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FREE OSCILLATIONS OF THE EARTH

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CONTENTS

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Summary: *Using general curvilinear coordinates, the fundamental relations of continuum mechanics have briefly been derived. Description in terms of Lagrange's and Euler's coordinates was distinguished on principle. This mathematical and physical method was used to derive the equations of motion and boundary conditions of elastic oscillations of a body prestressed by finite static stresses. It is assumed that the free oscillations cause small deviations from equilibrium position, such that the tensor of finite deformations can be approximated by the tensor of small deformations. The expression of boundary conditions at a fluid boundary, using Lagrange's description, is relatively complicated. From the point of view of this theory, the case of free elastic gravitational oscillations, treated for a general model of the Earth, is particular.*

1. INTRODUCTION

Considerable attention is being devoted to the problems of free oscillations of the Earth in geophysics from the point of view of observation techniques, recording and processing, as well as from the point of view of theoretical study of the free oscillations of the Earth. One of the main reasons for this is that the free oscillations of the Earth have expanded the interval of periods of seismic elastic waves roughly from periods of 5 mins to 1 hour. The interval of short periods already partly covers the interval of long-period seismic surface waves, whereas the longest eigenperiods are close to the values of periods of Earth tides.

By observing the free oscillations of the Earth, [5, 6, 16—20, 26, 27, 29, 47, 48, 53—55, 62, 63, 87, 92, 105, 108, 110, 120, 128, 129, 135, 138], new data have been obtained on the internal structure of the Earth, in particular on the radial density distribution within the Earth. By solving the inverse problem of free oscillations, the Earth model with a solid inner core has again been justified [49, 52, 54]. The positions of the inner core — outer core and core — mantle boundaries were determined more accurately [50]. The radial variation of the density and velocity of seismic waves was determined in the whole body, inclusive the low-velocity channel in the upper mantle [10, 32, 49, 50, 70, 75, 88, 93, 94, 106, 107, 122, 143].

The Earth's free oscillations are oscillations of the elastic Earth bounded, with respect to the propagation of elastic waves, by the outer boundary formed by the Earth's surface. The oscillations are generated by severe earthquakes. There are two fundamental types of free oscillations for the spherically symmetric, non-rotating, isotropic, linearly elastic model of the Earth (hereinafter referred to as the SNREI Earth model):

1. Toroidal oscillations which are characterized by the displacement vector having only horizontal components non-zero, and also by the volume dilatation being zero. From this it immediately follows that these oscillations do not perturb the gravitational field of the Earth and that they cannot be recorded by gravity meters. They are recorded only by means of long-period seismographs and strain-seismographs. These types of oscillations are only related to a solid elastic medium and, therefore, toroidal oscillations "propagate" only through the Earth's mantle and crust.

2. Spheroidal oscillations for which only the radial component of the rotation of the displacement vector is zero. These oscillations perturb the gravitational field of the Earth; therefore, they can be recorded not only by means of long-period seismographs and strain-seismographs, but also by means of tide gravity meters. These types of oscillations occur not only in a solid, but also a fluid elastic medium and, therefore, spheroidal oscillations "propagate" through the whole of the Earth's body.

The components of the displacement vector of both types of free oscillations and the additional gravitational potential can be described by the spherical function $Y_{nm}(\vartheta, \varphi)$, or by its gradient and a function of the coordinate r.

An important property of the free oscillations of the Earth is that, for a given mode, the maximum amplitudes of the free oscillations move from the central regions of the Earth to the surface with increasing n. This means that the various intervals of periods of free oscillations are determined by the properties of the various regions within the Earth. The free oscillations of the Earth, therefore, enable us to study the physical properties of the Earth not only integrally, like the Earth tides, but also differentially.

There is a direct relation between the free oscillations of higher orders and long-period surface waves. For example, it can be proved [9, 27, 127] that, for short-period free oscillations of the fundamental mode $25 < n < 200$ $(50 < T < 300)$, gravitational forces can be neglected and the Earth may be approximated by a layered halfspace. Spheroidal oscillations then reduce to Rayleigh waves and toroidal oscillations to Love waves.

The free oscillations of the Earth can be classified with regard to their amplitudes, on the one hand, and to their periods, on the other. The former aspect is prevalently determined by the properties of the source, the latter depends on the internal structure of the Earth.

The objective of this study is

a) to present a detailed theoretical interpretation of the free oscillations of the Earth, particularly as regards their periods;

b) to process the studied problem numerically on a computer;

c) to test the relevant programs by computing the free periods and eigenfunctions for the given theoretical SNREI Earth model.

This study has nine chapters, three supplements, a list of references and two appendices. Chapter 1 is devoted to the introduction into the problems of free oscillations of the Earth. Chapter 2 presents the equations of motion derived in general curvilinear coordinates and the boundary conditions of elastic oscillations of a body pre-stressed by finite static stresses. The relations which have been derived, are expressed in component as well as vectorial form.

Results specified for the free oscillations of a general model of the Earth are derived in Chapter 3. A rotating, inhomogeneous model of the Earth is considered, composed of solid and liquid regions.

Chapter 4 is devoted to the equations of motion and boundary conditions of the free oscillations of the SNREI Earth model. It is proved that the wave field of the free oscillations for this Earth model can be resolved precisely into two types of oscillations — toroidal and spheroidal. The systems of differential equations of the individual types of free oscillations are then derived together with the boundary conditions. It was found advantageous to introduce new variable functions. One special type of spheroidal oscillations with $n = 0$, so-called radial oscillations, is also discussed.

In Chapter 5, we deal with the free oscillations of a homogeneous Earth model. In this casse, the eigenfunctions of the oscillations can be expressed analytically in terms of Bessel's spherical functions, or their combinations. Both types of oscillations are again studied. These problems are particularly important for the matrix solution of free oscillations and for defining the initial values of the numerical integration of the systems of differential equations for free oscillations. The expansions of the eigenfunctions of spheroidal oscillations in the neighbourhood of the origin (Earth's centre) for the SNREI Earth model are derived in Chapter 6. The solutions derived for a solid and liquid medium are useful particularly in defining the initial values of numerical integration of equations of motions in the neighbourhood of the Earth's cfentre for radial or spheroidal oscillations of low orders.

The variation method used to determine the roots of the secular function for the SNREI Earth model is derived in Chapter 7. The result is a relation for computing an improved value of the eigenfrequency with the aid of the tested function and of the eigenfunctions computed for the tested frequency. Chapter 8 describes the method of numerical solution of the system of ordinary differential equations for the free oscillations of the SNREI Earth model. The numerical integration was carried out using the Runge—Kutta method of the 4th order. Some of the eigenperiods of model 1066A are tabulated in the second part of this chapter. The conclusion summarizes the most important results achieved in this study.

Supplement A is devoted to tensors. The tensors are introduced with the aid of the invariant properties of coordinate transformation. Some of the relations of tensor algebra have also been derived. Invariant differential operators have been introduced by means of covariant partial derivatives. The general relations are specified in orthogonal curvilinear coordinates. These coordinates have also been used to express the most important tensors occurring in the theory of elasticity. Supplement B gives a brief recapitulation of the fundamental relations of continuum mechanics. Strain geometry is described with the aid of differential geometry. The equations of motion of continuum mechanics are postulated in integral and differential form. On principle, the description in Lagrange coordinates is distinguished from that in Euler coordinates. The limiting quotient of two Bessel spherical functions is derived in Supplement C.

To conclude, I should like to thank Prof. K. Pěč for his valuable help and constant interest in this work.

