of the angles* of directions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ due to the motion

(B.54) $$\Gamma_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = \gamma_{(\boldsymbol{n}_1, \boldsymbol{n}_2)} = \Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} - \vartheta_{(\boldsymbol{n}_1, \boldsymbol{n}_2)}.$$

Here we have again dual representation, Γ and γ, for the same physical quantity, i.e. the change of angle of two directions is denoted differently in Lagrange's and Euler's representations. (B.53) and (B.54) yield

(B.55) $$\sin \Gamma_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = H \sin \Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} - (1 - H^2)^{1/2} \cos \vartheta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)},$$

which, for the orthogonal directions before deformation, $\Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = \frac{1}{2}\pi$, reduces to

(B.56) $$\sin \Gamma_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = H = \frac{C_{KL} N^K{}_1 N^L{}_2}{(E_{(\boldsymbol{N}_1)} + 1)(E_{(\boldsymbol{N}_2)} + 1)}.$$

If we eliminate directions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ from Eqs (B.52) and (B.53), we obtain

(B.57) $$\begin{aligned} \cos \Theta_{(KL)} &= G_{KL}/(G_{\underline{KK}} G_{\underline{LL}})^{1/2}, \\ \cos \vartheta_{(KL)} &= C_{KL}/(C_{\underline{KK}} C_{\underline{LL}})^{1/2} = \\ &= (G_{KL} + 2E_{KL})/[(G_{\underline{KK}} + 2E_{\underline{KK}})(G_{\underline{LL}} + 2E_{\underline{LL}})]^{1/2}. \end{aligned}$$

If X^K are Cartesian coordinates, (B.57) will simplify to

(B.58) $$\begin{aligned} \cos \Theta_{(KL)} &= \delta_{KL}, \\ \cos \vartheta_{(KL)} &= \sin \Gamma_{(KL)} = \\ &= (\delta_{KL} + 2E_{KL})/[(1 + 2E_{\underline{KK}})(1 + 2E_{\underline{LL}})]^{1/2}. \end{aligned}$$

By using (B.49) in $(\text{B.58})_2$ we may also write

(B.59) $$2E_{KL} = (1 + E_{(K)})(1 + E_{(L)}) \sin \Gamma_{(KL)} \quad \text{for} \quad K \neq L.$$

In the case of small strains, $E_{(K)} \ll 1$, $E_{(L)} \ll 1$, the following approximate relation applies

(B.60) $$2E_{KL} \doteq \sin \Gamma_{(KL)} \doteq \Gamma_{(KL)}.$$

B.1.6. Changes of areas and volumes

The element of area bounded by vectors $\boldsymbol{G}_1 \, \mathrm{d}X^1$ and $\boldsymbol{G}_2 \, \mathrm{d}X^2$ after deformation change to the area bounded by vectors $\boldsymbol{C}_1 \, \mathrm{d}X^1$ and $\boldsymbol{C}_2 \, \mathrm{d}X^2$. The deformed area is thus given by

(B.61) $$\mathrm{d}\boldsymbol{a}_3 = \boldsymbol{C}_1 \, \mathrm{d}X^1 \times \boldsymbol{C}_2 \, \mathrm{d}X^2 = x^k{}_{,1} x^l{}_{,2} \boldsymbol{g}_k \times \boldsymbol{g}_l \, \mathrm{d}X^1 \, \mathrm{d}X^2.$$

However,

(B.62) $$\boldsymbol{g}_k \times \boldsymbol{g}_l = \varepsilon_{klm} \boldsymbol{g}^m = g^{1/2} e_{klm} \boldsymbol{g}^m,$$

where e_{klm} is Levi-Civita's alternating symbol. By substituting (B.62) into (B.61) we obtain

(B.63) $$\mathrm{d}\boldsymbol{a}_3 = g^{1/2} x^k{}_{,1} x^l{}_{,2} e_{klm} \boldsymbol{g}^m \mathrm{d}X^1 \, \mathrm{d}X^2.$$

The element of area prior to deformation is

(B.64) $$\mathrm{d}\boldsymbol{A}_3 = \boldsymbol{G}_1 \times \boldsymbol{G}_2 \,\mathrm{d}X^1\,\mathrm{d}X^2 = G^{1/2}\boldsymbol{G}^3\,\mathrm{d}X^1\,\mathrm{d}X^2 .$$

Consequently,

(B.65) $$\mathrm{d}A_3 = G^{1/2}\,\mathrm{d}X^1\,\mathrm{d}X^2 .$$

By substituting (B.65) into (B.63) we obtain

(B.66) $$\mathrm{d}\boldsymbol{a}_3 = (g/G)^{1/2} x^k{}_{,1} x^l{}_{,2} e_{klm} \boldsymbol{g}^m \,\mathrm{d}A_3 .$$

Equation (B.24) yields

(B.67) $$jX^3{}_{,m} = e_{klm} x^k{}_{,1} x^l{}_{,2} ,$$

so that

(B.68) $$\mathrm{d}\boldsymbol{a}_3 = J X^3{}_{,m} \boldsymbol{g}^m \,\mathrm{d}A_3 ,$$

where

(B.69) $$J = (g/G)^{1/2} j .$$

Similar relations also hold for $\mathrm{d}\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_2$. Therefore,

(B.70) $$\mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{a}_1 + \mathrm{d}\boldsymbol{a}_2 + \mathrm{d}\boldsymbol{a}_3 = JX^K{}_{,k} \boldsymbol{g}^k \,\mathrm{d}A_K ,$$

the kth component of which yields the important relation

(B.71) $$\mathrm{d}a_k = JX^K{}_{,k} \,\mathrm{d}A_K .$$

Let us also determine the change of volume under deformation. The deformed volume element is

(B.72) $$\begin{aligned} \mathrm{d}v = \mathrm{d}\boldsymbol{a}_3 . \boldsymbol{C}_3 \,\mathrm{d}X^3 &= JX^3{}_{,k} \boldsymbol{g}^k . \boldsymbol{g}_m x^m{}_{,3} \,\mathrm{d}A_3\,\mathrm{d}X^3 = \\ &= JX^3{}_{,k} x^m{}_{,3} \delta^k{}_m \,\mathrm{d}A_3\,\mathrm{d}X^3 = J\,\mathrm{d}A_3\,\mathrm{d}X^3 . \end{aligned}$$

The undeformed volume element is

(B.73) $$\mathrm{d}V = \mathrm{d}\boldsymbol{A}_3 . \boldsymbol{G}_3 \,\mathrm{d}X^3 = \boldsymbol{G}^3 . \boldsymbol{G}_3 \,\mathrm{d}A_3\,\mathrm{d}X^3 = \mathrm{d}A_3\,\mathrm{d}X^3 .$$

Finally, (B.72) and (B.73) yield the following important relation:

(B.74) $$\mathrm{d}v = J\,\mathrm{d}V .$$

B.2. Stress Tensor

B.2.1. Stress vector and tensor

We shall denote the surface force per unit surface in the deformed body with external normal $\boldsymbol{n}$ by $\boldsymbol{t}_{(n)}$ and refer to it as the *stress vector*. In particular, the stress vector which acts on the kth unit coordinate surface from the side of the external normal, will be denoted by $\boldsymbol{t}_k$; we shall refer to its lth component, t_{kl}, as the *stress tensor*:

(B.75) $$\boldsymbol{t}_k = t_{kl}\boldsymbol{g}^l .$$

To be able to find the relation between the components of the stress tensor t_{kl} and the components of the stress vector $\boldsymbol{t}_{(n)}$, acting on any surface in any point of the continuum, let us consider the condition of equilibrium of an infinitesimal tetrahedron, volume Δv whose three sides $\Delta a^{(k)}$ lie in the coordinate surfaces passing through point p, and the fourth side Δa is perpendicular to $\boldsymbol{n}$ (see Fig. B4). The equation of equilibrium of the acting forces can be estimated with the aid of the mean-value theorem,

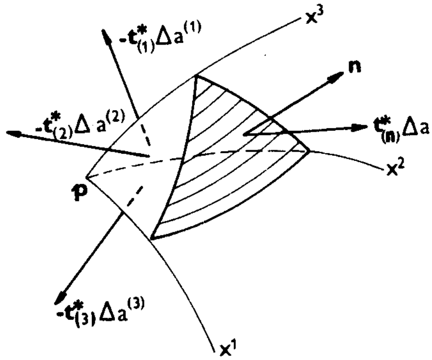

Fig. B4. Tetrahedron.