2. SMALL ELASTIC MOTIONS IN A MEDIUM WITH FINITE STATIC STRESSES

In this Section we derive the fundamental equations, i.e. the equations of motion and boundary configurations of small elastic vibrations in a medium with finite static stress. The general problem of infinitesimal displacements superimposed on a large elastic deformation has been tackled by many investigators. Here we mention only a few: Biot [25], Dahlen [42], Eringen and Suhubi [59], Truesdell and Noll [139].

2.1. Equations of motion

Consider a general inhomogeneous elastic body. Assume that, in the initial unstrained configuration, no forces act on the body, and that no elastic displacements are generated within it or on its surface. We shall say that the body is in its natural configuration $\boldsymbol{B}_0$. Its volume and surface will be denoted by V_0 and S_0, respectively.

Let us also assume that, due to finite static stress which will begin to act a particular instant, the body will change from its natural configuration $\boldsymbol{B}_0$ to strained configuration $\boldsymbol{B}$. Assume that in this configuration equilibrium is again established between the acting static forces and the internal stresses. In the strained configuration $\boldsymbol{B}$ we shall denote the volume of the elastic body by V and its surface by S. Let us consider configuration $\boldsymbol{B}$ as reference, i.e. the equations of motion and boundary conditions are referred to this configuration.

We shall describe the position of a mass particle in natural configuration $\boldsymbol{B}_0$ in terms of the curvilinear coordinates X^1, X^2, X^3, and in the reference configuration $\boldsymbol{B}$ in terms of the curvilinear coordinates x^1, x^2, x^3. The static deformation from configuration $\boldsymbol{B}_0$ to configuration $\boldsymbol{B}$ is described by the equations

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}), \quad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}). \tag{2.1}$$

Assume that the Jacobian

$$j_0 = \det(\partial x^k / \partial X^K) \tag{2.2}$$

is continuous and different from zero within a certain neighbourhood of point X^K.

The assumption that the strained configuration $\boldsymbol{B}$ is equilibrium, yields the following equilibrium conditions:

$$\begin{aligned} \operatorname{div} \mathbf{t}_0 + \varrho_0 \boldsymbol{f}_0 &= 0 \qquad \text{in } V, \\ t^k_{0\,l;k} + \varrho_0 f_{0l} &= 0 \qquad \text{in } V, \end{aligned} \tag{2.3}$$

where $\mathbf{t}_0$ is Cauchy's tensor of initial static stress, ϱ_0 is the density of material in configuration $\boldsymbol{B}$, $\boldsymbol{f}_0$ is the initial body force per unit mass in configuration $\boldsymbol{B}$. Note that the indices following the semi-colon or comma are used to denote covariant partial derivatives or "ordinary" partial derivatives with respect to x^k, if the indices are lower-case letters, and with respect to X^K, if the indices are capital letters.

We now consider a time- and space-dependent field of additional elastic stress which is superimposed on the initial static stress. Assume that the additional elastic stress is small compared to the initial static stress. We restrict ourselves to the linear theory of elasticity, i.e. we assume that the field of displacement and strain of configuration $\boldsymbol{B}$ due to the elastic stress are small. In order to carry out the linearization of the problem properly, we shall express the displacement vector as $\varepsilon\boldsymbol{u}(\boldsymbol{x}, t)$, where ε is a small parameter.

The displacement field deforms the body from reference configuration B into another time-dependent configuration $B'(t)$ with volume $V'(t)$ and surface $S'(t)$. Assuming that the deformations are small, volumes V and $V'(t)$ will be similar. The coordinates of the mass particle in configuration $B'(t)$ with respect to the reference configuration B are (see Fig. 1)

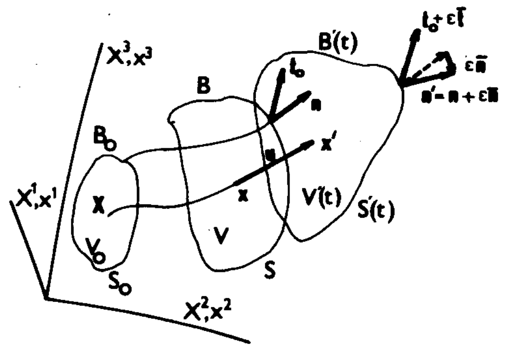

Fig. 1. Small motion superimposed on large static pre-stress.

(2.4) $$\boldsymbol{x}' = \boldsymbol{x} + \varepsilon\boldsymbol{u}(\boldsymbol{x}, t).$$

We shall not introduce a new curvilinear coordinate system for the strained configuration $B'(t)$, but we shall refer all the quantities to the curvilinear system describing configuration B, i.e.

(2.5) $$x'^{k'} = x'^{k}, x^{k'} = x^{k}.$$

This also means that the basis vectors, the determinants of metric tensors and Jacobians in configurations B and $B'(t)$ are identical,

(2.6) $$\boldsymbol{g}'_{k'} = \boldsymbol{g}_k,\ g' = g\,,\ j' = j\,.$$

Equation (2.4) in terms of components is expressed as

(2.7) $$x'^{k'} = x'^{k} = x^{k} + \varepsilon u^{k}(\boldsymbol{x}, t) = x^{k'} + \varepsilon u^{k'}(\boldsymbol{x}, t) = x^{k'} + \varepsilon u'^{k'}(\boldsymbol{x}', t).$$

The stress vector $\boldsymbol{T}^k$ in $B'(t)$, referred to unit area in B, is characterized by the first Piola—Kirchhoff pseudo-tensor $T^{k}{}_{r}$. With a view to $(\mathrm{B}111)_1$ and (2.5),

(2.8) $$\boldsymbol{T}^k = T^{k}{}_{r}\boldsymbol{g}'^{r} = T^{k}{}_{l}\boldsymbol{g}^{l},$$

which means that

(2.9) $$T^{k}{}_{r} = T^{k}{}_{l}.$$

According to our assumption, the additional elastic stress is small compared with the initial static stress. Consequently, with an accuracy of the quantities of the order of ε we may put

$$\text{(2.10)} \qquad \mathbf{T}(\boldsymbol{x}, t) = \mathbf{t}_0(\boldsymbol{x}) + \varepsilon\bar{\mathbf{T}}(\boldsymbol{x}, t), \quad T^k{}_l(\boldsymbol{x}, t) = t^k_{0l}(\boldsymbol{x}) + \varepsilon\bar{T}^k{}_l(\boldsymbol{x}, t),$$

where $\varepsilon\bar{\mathbf{T}}(\boldsymbol{x},t)$ is the increment of the Piola—Kirchhoff tensor of stress $\mathbf{T}(\boldsymbol{x}, t)$ with respect to configuration B due to the infinitesimal displacement $\varepsilon\boldsymbol{u}$.