(B.76) $$\frac{\mathrm{d}}{\mathrm{d}t}(\varrho^*\boldsymbol{v}^*\Delta \boldsymbol{v}) = \boldsymbol{t}^*_{(n)}\Delta a - \boldsymbol{t}^*_{(k)}\Delta a^{(k)} + \varrho^*\boldsymbol{f}^*\Delta \boldsymbol{v},$$

where ϱ^*, $\boldsymbol{v}^*$ and $\boldsymbol{f}^*$ are the density, velocity and body force per unit mass at some interior point of the tetrahedron, $\boldsymbol{t}^*_{(n)}$ and $\boldsymbol{t}^*_{(k)}$ are the values of the stress vector $\boldsymbol{t}_{(n)}$ on surface Δa and on the coordinate surfaces $\Delta a^{(k)}$. The limiting transition for $\Delta v \to 0$ yields

(B.77) $$\boldsymbol{t}_{(n)}\,\mathrm{d}a = \boldsymbol{t}_{(k)}\,\mathrm{d}a^{(k)} .$$

However,

(B.78) $$\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\,\mathrm{d}a = \sum_k \mathrm{d}a^{(k)}\boldsymbol{g}_k/(g_{\underline{kk}})^{1/2} = \mathrm{d}a^k\boldsymbol{g}_k .$$

The last equation yields

(B.79) $$\mathrm{d}a^{(k)}/(g_{\underline{kk}})^{1/2} = \mathrm{d}a^k = n^k\,\mathrm{d}a .$$

(B.80) $$\boldsymbol{t}_{(n)} = \sum_k \boldsymbol{t}_{(k)} (g_{\underline{kk}})^{1/2} n^k = \boldsymbol{t}_{(k)} n^{(k)} = \boldsymbol{t}_k n^k = \boldsymbol{t}^k n_k ,$$

where $n^{(k)}$ is the physical component of the vector of the external normal $\boldsymbol{n}$ and

(B.81) $$\boldsymbol{t}_k = \boldsymbol{t}_{(k)} (g_{\underline{kk}})^{1/2} , \quad \boldsymbol{t}^k = g^{kl} \boldsymbol{t}_l , \quad n^{(k)} = n^k (g_{\underline{kk}})^{1/2} .$$

Substituting (B.75) into (B.80) leads to

(B.82) $$\boldsymbol{t}_{(n)} = t_{kl} n^k \boldsymbol{g}^l , \quad \text{or} \quad t_{(n)l} = t_{kl} n^k .$$

We can see that the stress vector, acting on any surface, is fully described by the components of the stress tensor at this point. Equation (B.80) also yields

(B.83) $$\boldsymbol{t}_{(-n)} = -\boldsymbol{t}_{(n)} .$$

B.2.2. Equations of motion in integral form

Independently of the geometry of strain and rheological relations, the following laws of conservation are postulated in continuum mechanics.
Axiom 1 (Conservative of Mass): The total mass of a body does not change with motion.

The existence of a continuous function of mass density ϱ is postulated in continuum mechanics. The total mass is given by the expression

(B.84) $$M = \int_V \varrho \, \mathrm{d}V , \qquad 0 \leq \varrho < \infty ,$$

where the integration is taken over the material volume of the body.

The law of mass conservation in turn postulates that the initial total mass of a body is equal to the total mass of the body at any other time, i.e.

(B.85) $$\int_V \varrho_0 \, \mathrm{d}V = \int_v \varrho \, \mathrm{d}v .$$

By using the transformation relation $\mathrm{d}v = J \mathrm{d}V$, we may write

(B.86) $$\int_V (\varrho_0 - \varrho J) \, \mathrm{d}V = 0 .$$

Alternatively, we may take the material derivative of (B.85). Thus

(B.87) $$\frac{\mathrm{d}}{\mathrm{d}t} \int_v \varrho \, \mathrm{d}v = 0 .$$

The law of mass conservation may thus be mathematically expressed as either Eq. (B.86) or Eq. (B.87).

Axiom 2 (Balance of Momentum): The time rate of change of the total momentum of a body is equal to the resultant of external forces $\boldsymbol{F}$ *acting on the body.*

Mathematically,

$$\text{(B.88)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{v} \mathrm{d}v = \boldsymbol{F},$$

where the l.h.s. represents the time rate of change of the total momentum of the body. The external forces acting on a body are the body forces such as gravity, on the one hand, and surface forces, generated by contact of the body with other bodies, on the other. Consequently,

$$\text{(B.89)} \qquad \boldsymbol{F} = \int_s \boldsymbol{t}_{(n)} \mathrm{d}a + \int_v \varrho \boldsymbol{f} \mathrm{d}v,$$

where $\boldsymbol{t}_{(n)}$ is the stress vector per unit area of the surface s with external normal $\boldsymbol{n}$. The body force $\boldsymbol{f}$ refers to unit mass. The balance of momentum thus takes the form

$$\text{(B.90)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{v} \mathrm{d}v = \int_s \boldsymbol{t}_{(n)} \mathrm{d}a + \int_v \varrho \boldsymbol{f} \mathrm{d}v.$$

Axiom 3 (Balance of Moment of Momentum): The time rate of change of the moment of momentum of a body is equal to the resultant moment of all external forces.

Mathematically,

$$\text{(B.91)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{p} \times \boldsymbol{v} \mathrm{d}v = \int_s \boldsymbol{p} \times \boldsymbol{t}_{(n)} \mathrm{d}a + \int_v \varrho \boldsymbol{p} \times \boldsymbol{f} \mathrm{d}v,$$

where the l.h.s. is the time rate of change of the total moment of momentum of the body about the origin. The surface integral on the r.h.s. of (B.91) is the resultant moment of the surface forces about the origin, and the volume integral is the resultant moment of the body forces about the origin.

Let us emphasize that these relations do not follow from similar equations for a system of mass points and a rigid body, but that they are independent physical laws.

B.2.3. Equations of motion in differential form

The two following integral theorems [57, 59] are important for deriving the equations of motion in differential form.

Consider a continuum, volume v, intersected by surface of discontinuity $\sigma(t)$

moving at velocity $\boldsymbol{v}$ (see Fig. B.5). The material derivative of the volume integral of tensor field $\boldsymbol{\Phi}$ then reads

$$\text{(B.92)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{v-\sigma} \boldsymbol{\Phi}\,\mathrm{d}v = \int_{v-\sigma}\left[\frac{\partial \boldsymbol{\Phi}}{\partial t} + \operatorname{div}(\boldsymbol{\Phi}\boldsymbol{v})\right]\mathrm{d}v + \int_{\sigma}[\boldsymbol{\Phi}(\boldsymbol{v}-\nu)]_{-}^{+}\,.\,\mathrm{d}\boldsymbol{a}\,.$$

The Green-Gauss theorem generalized for a 2nd-order tensor field, $\boldsymbol{\tau} = \tau^{kl}\boldsymbol{g}_k\boldsymbol{g}_l$ is

$$\text{(B.93)} \qquad \int_{v-\sigma} \operatorname{div}\boldsymbol{\tau}\,\mathrm{d}v + \int_{\sigma}[\boldsymbol{\tau}]_{-}^{+}\,.\,\boldsymbol{n}\,\mathrm{d}a = \int_{s-\sigma}\boldsymbol{\tau}\,.\,\boldsymbol{n}\,\mathrm{d}a\,.$$

By volume integral over $v - \sigma$ we understand the volume integral over volume v excluding the material points lying on the surface of discontinuity σ. The same applies to the surface integral over $s - \sigma$. Therefore (see Fig. B5).