The stress vector in $B'(t)$, referred to unit area in $B'(t)$ is characterized by Cauchy's stress tensor $\mathbf{t}$. The expression for it in terms of tensor $\mathbf{T}$ follows from (B.113)$_1$:

$$\text{(2.11)} \qquad t^{k'}{}_{l'} = (j')^{-1} (\partial x'^{k'}/\partial x^m)\, T^m{}_{l'}.$$

Taking into consideration Eqs (2.5) and (2.9),

$$\text{(2.12)} \qquad t^k{}_l = j^{-1} (\partial x'^k/\partial x^m)\, T^m{}_l,$$

where j is the small strain Jacobian, i.e.

$$\text{(2.13)} \qquad j = j' = \det(\partial x'^{k'}/\partial x^l) = \det(\partial x'^k/\partial x^l).$$

If we use (2.7) in (2.13) and neglect quadratic and higher terms in ε, we arrive at

$$\text{(2.14)} \qquad j = 1 + \varepsilon u^m{}_{;m}, \quad j^{-1} = 1 - \varepsilon u^m{}_{;m},$$

because

$$\text{(2.15)} \qquad \partial x'^k/\partial x^l = \delta^k{}_l + \varepsilon u^k{}_{;l}.$$

If, according to (A113), we defined the gradient of the displacement vector $\boldsymbol{u}$ in terms of tensor $\mathbf{H}$,

$$\text{(2.16)} \qquad H^k{}_l = u_{l;}{}^k, \quad \mathbf{H} = \operatorname{grad}\boldsymbol{u} = u_{l;}{}^k\, \boldsymbol{g}_k\boldsymbol{g}^l = (\operatorname{grad} u)^k{}_l\, \boldsymbol{g}_k\boldsymbol{g}^l = H^k{}_l\, \boldsymbol{g}_k\boldsymbol{g}^l,$$

where $\boldsymbol{g}_k\boldsymbol{g}^l$ is the dyadic product of the basis vectors in configuration B, Eq. (2.15) can be expressed as

$$\text{(2.17)} \qquad \partial x'^k/\partial x^l = \delta^k{}_l + \varepsilon H_l{}^k.$$

If we use Eqs (2.14)$_2$ and (2.17), Eq. (2.12) can be approximated by the relations

$$\text{(2.18)} \qquad t^k{}_l = t^k_{0l} + \varepsilon\bar{t}^k{}_l, \quad \mathbf{t} = \mathbf{t}_0 + \varepsilon\bar{\mathbf{t}},$$

in which the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ is given by

$$\text{(2.19)} \qquad \bar{t}^k{}_l = \bar{T}^k{}_l + H_m{}^k t^m_{0l} - t^k_{0l} H_m{}^m, \quad \bar{\mathbf{t}} = \bar{\mathbf{T}} + \mathbf{H}^{\mathrm{T}}.\mathbf{t}_0 - (\operatorname{tr}\mathbf{H})\mathbf{t}_0.$$

If we express the latter relations in terms of the displacement vector $\boldsymbol{u}$, we obtain

(2.20)
$$\bar{t}^k{}_l = \bar{T}^k{}_l + u^k{}_{;m}\, t_{0l}^m - t_{0l}^k u^m{}_{;m}\,,$$
$$\bar{\mathbf{t}} = \bar{\mathbf{T}} + (\operatorname{grad} \boldsymbol{u})^{\mathrm{T}} . \mathbf{t}_0 - (\operatorname{div} \boldsymbol{u})\, \mathbf{t}_0\,.$$

According to (B.115), the equations of motion in reference system B read

(2.21)
$$T^k{}_{l':k}(\boldsymbol{x}, t) + \varrho_0[f'_{l'}(\boldsymbol{x}', t) - a'_{l'}(\boldsymbol{x}', t)] = 0\,,$$

in which the linearized relation for acceleration (B.101) is

(2.22)
$$a'_{l'} = \partial^2 u'_{l'}/\partial t^2 = \partial^2 u_l/\partial t^2\,.$$

According to (A.168),

(2.23)
$$T^k{}_{l':k} = T^k{}_{l';k} + T^k{}_{l';m}\, x'^{m'}{}_{,k}\,.$$

Since the second term on the r.h.s. of Eq. (2.23) is identically equal to zero and since (2.9) holds true,

(2.24)
$$T^k{}_{l':k} = T^k{}_{l;k}\,.$$

If we now consider Eq. (2.5), the equation of motion (2.21) will become

(2.25)
$$T^k{}_{l;k} + \varrho_0 f'_l = \varepsilon \varrho_0\, \partial^2 u_l/\partial t^2\,,$$
$$\operatorname{div} \mathbf{T} + \varrho_0\, \boldsymbol{f}' = \varepsilon \varrho_0\, \partial^2 \boldsymbol{u}/\partial t^2\,.$$

Taylor's expansion of body force $f_l(\boldsymbol{x}', t)$ in the neighbourhood of point $\boldsymbol{x}$, retaining terms of the order of ε, reads

(2.26)$_1$
$$f_l(\boldsymbol{x}', t) = f_l(\boldsymbol{x}, t) + \varepsilon\, \partial f_l/\partial x^k|_{\boldsymbol{x}}\, u^k(\boldsymbol{x}, t) + \varepsilon \bar{f}_l(\boldsymbol{x}', t) =$$
$$= f_{0l}(\boldsymbol{x}) + \varepsilon[(\operatorname{grad} \boldsymbol{f}_0)_{kl}|_{\boldsymbol{x}}\, u^k(\boldsymbol{x}, t) + \bar{f}_l(\boldsymbol{x}, t)]\,,$$

where $\bar{f}_l(\boldsymbol{x}, t)$ is the additional body force per unit mass which may be generated by elastic oscillations, e.g. disturbances of the gravitional potential due to the displacement of masses under elastic oscillations. In vectorial for (2.26)$_1$ reads

(2.26)$_2$
$$\boldsymbol{f}(\boldsymbol{x}', t) = \boldsymbol{f}_0(\boldsymbol{x}) + \varepsilon[\boldsymbol{u}(\boldsymbol{x}, t) . \operatorname{grad} \boldsymbol{f}_0(\boldsymbol{x}) + \bar{\boldsymbol{f}}(\boldsymbol{x}, t)]\,.$$

If we substitute from (2.10) and (2.26) into the equation of motion (2.25) and apply the equilibrium condition (2.3), we arrive at the equations of motion in reference system B for the increment of the Piola—Kirchhoff tensor $\bar{\mathbf{T}}$,

(2.27)
$$\bar{T}^k{}_{l;k} + \varrho_0(f_{0l;k}\, u^k + \bar{f}_l) = \varrho_0\, \partial^2 u_l/\partial t^2\,,$$
$$\operatorname{div} \bar{\mathbf{T}} + \varrho_0(\boldsymbol{u} . \operatorname{grad} \boldsymbol{f}_0 + \bar{\boldsymbol{f}}) = \varrho_0\, \partial^2 \boldsymbol{u}/\partial t^2\,.$$

By using Eqs (2.3), (2.20) and (A.113), we can express the equations of motion (2.27) in terms of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ and of the initial static stress $\mathbf{t}_0$,

(2.28)
$$\operatorname{div} \bar{\mathbf{t}} - \operatorname{div}[(\operatorname{grad} \boldsymbol{u})^{\mathrm{T}} . \mathbf{t}_0] + \operatorname{grad} \operatorname{div} \boldsymbol{u} . \mathbf{t}_0 +$$
$$+ \varrho_0(\boldsymbol{u} . \operatorname{grad} \boldsymbol{f}_0 - \operatorname{div} \boldsymbol{u}\, \boldsymbol{f}_0 + \bar{\boldsymbol{f}}) = \varrho_0\, \partial^2 \boldsymbol{u}/\partial t^2\,.$$

The equations of motion derived above do not depend on the type of material. However, the stress increments $\bar{\mathbf{T}}$ and $\bar{\mathbf{t}}$ can only be determined after the rheological model of the material has been defined. Below we shall study so-called hyperelastic materials for which [56, 59]

(2.29)
$$\bar{T}^k{}_l = B_0{}^k{}_l{}^m{}_n u^n{}_{;m}, \quad B_0{}^k{}_l{}^m{}_n = C_0{}^k{}_l{}^m{}_n + \delta_l{}^m t_0{}^k{}_n,$$

here, $C_0{}^k{}_l{}^m{}_n$ are the elastic coefficients of the body in natural configuration B_0, for which the following symmetry holds:

(2.30)
$$C_0{}^k{}_l{}^m{}_n = C_{0l}{}^{km}{}_n = C_0{}^k{}_{ln}{}^m = C_0{}^m{}_n{}^k{}_l.$$

Thus, the elastic 4th-order tensor $\mathbf{C}_0$ only has 21 independent components. By substituting (2.29) into (2.20) we obtain the increment of Cauchy's stress tensor in configuration $B'(t)$,

(2.31)
$$\bar{t}^k{}_l = \bar{C}^k{}_l{}^m u^n{}_{;m}, \quad \bar{C}^k{}_l{}^m{}_n = C_0{}^k{}_l{}^m{}_n + \delta_l{}^m t_0{}^k{}_n + \delta^k{}_n t_{0l}{}^m - \delta^m{}_n t_0{}^k{}_l,$$

where $\bar{C}_0{}^k{}_l{}^m{}_m$ are the elastic coefficients of the body in configuration B. For a general anisotropic elastic body with the initial anisotropic static stress, tensor $\bar{\mathbf{C}}$ has 27 independent components: 21 independent components of tensor $\mathbf{C}_0$ and 6 independent components of tensor $\mathbf{t}_0$.

Let us investigate the special case of an elastic body which is isotropic in natural configuration B_0, i.e. $\mathbf{C}_0$ is an isotropic 4th-order tensor,

(2.32)
$$C_0{}^k{}_l{}^m{}_n = \lambda_0 \delta^k{}_l \delta^m{}_n + \mu_0(\delta^k{}_n \delta_l{}^m + g^{km} g_{ln}),$$

where λ_0 and μ_0 are Lamé's elastic coefficients of the body in natural configuration B_0. If this body is transformed to configuration B by an anisotropic finite static stress, Eqs $(2.31)_2$ and (2.32) indicate that it loses its isotropic properties and begins to manifest itself as an anisotropic body. If, in particular, the initial static stres is hydrostatic,

(2.33)
$$\mathbf{t}_0 = -p_0(\boldsymbol{X})\mathbf{I}, \quad t_0{}^k{}_l = -p_0(\boldsymbol{X})\delta^k{}_l,$$

the body will preserve its isotropy even in strained configuration B, because $\bar{\mathbf{C}}$ is an isotropic 4th-order tensor.

In the next part of this study, we shall study the free oscillations of gravitating bodies, i.e. of bodies whose oscillations do not allow its inherent gravitational field to be neglected. In these cases, in which the initial statis stress tensor $\mathbf{t}_0$ is given by the inherent gravitation of a body, it is meaningless to refer the equation of motions to the natural configuration B_0 because it does not exist. It is more expedient and more advantageous, especially for the boundary conditions, to refer the equations of motion to reference configuration B, and to consider the coefficients of tensor $\bar{\mathbf{C}}$ as independent. Equation (2.20) yields the relation for the increment of the Piola—Kirchhoff stress tensor,

$$\bar{T}^k{}_l = \Lambda^k{}_l{}^m{}_n u^n{}_{;m}, \qquad (2.34)$$
$$\Lambda^k{}_l{}^m{}_n = \bar{C}^k{}_l{}^m{}_n + \delta^m{}_n t_0{}^k{}_l - \delta^k{}_n t_{0l}{}^m,$$

where $\bar{C}^k{}_l{}^m{}_n$ are the independent elastic coefficients of the body in configuration B.

In the special case when the body is isotropic in configuration B,

$$\bar{C}^k{}_l{}^m{}_n = \lambda\delta^k{}_l\delta^m{}_n + \mu(\delta^k{}_n\delta^m{}_l + g^{km}g_{ln}), \qquad (2.35)$$

λ and μ being Lamé's coefficifents of body B, the increment of Cauchy's stress tensor in $B'(t)$

$$\bar{t}^k{}_l = \lambda u^m{}_{;m}\delta^k{}_l + \mu(u^k{}_{;l} + u_{l;}{}^k). \qquad (2.36)$$

If we now introduce the tensor of small strains $\mathbf{e} = e^k{}_l \mathbf{g}_k \mathbf{g}^l$ by the relations

$$\mathbf{e} = \tfrac{1}{2}[\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}], \qquad (2.37)$$
$$e^k{}_l = \tfrac{1}{2}(u^k{}_{;l} + u_{l;}{}^k),$$

the increment of Cauchy's stress tensor becomes

$$\bar{\mathbf{t}} = \lambda \operatorname{div}\boldsymbol{u}\mathbf{I} + 2\mu\mathbf{e}, \qquad (2.38)$$
$$\bar{t}^k{}_l = \lambda u^m{}_{;m}\delta^k{}_l + 2\mu e^k{}_l.$$

2.2. Boundary conditions

Assume that the physical properties of an elastic body vary discontinuously over surface σ_0 in natural configuration. Assume that, due to the initial stress $\mathbf{t}_0$, surface σ_0 will change to surface σ and, due to elastic motions, surface σ will change to surface $\sigma'(t)$. Surface $\sigma'(t)$ is thus moving with velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$. In Supplement B it is shown that equations of motion (2.27) and/or (2.28) must be complemented with boundary conditions on this surface of discontinuity. We shall again express the boundary conditions in the reference configuration, i.e. on surface σ.

Three types of boundaries are important for the problems being studied:
a) solid elastic medium — solid elastic medium, the so-called solid boundary;
b) solid elastic medium — liquid elastic medium, the so-called liquid boundary;
c) solid elastic medium or liquid elastic medium — free space, the so-called free boundary.

Boundary condition $(B.100)_2$ for Cauchy's initial static stress tensor $\mathbf{t}_0$ reads

$$\boldsymbol{n}.[\mathbf{t}_0]^+_- = 0 \quad \text{at } \sigma, \qquad (2.39)$$

if surface σ represents a solid or liquid boundary. The symbol $[\,]^+_-$ indicates a jump of the function in brackets at the boundary, $\boldsymbol{n}$ is the unit vector of the normal external to surface σ. There is yet another boundary condition for vector $\boldsymbol{n}.\mathbf{t}_0$ at the liquid boundary σ. Let us assume that the shear motion along the liquid boundary σ takes place without friction. It follows that the tangential component of the stress vector $\boldsymbol{n}.\mathbf{t}_0$ must be zero,

$$\boldsymbol{n}.\mathbf{t}_0.(\mathbf{I}-\boldsymbol{nn})=0. \tag{2.40}$$

If we denote the projection of the stress vector $\boldsymbol{n}.\mathbf{t}_0$ along the normal $\boldsymbol{n}$ by the symbol π_0,

$$\pi_0=\boldsymbol{n}.\mathbf{t}_0.\boldsymbol{n}, \tag{2.41}$$

condition (2.40) can be expressed as

$$\boldsymbol{n}.\mathbf{t}_0=\pi_0\boldsymbol{n}. \tag{2.42}$$

The two following boundary conditions must be satisfied for elastic oscillations:

1. For the axiom of body continuity to be preserved, i.e. to avoid creating cavities at the boundary or, on the contrary, to avoid concentrations of material at the boundary, on the deformed surface $\sigma'(t)$

a) the displacement vector $\boldsymbol{u}$ must be continuous, if surface $\sigma'(t)$ is a solid boundary,

b) the normal component of displacement must be continuous, if $\sigma'(t)$ is a liquid boundary. Moreover, both tangential components of the displacement vector may, of course, be discontinuous across the boundary, because the liquid medium may slip along the boundary. Moreover, the effect of friction is not considered under slipping. With a view to (2.4) and with an accuracy of the quantities of the order of ε, we may adopt the approximation

$$\varepsilon\boldsymbol{u}(\boldsymbol{x}')=\varepsilon\boldsymbol{u}(\boldsymbol{x}). \tag{2.43}$$

The conditions given for the displacement vector $\boldsymbol{u}$ may be expressed mathematically as

$$\begin{aligned} &[\boldsymbol{u}]^+_-=0 && \text{at the solid boundary } \sigma,\\ &[\boldsymbol{n}.\boldsymbol{u}]^+_-=0 && \text{at the liquid boundary } \sigma. \end{aligned} \tag{2.44}$$

No boundary conditions are imposed on the displacement vector at a free boundary.