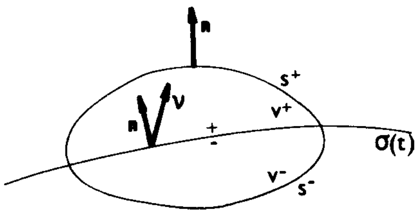

Fig. B5. Region with discontinuity surface.

$$v - \sigma = v^+ + v^-\,, \quad s - \sigma = s^+ + s^-\,.$$

The symbol $[\ ]_{-}^{+}$ indicates a jump of the function in brackets at boundary σ,

$$[f]_{-}^{+} = f^+ - f^-\,.$$

Let us apply these two theorems to balance laws postulated in the preceding section. If we put $\boldsymbol{\Phi} = \varrho$ in (B.92), we shall obtain the law of mass conservation in the following form:

$$\text{(B.94)} \qquad \int_{v-\sigma}\left[\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v})\right]\mathrm{d}\boldsymbol{v} + \int_{\sigma}[\varrho(\boldsymbol{v}-\nu)]_{-}^{+}\,.\,\mathrm{d}\boldsymbol{a} = 0\,.$$

For the last equation to hold in any part of the body and on any surface of discontinuity, the integrands in both integrals must be equal to zero:

$$\text{(B.95)} \qquad \begin{aligned} &\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0 \quad \text{in } v - \sigma\,,\\ &[\varrho(\boldsymbol{v}-\nu)]_{-}^{+}\,.\,\boldsymbol{n} = 0 \quad \text{on } \sigma\,. \end{aligned}$$

These equations express "locally" the law of mass conservation in continuum together with the boundary condition. Equation (B.95)$_1$ is called the *equation of continuity*. It is none other than the material derivative of

(B.96) $$\varrho_0 = \varrho J\,.$$

In virtue of Eq. (B.80), the equation of global balance of momentum now reads

(B.97) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v-\sigma} \varrho \boldsymbol{v}\,\mathrm{d}v = \int_{s-\sigma} \boldsymbol{t}^k n_k\,\mathrm{d}a + \int_{v-\sigma} \varrho \boldsymbol{f}\,\mathrm{d}v\,.$$

However,

(B.98) $$\boldsymbol{t}^k n_k = t^{kl} n_k \boldsymbol{g}_l = (\boldsymbol{n}\,.\,\mathbf{t})^l \boldsymbol{g}_l = \boldsymbol{n}\,.\,\mathbf{t} = \mathbf{t}\,.\,\boldsymbol{n}\,,$$

since, as we shall show in the next, the stress tensor is symmetric. Using Eqs (B.92) and (B.93) $\boldsymbol{\Phi} = \varrho \boldsymbol{v}$ and $\boldsymbol{\tau} = \mathbf{t}$, we obtain

(B.99) $$\int_{v-\sigma}\left[\frac{\partial(\varrho \boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}\boldsymbol{v}) - \operatorname{div}\mathbf{t} - \varrho \boldsymbol{f}\right]\mathrm{d}v + \\ + \int_{\sigma}[\varrho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \mathbf{t}]^{+}_{-}\,.\,\boldsymbol{n}\,\mathrm{d}a = 0$$

This is postulated to be valid for all parts of the body. Thus the integrounds vanish separately.

(B.100) $$\begin{aligned} &\operatorname{div}\mathbf{t} + \varrho(\boldsymbol{f} - \boldsymbol{a}) = 0 \qquad &&\text{in } v - \sigma\,, \\ &[\varrho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \mathbf{t}]^{+}_{-}\,.\,\boldsymbol{n} = 0 &&\text{on } \sigma\,, \end{aligned}$$

where

(B.101) $$\boldsymbol{a} = \partial \boldsymbol{v}/\partial t + \boldsymbol{v}\,.\operatorname{grad}\boldsymbol{v}\,.$$

These equations express "locally" the balance of momentum together with the boundary condition. Equation (B.100)$_1$ is frequently referred to as *Cauchy's first law of motion*, and the stress tensor $\mathbf{t}$, which occurs in it and which is referred to the deformed body, as *Cauchy's stress tensor*.
Equation (B.100)$_1$ in component form reads

(B.102) $$t^{lk}{}_{;l} + \varrho(f^k - a^k) = 0\,.$$

By lowering the indices we obtain the associated equation

(B.103) $$\begin{aligned} t^{l}{}_{k;l} + \varrho(f_k - a_k) = 0\,, \\ t_{lk;}{}^{l} + \varrho(f_k - a_k) = 0\,. \end{aligned}$$

By substituting Eq. (B.80) into the equation of balance of the moment of momentum (B.91) and using Eqs. (B.92), (B.93) and (B.100), we arrive at

(B.104) $$\boldsymbol{g}_k \times \boldsymbol{t}^k = 0 \quad \text{in } v - \sigma .$$

The associated jump conditions have already been expressed by Eqs (B.95)$_2$ and (B.100)$_2$. The substitution of (B.75) and (B.62) into (B.104) yields

(B.105) $$t^{kl} = t^{lk}, \ t^k{}_l = t_l{}^k ,$$

which is the expression for *Cauchy's second law of motion.*

To conclude, let us express Cauchy's first law of motion in terms of the physical components of vectors and tensors. Equation (B.103)$_1$ will read

(B.106) $$t^l{}_{k,l} + \begin{Bmatrix} l \\ m \ l \end{Bmatrix} t^m{}_k - \begin{Bmatrix} m \\ k \ l \end{Bmatrix} t^l{}_m + \varrho g_{kl}(f^l - a^l) = 0 .$$

Vectors f^l and a^l are expressed in terms of the physical components $f^{(l)}$ and $a^{(l)}$ in Eq. (A.70)$_1$, the stress tensor $t^k{}_l = t_l{}^k$ in terms of the physical components $t^{(k)}{}_{(l)}$ in Eq. (A.76)$_1$. If we now use Eq. (A.94), Eq. (B.106) can be modified to read

(B.107) $$\sum_{k=1}^{3} \left\{ \frac{\partial}{\partial x^k} \left[t_{(l)}{}^{(k)} \frac{(g_{ll})^{1/2}}{(g_{kk})^{1/2}} \right] + t_{(l)}{}^{(k)} \frac{(g_{ll})^{1/2}}{(g_{kk})^{1/2}} \frac{\partial}{\partial x^k} [\log (g)^{1/2}] - \right.$$
$$\left. - \sum_{m=1}^{3} \left[\begin{Bmatrix} k \\ l \ m \end{Bmatrix} t_{(k)}{}^{(m)} \frac{(g_{kk})^{1/2}}{(g_{mm})^{1/2}} \right] + \varrho \frac{g_{kl}}{(g_{kk})^{1/2}} [f^{(k)} - a^{(k)}] \right\} = 0 .$$

This equation is valid in any curvilinear coordinate system provided the stress tensor is symmetric. If the curvilinear coordinates are orthogonal, Eq. (B.107) converts to Eq. (A.160).

B.2.4. Equations of motion in the reference coordinate system

Cauchy's equations of motion have been expressed in terms of Euler's coordinates. However, in many cases it is convenient to formulate the problem in the reference (Lagrange's) coordinate system.

Let us now, therefore, express the equations of motion in the reference system X^K. Equation (B.96) followed from the law of mass conservation:

(B.108) $$\varrho_0 = \varrho J , \ J = (g/G)^{1/2} j , \ j = \det (x^k_{,K}) .$$

Let us introduce the stress vector $\boldsymbol{T}^K$ at spatial point $\boldsymbol{x}$ and time t relative to the underformed surface $\mathrm{d}A_K$, located at point $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$:

(B.109) $$\boldsymbol{t}_{(n)} \, \mathrm{d}a = \boldsymbol{t}^k \, \mathrm{d}a_k = \boldsymbol{T}^K \mathrm{d}A_K .$$

By using Eq. (B.71) we obtain

(B.110) $$\boldsymbol{t}^k = J^{-1} x^k_{,K} \boldsymbol{T}^K , \ \boldsymbol{T}^K = J X^K_{,k} \boldsymbol{t}^k .$$

Let us introduce the *Piola-Kirchhoff pseudostress tensor* T^{Kl} and T^{KL} by

(B.111)
$$\boldsymbol{T}^K = T^{Kl}\boldsymbol{g}_l = T^{KL}x^l_{,L}\boldsymbol{g}_l .$$