2. For boundary condition $(B.100)_2$ to be satisfied, i.e. for the pressure to be continuous at the boundary and to avoid the accumulation of surface shear forces at the boundary, Cauchy's stress vector $\boldsymbol{n}'.\mathbf{t}$ must be continous on the surface of discontinuity $\sigma'(t)$, $\boldsymbol{n}'$ being the unit vector of the normal external to

surface $\sigma'(t)$ and $\mathbf{t}$ Cauchy's stress tensor in strained configuration $B'(t)$. The boundary condition at the solid and liquid boundary is

(2.45) $$\boldsymbol{n}' . [\mathbf{t}]_-^+ = 0 \qquad \text{at } \sigma'(t) .$$

As proved in Section B.2.4, boundary condition (2.45) is equivalent to the condition

(2.46) $$\begin{aligned} [\boldsymbol{T}^k]_-^+ n_k &= 0 && \text{at the solid boundary } \sigma, \\ [\boldsymbol{T}^k n_k \, da]_-^+ &= 0 && \text{at the liquid boundary } \sigma. \end{aligned}$$

By applying (2.8) these conditions can be changed to read

(2.47) $$\begin{aligned} \boldsymbol{n} . [\mathbf{T}]_-^+ &= 0 && \text{at the solid boundary } \sigma, \\ [\boldsymbol{n} . \mathbf{T} \, da]_-^+ &= 0 && \text{at the liquid boundary } \sigma. \end{aligned}$$

If we substitute for the Piola—Kirchhoff stress tensor $\mathbf{T}$ from (2.10) and make use of condition (2.39), boundary condition $(2.47)_1$ will adopt the final form

(2.48) $$\boldsymbol{n} . [\bar{\mathbf{T}}]_-^+ = 0 \qquad \text{at the solid boundary } \sigma .$$

At the liquid boundary the situation is more complicated. Boundary condition $(2.47)_2$ now reads

(2.49) $$[(\boldsymbol{n} . \mathbf{t}_0 + \boldsymbol{n} . \bar{\mathbf{T}}) \, da]_-^+ = 0 .$$

Let us select elementary surfaces $da^{\pm}$ with the radius-vectors $\boldsymbol{x}^{\pm}$ so that they are located on opposite sides of the liquid boundary, and so that they are equal at the strained boundary $\sigma'(t)$ (see Fig. B.6),

(2.50) $$\begin{aligned} da'^+(t) &= da'^-(t) = da', \\ \boldsymbol{x}' &= \boldsymbol{x}^+ + \varepsilon \boldsymbol{u}^+ = \boldsymbol{x}^- + \varepsilon \boldsymbol{u}^- . \end{aligned}$$

Since vector $\boldsymbol{n} . \mathbf{t}_0$ is continuous on surface σ, Taylor's expansion up to the order of magnitude of the displacement vector can be expressed as

(2.51) $$\begin{aligned} \boldsymbol{n}^+ . \mathbf{t}_0^+ - \boldsymbol{n}^- . \mathbf{t}_0^- &= -\varepsilon [\boldsymbol{u}]_-^+ . \operatorname{grad}_s (\boldsymbol{n}^+ . \mathbf{t}_0^+) , \\ \boldsymbol{n}^+ . \mathbf{t}_0^+ - \boldsymbol{n}^- . \mathbf{t}_0^- &= -\varepsilon [\boldsymbol{u}]_-^+ . \operatorname{grad}_s (\boldsymbol{n}^- . \mathbf{t}_0^-) , \end{aligned}$$

where grad_s is the surface gradient, i.e. the projection of operator grad onto surface σ,

(2.52) $$\operatorname{grad}_s = (\mathbf{I} - \boldsymbol{n}\boldsymbol{n}) . \operatorname{grad} = \operatorname{grad} - \boldsymbol{n}(\boldsymbol{n} . \operatorname{grad}) .$$

Further properties of operator grad_s are derived in Section A.11.

From Eq. (2.51) it follows that

(2.53) $$[\boldsymbol{n} . \mathbf{t}_0]_-^+ = -\varepsilon [\boldsymbol{u}]_-^+ . \operatorname{grad}_s (\boldsymbol{n} . \mathbf{t}_0) ,$$

where $\operatorname{grad}_s (\boldsymbol{n} . \mathbf{t}_0)$ is considered either at point $\boldsymbol{x}^+$ or at point $\boldsymbol{x}^-$.

To be able to modify condition (2.49), we must first derive the geometric relation between the elementary surfaces at the unstrained and strained boundaries. According to (B.121),

(2.54) $$da/da' = (j')^{-1} g^{k'l'} x^k_{,k'} x^l_{,l'} n_k n_l)^{-1/2}.$$

With a view to Eqs (2.5) and (2.6),

(2.55) $$da/da' = j^{-1} [g^{kl} (\partial x^r/\partial x'^k)(\partial x^s/\partial x'^l) n_r n_s]^{-1/2}$$

and by substituting from $(2.14)_2$ and (2.15) we arrive at

(2.56) $$da/da' = (1 - \varepsilon u^m{}_{;m}) [g^{kl} (\delta^r{}_k - \varepsilon u^r{}_{;k})(\delta^s{}_l - \varepsilon u^s{}_{;l}) n_r n_s]^{-1/2}.$$

After some minor algebra we obtain the relations

(2.57) $$da/da' = 1 - \varepsilon(u^k{}_{;k} - u^k{}_{;l} n_k n^l),$$
$$da/da' = 1 - \varepsilon(\operatorname{div} \boldsymbol{u} - \boldsymbol{n} . \operatorname{grad} \boldsymbol{u} . \boldsymbol{n}),$$

in which we have neglected the quadratic and higher terms in ε. We now rewrite Eq. (2.57) in terms of tensor $\mathbf{H} = \operatorname{grad} \boldsymbol{u}$ and its normal component h_0,

(2.58) $$h_0 = \boldsymbol{n} . \mathbf{H} . \boldsymbol{n},$$

in the following form,

(2.59) $$da/da' = 1 - \varepsilon(\operatorname{tr} \mathbf{H} - h_0),$$

an then in terms of the surface divergence of the displacement vector $\boldsymbol{u}$,

(2.60) $$\operatorname{div}_s \boldsymbol{u} = \operatorname{div} \boldsymbol{u} - \boldsymbol{n} . \operatorname{grad} \boldsymbol{u} . \boldsymbol{n} = \operatorname{tr} \mathbf{H} - h_0,$$

in the following form,

(2.61) $$da/da' = 1 - \varepsilon \operatorname{div}_s \boldsymbol{u}.$$

Let us now go back to condition (2.49). If we substitute into this condition from (2.53) and (2.61) and if we neglect the quadratic terms in ε, we arrive at

(2.62) $$[\boldsymbol{n} . \bar{\mathbf{T}} - \operatorname{div}_s \boldsymbol{u}(\boldsymbol{n} . \mathbf{t}_0) - \boldsymbol{u} . \operatorname{grad}_s (\boldsymbol{n} . \mathbf{t}_0)]_-^+ = 0.$$

The linearized boundary condition (2.62) we have just derived expresses the continuity of Cauchy's stress vector at any displacement boundary and was also derived in [42]. We are interested in a special type of displacement boundary at which no friction occurs, i.e. an ideal liquid boundary. If no friction occurs along this boundary, Cauchy's stress vector has the direction of the vector of normal $\boldsymbol{n}'$ at point $\boldsymbol{x}'$, i.e.

(2.63) $$(\boldsymbol{n}' . \mathbf{t}) . (\mathbf{I} - \boldsymbol{n}'\boldsymbol{n}') = 0.$$

To be able to express this condition at the unstrained boundary σ, we must first derive the geometric relation between the unit normal $\boldsymbol{n}$ and $\boldsymbol{n}'$ at the unstrained and strained boundaries. According to (B.120),

(2.64)
$$n'_{k'} = j' \, x^k{}_{,k'} \, n_k \, \mathrm{d}a/\mathrm{d}a' .$$

With a view to Eqs (2.5) and (2.6),

(2.65)
$$n'_k = j(\partial x^r/\partial x'^k) \, n_r \, \mathrm{d}a/\mathrm{d}a'$$

and, by substituting from $(2.14)_1$, (2.15) and $(2.57_1$, we obtain

(2.66)
$$n'_k = (1 + \varepsilon u^m{}_{;m})(\delta^l{}_k - \varepsilon u^l{}_{;k}) \quad [1 - \varepsilon(u^r{}_{;r} - u^r{}_{;s} n_r n^s)] \, n_l .$$

After some minor algebra we arrive at the relations

(2.67)
$$n'_k = n_k - \varepsilon(u^l{}_{;k} n_l - u^r{}_{;s} n_r n^s n_k) ,$$
$$\boldsymbol{n}' = \boldsymbol{n} - \varepsilon(\mathbf{H} . \boldsymbol{n} - \mathrm{h}_0 \boldsymbol{n}) ,$$

in which we have neglected the quadratic and higher terms in ε. We can now rewrite Eqs (2.67) in terms of the surface gradient of the displacement vector $\boldsymbol{u}$,

(2.68)
$$\mathrm{grad}_s \boldsymbol{u} \doteq (\mathbf{I} - \boldsymbol{nn}) . \mathrm{grad}\, \boldsymbol{u} =$$
$$= \mathrm{grad}\, \boldsymbol{u} - \boldsymbol{nn} . \mathrm{grad}\, \boldsymbol{u} = \mathrm{grad}\, \boldsymbol{u} - \boldsymbol{n}(\boldsymbol{n} . \mathrm{grad}\, \boldsymbol{u}) ,$$

in the following form:

(2.69)
$$\boldsymbol{n}' = \boldsymbol{n} - \varepsilon \, \mathrm{grad}_s \boldsymbol{u} . \boldsymbol{n} .$$

Let us go back to condition (2.63). Let us express the terms occuring in this condition, using (2.18), (2.20) and (2.67) with the accuracy of the quantities of the order of ε:

(2.70)
$$\boldsymbol{n}' . \mathbf{t} = \boldsymbol{n} . \mathbf{t}_0 + \varepsilon[\boldsymbol{n} . \bar{\mathbf{T}} - (tr\,\mathbf{H} - h_0)(\boldsymbol{n} . \mathbf{t}_0)] ,$$
$$\boldsymbol{n}'\boldsymbol{n}' = \boldsymbol{nn} - \varepsilon[(\mathbf{H} . \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n}(\mathbf{H} . \boldsymbol{n}) - 2 h_0 \boldsymbol{nn}] .$$

By substituting into (2.63) from (2.70) and using condition (2.40) and (2.42), we obtain condition (2.63) expressed at the unstrained boundary,

(2.71)
$$[\boldsymbol{n} . \bar{\mathbf{T}} + \pi_0(\mathbf{H} . \boldsymbol{n} - h_0 \boldsymbol{n})] . (\mathbf{I} - \boldsymbol{nn}) = 0 ,$$

which can further be modified using the operator grad_s,

(2.72)
$$[\boldsymbol{n} . \bar{\mathbf{T}} + \pi_0 \, \mathrm{grad}_s \boldsymbol{u} . \boldsymbol{n}] . (\mathbf{I} - \boldsymbol{nn}) = 0 .$$

By making use of condition (2.42) in the linearized condition (2.62), we obtain

(2.73)
$$[\boldsymbol{n} . \bar{\mathbf{T}} - \pi_0 (\mathrm{div}_s \boldsymbol{u}) \, \boldsymbol{n} - \boldsymbol{u} . \mathrm{grad}_s (\pi_0 \, \boldsymbol{n})]^+_- = 0 .$$

Modification of the last equation with the aid of Eqs (A.184) yields

(2.74)
$$[\boldsymbol{n} . \bar{\mathbf{T}} - \mathrm{div}_s (\pi_0 \, \boldsymbol{u}) \, \boldsymbol{n} - \pi_0 \, \boldsymbol{u} . \mathrm{grad}_s \boldsymbol{n}]^+_- = 0 .$$

With a view to condition $(2.44)_2$, which requires the normal component of the displacement vector $\boldsymbol{u}$ to be continuous, the changes in function $\boldsymbol{n} . \boldsymbol{u}$ must be equally large along the boundary, i.e.

(2.75) $$[\mathrm{grad}_s(\boldsymbol{n}\,.\,\boldsymbol{u})]_-^+ = 0\,.$$

If we expand (2.75) according to (A.184) and make use of the symmetry of tensor $\mathrm{grad}_s\boldsymbol{n}$, (2.75) can be expressed as

(2.76) $$[\boldsymbol{u}\,.\,\mathrm{grad}_s\,\boldsymbol{n} + \mathrm{grad}_s\boldsymbol{u}\,.\,\boldsymbol{n}]_-^+ = 0\,.$$

If we substitute for the last term in condition (2.74) form (2.76) and make use of the continuity of the normal stress π_0, condition (2.74) becomes

(2.77) $$[\boldsymbol{n}\,.\,\bar{\mathbf{T}} - \mathrm{div}_s(\pi_0\,\boldsymbol{u})\,\boldsymbol{n} + \pi_0\,\mathrm{grad}_s\boldsymbol{u}\,.\,\boldsymbol{n}]_-^+ = 0\,,$$

By comparing conditions (2.72) and (2.77) we find that the vector

(2.78) $$\boldsymbol{b} = \boldsymbol{n}\,.\,\bar{\mathbf{T}} - \mathrm{div}_s(\pi_0\,\boldsymbol{u})\,\boldsymbol{n} + \pi_0\,\mathrm{grad}_s\boldsymbol{u}\,.\,\boldsymbol{n}$$

must be continuous at and normal to the unstrained liquid boundary σ,

(2.79) $$[\boldsymbol{b}]_-^+ = 0,\ \ \boldsymbol{b} = \boldsymbol{n}(\boldsymbol{n}\,.\,\boldsymbol{b})\,.$$

To conclude, we shall review the boundary conditions expressed in terms of the vector $\boldsymbol{b}$ at the individual types of boundaries:

(2.80)

solid boundary:	$[\boldsymbol{u}]_-^+ = 0\,,$
	$[\boldsymbol{b}]_-^+ = 0\,,$
liquid boundary:	$[\boldsymbol{n}\,.\,\boldsymbol{u}]_-^+ = 0\,,$
	$[\boldsymbol{b}]_-^+ = 0\,,$
	$\boldsymbol{b} = \boldsymbol{n}(\boldsymbol{n}\,.\,\boldsymbol{b})\,,$
free boundary:	$\boldsymbol{b} = 0\,.$

3. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS OF FREE OSCILLATIONS OF A GENERAL MODEL OF THE EARTH

In this Section we derive the Lagrangian equations of motion and boundary conditions of free oscillations of a completely general Earth model with interior fluid-solid boundaries. This problem was also discussed in [40—42, 45, 145, 146].