Equations (B.110) and (B.75) then yield

(B.112)
$$T^{Kl} = JX^K{}_{,k}t^{kl} ,$$
$$T^{KL} = T^{Kl}X^L{}_{,l} = JX^K{}_{,k}X^L{}_{,l}t^{kl} .$$

Equations (B.109) and (B.111)$_1$ indicate that T^{Kl} expresses the stress at $\boldsymbol{x}$ measured per unit undeformed area at $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$. From (B.112) it also follows that

(B.113)
$$t^{kl} = J^{-1}x^k{}_{,K}T^{Kl} = J^{-1}x^k{}_{,K}x^l{}_{,L}T^{KL} .$$

The equations of motion (B.102) can be expressed in terms of the components T^{Kl} as

(B.114)
$$T^{Kk}{}_{,K} + T^{Km}\begin{Bmatrix} k \\ m\ l \end{Bmatrix}x^l{}_{,K} + T^{Kk}\begin{Bmatrix} L \\ L\ K \end{Bmatrix} + \varrho_0(f^k - a^k) = 0 .$$

If we introduce total covariant derivatives of the two-point tensor field $T^{Kk}(\boldsymbol{X}, \boldsymbol{x})$ — refer to Supplement A — Eq. (B.114) can be expressed in a more consise form

(B.115)
$$T^{Kk}{}_{:K} + \varrho_0(f^k - a^k) = 0 .$$

Cauchy's second law of motion now has a more complicated form,

(B.116)
$$T^{Kk}x^l{}_{,K} = T^{Kl}x^k{}_{,K} .$$

The equations of motion, expressed in terms of the components T^{KL}, now read

(B.117)
$$(T^{KL}x^k{}_{,L})_{,K} + \left(\begin{Bmatrix} k \\ m\ l \end{Bmatrix}x^m{}_{,L}x^l{}_{,K} + \right.$$
$$\left. + \begin{Bmatrix} M \\ M\ K \end{Bmatrix}x^k{}_{,L}\right)T^{KL} + \varrho_0(f^k - a^k) = 0 ,$$
$$T^{KL} = T^{LK} .$$

It is easy to prove that, if the deformations are small, there is no difference between the equations of motion expressed in Euler's and Lagrange's coordinates.

To be able to express the jump conditions in the reference system, we shall first derive the relation for the external normals $\boldsymbol{n}$ and $\boldsymbol{N}$ of the deformed and undeformed surfaces s and S. With a view to (B.71) we have

(B.118)
$$\mathrm{d}a_k = JX^K{}_{,k}\mathrm{d}A_K .$$

However,

(B.119)
$$n_k = \mathrm{d}a_k/\mathrm{d}a = \mathrm{d}a_k/(\mathrm{d}a^l\,\mathrm{d}a_l)^{1/2} ,$$
$$N_K = \mathrm{d}A_K/\mathrm{d}A = \mathrm{d}A_K/(\mathrm{d}A^L\,\mathrm{d}A_L)^{1/2} ,$$

and, therefore,

(B.120) $$n_k = J X^K{}_{,k} N_K \mathrm{d}A/\mathrm{d}a\,.$$

By using (B.118) we obtain

(B.121) $$\mathrm{d}A/\mathrm{d}a = J^{-1}(C^{KL}N_K N_L)^{-1/2},$$

where

(B.122) $$C^{KL} = g^{kl} X^K{}_{,k} X^L{}_{,l}$$

is *Piola's deformation tensor*. Finally, we obtain

(B.123) $$n_k = (C^{KL}N_K N_L)^{-1/2} X^M{}_{,k} N_M\,.$$

By substituting Eqs (B.109) and (B.120) into $(B.95)_2$ and $(B.100)_2$, we arrive at the jump conditions in the reference system:

(B.124) $$\left[\varrho_0(v^k - \nu^k)\, X^K{}_{,k} N_K \frac{\mathrm{d}A}{\mathrm{d}a}\right]_-^+ = 0 \quad \text{on } \Sigma,$$