Consider a general, inhomogeneous, elastic model of the Earth, consisting of solid and liquid regions. The physical properties of each region are described by continuous functions of position. These regions are separated by internal boundaries, which are simple, smooth, closed surfaces which do not intersect. The internal boundaries are of two types, solid and liquid. The union of the internal boundaries will be denoted by σ. The model is bounded by an external boundary S — a free surface. The volume of the model is V. In equilibrium configuration, the model of the Earth rotates uniformly with the angular frequency $\boldsymbol{\Omega}$ about the origin located at the Earth's centre of gravity. The position of any particle in V or on S will be described by radius-vector $\boldsymbol{x}$ relative to this rotating system. The rotating equilibrium configuration B (volume of the model V, surface of the boundary S) is considered to be the reference configuration, i.e. the equations of motion and boundary conditions will be expressed relative to this system.

Assume $\varrho_0(\boldsymbol{x})$, $\varphi_0(\boldsymbol{x})$ and $\mathfrak{t}_0(\boldsymbol{x})$ to be the initial density, initial gravitational potential and Cauchy's tensor of the initial static stress of the Earth model in configuration B. Let $\psi(\boldsymbol{x})$ be the potential of the centrifugal force,

$$\psi(\boldsymbol{x}) = -\tfrac{1}{2}[\Omega^2 \boldsymbol{x}^2 - (\boldsymbol{\Omega}.\boldsymbol{x})^2]. \tag{3.1}$$

The geopotential in configuration B will be denoted by Φ_0,

$$\Phi_0 = \varphi_0 + \psi. \tag{3.2}$$

In volume V, the initial equilibrium state of the Earth is described by the equation

$$\operatorname{div} \mathfrak{t}_0 + \varrho_0 \boldsymbol{f}_0 = 0, \tag{3.3}$$

where

$$\boldsymbol{f}_0 = -\operatorname{grad} \Phi_0 \tag{3.4}$$

is the initial body force per unit mass, and by Poisson's equation for the gravitational potential φ_0,

$$\nabla^2 \varphi_0 = 4\pi G \varrho_0, \tag{3.5}$$

where G is the gravitational constant. On surface S and at the boundaries σ, these equations must be supplemented with the conditions

$$\begin{aligned} &\boldsymbol{n}.[\mathfrak{t}_0]_-^+ = 0, \\ &[\varphi_0]_-^+ = 0, \\ &\boldsymbol{n}.[\operatorname{grad} \varphi_0]_-^+ = 0, \end{aligned} \tag{3.6}$$

where $\boldsymbol{n}$ is the unit vector of the normal external to surface S or boundary σ. Functions $\varrho_0(\boldsymbol{x})$ and $\mathfrak{t}_0(\boldsymbol{x})$ are zero outside the Earth model. It is also required that the gravitational potential φ_0 vanish in infinity. If we denote the normal component of the initial static stress by

$$\pi_0 = \boldsymbol{n}.\mathfrak{t}_0.\boldsymbol{n}, \tag{3.7}$$

at the liquid boundary also condition (2.42) must hold:

$$\boldsymbol{n}.\mathfrak{t}_0 = \pi_0 \boldsymbol{n}. \tag{3.8}$$

If an external force now begins to act on the Earth model at time $t = t_0$, just for a certain interval of time after which it stops doing so, the Earth model will begin to perform free elastic gravitational oscillations. The characteristic is then the time-variable field of displacement, $\varepsilon \boldsymbol{u}(\boldsymbol{x}, t)$, which describes the transfer of the particle from equilibrium configuration B to strained configuration $B'(t)$ — see Eq. (2.4),

$$\mathbf{x}' = \mathbf{x} + \varepsilon \mathbf{u}(\mathbf{x}, t)\,. \tag{3.9}$$

We restrict ourselves to the linear theory of elasticity, i.e. we assume that the strains of configuration B are small, ε is then a small parameter. The displacement field will change the volume $V'(t)$, as well as the surface $S'(t)$. At a fixed point of the body, the oscillations will be characterized by a change of density, of the gravity potential and of the stress tensor.

Let us first derive the relation for the change of density. From the law of mass conservation, i.e. from the equation of continuity, follows Eq. (B.96),

$$\varrho(\mathbf{x}') = \varrho_0(\mathbf{x})/j \doteq \varrho_0(1 - \varepsilon u^m{}_{;m}) = \varrho_0\,[1 - \varepsilon\,\Delta(\mathbf{x}, t)]\,, \tag{3.10}$$

in which we have made use of Eq. $(2.14)_2$ and $\varrho_0(\mathbf{x})$ is the density prior to the deformation in configuration B, $\varrho(\mathbf{x}')$ the density after the deformation in configuration $B'(t)$, and $\Delta(\mathbf{x}, t)$ is the relative change of volume. Taylor's expansion of function $\varrho(\mathbf{x}')$ in the neighbourhood of point $\mathbf{x}$ reads

$$\varrho(\mathbf{x}') = \varrho(\mathbf{x}) + \varepsilon\,\mathrm{grad}\,\varrho\,.\,\mathbf{u}|_{\mathbf{x}} \doteq \varrho(\mathbf{x}) + \varepsilon\,\mathrm{grad}\,\varrho_0\,.\,\mathbf{u}|_{\mathbf{x}}\,, \tag{3.11}$$

where we have retained the absolute term and the terms linear in ε. If we compare Eqs (3.10) and (3.11), under the given linearization the density after deformation at point $\mathbf{x}$ may be expressed as

$$\varrho(\mathbf{x}, t) = \varrho_0(\mathbf{x}) + \varepsilon\varrho_1(\mathbf{x}, t)\,, \tag{3.12}$$

where

$$\varrho_1(\mathbf{x}, t) = -\varrho_0(\mathbf{x})\,\Delta(\mathbf{x}, t) - \mathrm{grad}\,\varrho_0\,.\,\mathbf{u}|_{\mathbf{x}} = -\mathrm{div}\,(\varrho_0\,\mathbf{u}) \tag{3.13}$$

is the increment of density after deformation at point $\mathbf{x}$.