(B.125) $$\left[[\varrho_0 \boldsymbol{v}(v^k - \nu^k)\, X^K{}_{,k} - \boldsymbol{T}^K] N_K \frac{\mathrm{d}A}{\mathrm{d}a}\right]_-^+ = 0 \quad \text{on } \Sigma.$$

At a solid surface of discontinuity (solid elastic substance — solid elastic substance boundary) it also holds that

(B.126) $$[\mathrm{d}a]_-^+ = [\mathrm{d}A]_-^+ = 0$$

and conditions (B.124) and (B.125) can be expressed as

(B.127) $$[\varrho_0(v^k - \nu^k)\, X^K{}_{,k}]_-^+ N_K = 0 \quad \text{on } \Sigma,$$

(B.128) $$[\varrho_0 \boldsymbol{v}(v^k - \nu^k)\, X^K{}_{,k} - \boldsymbol{T}^K]_-^+ N_K = 0 \quad \text{on } \Sigma.$$

However, at a liquid surface of discontinuity (solid elastic substance — liquid boundary) only the following holds (see Fig. B6):

(B.129) $$[\mathrm{d}a]_-^+ = 0$$

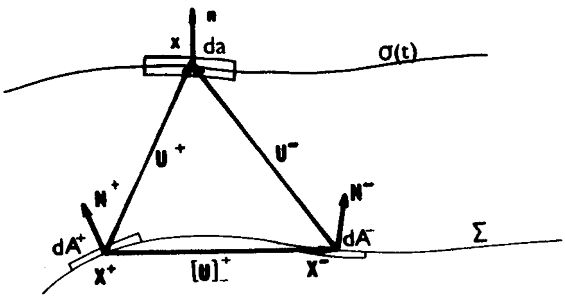

Fig. B6. Liquid boundary before and after deformation.

and conditions (B.124) and (B.125) can be expressed as

(B.130) $$[\varrho_0(v^k - \nu^k) X^K{}_{,k} \mathrm{d}A]^+_- = 0 \quad \text{on } \Sigma,$$

(B.131) $$[[\varrho_0 \nu(v^k - \nu^k) X^K{}_{,k} - t^K] N_K \mathrm{d}A]^+_- = 0 \quad \text{on } \Sigma.$$

SUPPLEMENT C. LIMITING VALUE OF FUNCTION $z_n(x)$

Equation (8.10) defines function $z_n(x)$,

(C.1) $$z_n(x) = x j_{n+1}(x) / j_n(x),$$

where $j_n(x)$ is a spherical Bessel function of the 1st kind,

(C.2) $$j_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} J_{n+\frac{1}{2}}(x)$$

and $J_n(x)$ is Bassel's function of the 1st kind. Let us seek to determine the limiting value of function $z_n(x)$ for $n \to \infty$ for a fixed value of x. According to [1],

(C.3) $$\lim_{n\to\infty} J_n(x) = \frac{1}{\sqrt{(2\pi x)}} \left(\frac{\mathrm{e}x}{2n}\right)^n \quad \text{for fixed } x,$$

where e $\doteq$ 2.718 281 828. This yields the limiting value of function $z_n(x)$ for a fixed x,

(C.4) $$\lim_{n\to\infty} z_n(x) = \frac{\mathrm{e}x^2}{2n+3} \left(\frac{n+\frac{1}{2}}{n+\frac{3}{2}}\right)^{n+1}.$$

However, according to [125], for any finite number a

(C.5) $$\lim_{n\to\infty} (1 + a/\mathrm{n})^{\mathrm{n}} = e^a.$$

Equation (C.4) can then be modified to read

(C.6) $$\lim_{n\to\infty} z_n(x) = \frac{\mathrm{e}x^2}{2n+3} \lim_{n\to\infty} \frac{1+\frac{1}{2n}}{1+\frac{3}{2n}} \lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^n}{\left(1+\frac{3}{2n}\right)^n}.$$

The first limiting value on the r.h.s. of (C.6) is equal to 1, the second limit is 1/e. Finally,

(C.7) $$\lim_{n\to\infty} z_n(x) = \frac{x^2}{2n+3}.$$

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