The geopotential $\Phi(\mathbf{x}, t)$ after deformation at point $\mathbf{x}$ can be expressed as the sum of the geopotential $\Phi_0(\mathbf{x})$ before the deformation at point $\mathbf{x}$ and of the increment of the geopotential $\Phi_1(\mathbf{x}, t)$ after the deformation at point $\mathbf{x}$,

$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \varepsilon\Phi_1(\mathbf{x}, t)\,, \tag{3.14}$$

where $\Phi_1(\mathbf{x}, t)$ is the sum of the increments of the gravitational potential and of the potential of the centrifugal force. However, the increment of the potential of the centrifugal force is of the order of ε^2, i.e. a term which is not considered in our linearization. Consequently, the increment of the geopotential is equal to the increment of the gravitational potential,

$$\Phi_1(\mathbf{x}, t) = \varphi_1(\mathbf{x}, t)\,. \tag{3.15}$$

The resultant potential after the deformation at point $\mathbf{x}$ is then

$$\Phi(\mathbf{x}, t) = \varphi(\mathbf{x}, t) + \psi(\mathbf{x})\,, \tag{3.16}$$

where

$$\varphi(\boldsymbol{x}, t) = \varphi_0(\boldsymbol{x}) + \varepsilon\varphi_1(\boldsymbol{x}, t) \tag{3.17}$$

is the resultant gravitational potential after deformation at point $\boldsymbol{x}$. The increment of the gravitational potential $\varphi_1(\boldsymbol{x}, t)$ is created as a result of the gravitional attraction of the strained volume of density $\varrho_1(\boldsymbol{x}, t)$ and, therefore, Poisson's equation

$$\nabla^2 \varphi_1 = 4\pi G \varrho_1 \tag{3.18}$$

must hold. The body force per unit mass after deformation at point $\boldsymbol{x}$ is

$$\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{f}_0(\boldsymbol{x}) + \varepsilon \boldsymbol{f}_1(\boldsymbol{x}, t), \tag{3.19}$$

where

$$\boldsymbol{f}_1(\boldsymbol{x}, t) = -\operatorname{grad} \varphi_1(\boldsymbol{x}, t) \tag{3.20}$$

is the increment of the body force per unit mass after deformation at point $\boldsymbol{x}$. If we substitute (3.19) and (3.20) into the linearized equation of motion (2.27), we arrive at

$$\varrho_0\left(\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + 2\boldsymbol{\Omega} \times \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{grad} \varphi_1 + \operatorname{grad}\operatorname{grad} \Phi_0 \,.\, \boldsymbol{u}\right) = \operatorname{div} \bar{\mathbf{T}}, \tag{3.21}$$

where the non-symmetric 2nd-order tensor $\bar{\mathbf{T}}$ is the increment of the Piola—Kirchhoff stress tensor. In Eq. (3.21) we have also made use of the symmetry of tensor grad grad Φ_0, i.e.

$$\boldsymbol{u} \,.\, \operatorname{grad}\operatorname{grad} \Phi_0 = \operatorname{grad}\operatorname{grad} \Phi_0 \,.\, \boldsymbol{u}. \tag{3.22}$$

By using (2.28) we can express the equations of motion in terms of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$,

$$\begin{aligned} &\varrho_0\left(\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + 2\boldsymbol{\Omega} \times \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{grad} \varphi_1 + \right. \\ &\left. + \operatorname{grad}\operatorname{grad} \Phi_0 \,.\, \boldsymbol{u} - \operatorname{div} \mathbf{u} \operatorname{grad} \Phi_0\right) = \\ &= \operatorname{div} \bar{\mathbf{t}} - \operatorname{div}\left[(\operatorname{grad} \boldsymbol{u})^{\mathrm{T}} \,.\, \mathbf{t}_0\right] + \operatorname{grad} \operatorname{div} \boldsymbol{u} \,.\, \mathbf{t}_0 . \end{aligned} \tag{3.23}$$

The equations of motion, derived above, are independent of the rheological model of the Earth. We shall assume the hyperelastic rheological model of the Earth for which the increment of Cauchy's stress tensor is given by Eq. (2.31),

$$\bar{t}^k{}_l = C^k{}_l{}^m{}_n u^n{}_{;m}, \tag{3.24}$$

where $C^k{}_l{}^m{}_n$ are the elastic coefficients of the Earth model in configuration B, which we shall consider to be independent. According to (2.34), the increment of the Piola—Kirchhoff stress tensor $\bar{\mathbf{T}}$ is given by the formula

(3.25)
$$\bar{T}^k{}_l = \Lambda^k{}_l{}^m{}_n u^n{}_{;m},$$
$$\Lambda^k{}_l{}^m{}_n = \bar{C}^k{}_l{}^m{}_n + \delta^m{}_n t_0{}^k{}_l - \delta^k{}_n t_{0l}{}^m .$$

The boundary conditions for the increment of the gravitational potential $\varphi_1(\boldsymbol{x}, t)$ can be obtained from the condition of continuity of the total gravitational potential and of the normal component of its gradient in the Earth model as a whole, i.e. also on the deformed surface $S'(t)$ and surface of discontinuity $\sigma'(t)$,

(3.26)
$$[\varphi(\boldsymbol{x}')]^+_- = 0,$$
$$[\boldsymbol{n}' . \text{grad}' \varphi(\boldsymbol{x}')]^+_- = 0,$$

where $\varphi(\boldsymbol{x}')$ is the resultant gravitational potential after deformation at point $\boldsymbol{x}'$, given by Eq. (3.17). With the accuracy of the quantities of the order of the displacement vector it holds that

(3.27)
$$\varphi(\boldsymbol{x}') = \varphi(\boldsymbol{x}) + \varepsilon \boldsymbol{u} . \text{grad}\, \varphi|_{\boldsymbol{x}} \doteq \varphi_0(\boldsymbol{x}) + \varepsilon[\boldsymbol{u} . \text{grad}\, \varphi_0|_{\boldsymbol{x}} + \varphi_1(\boldsymbol{x})],$$

(3.28)
$$\text{grad}' \varphi(\boldsymbol{x}') = \text{grad}\, \varphi(\boldsymbol{x}) + \varepsilon \boldsymbol{u} . \text{grad grad}\, \varphi|_{\boldsymbol{x}} \doteq$$
$$\doteq \text{grad}\, \varphi_0(\boldsymbol{x}) + \varepsilon[\boldsymbol{u} . \text{grad grad}\, \varphi_0|_{\boldsymbol{x}} + \text{grad}\, \varphi_1(\boldsymbol{x})] .$$

By taking the scalar product of Eq. (3.28) and Eq. (2.69), we obtain the normal component of the gradient of the gravitational potential,

(3.29)
$$\boldsymbol{n}' . \text{grad}' \varphi(\boldsymbol{x}') = \boldsymbol{n} . \text{grad}\, \varphi_0(\boldsymbol{x}) + \varepsilon[\boldsymbol{n} . \boldsymbol{u} . \text{grad grad} \varphi_0|_{\boldsymbol{x}} +$$
$$+ \boldsymbol{n} . \text{grad}\, \varphi_1(\boldsymbol{x}) - \text{grad}_s \boldsymbol{u} . \boldsymbol{n} . \text{grad}\, \varphi_0(\boldsymbol{x})] .$$

For the real Earth, the approximate relation

(3.30)
$$\text{grad}\, \varphi_0 \doteq (\boldsymbol{n} . \text{grad}\, \varphi_0)\, \boldsymbol{n} .$$

holds true. If we substitute (3.27) and (3.29) into (3.26) and if we use conditions (3.30), (3.6) and (2.44), i.e. also the continuity of tensor $\text{grad}_s \boldsymbol{u}$ and Poison's equation (3.5), we can derive boundary conditions for the increment of the gravitational potential $\varphi_1(\boldsymbol{x}, t)$ on surface S and boundary σ,

(3.31)
$$[\varphi_1]^+_- = 0,$$
$$[\boldsymbol{n} . \text{grad}\, \varphi_1 + 4\pi G \varrho_0 \boldsymbol{n} . \boldsymbol{u}]^+_- = 0 .$$

To these two boundary conditions for the increment of the gravitational potential φ_1, we also add the boundary conditions for the displacement vector and stress tensor which we derived in Chapter 2. To conclude, let us review the boundary conditons at the individual boundaries (compare with [146]):

(3.32) solid boundary:
$$[\boldsymbol{u}]^+_- = 0,$$
$$[\boldsymbol{b}]^+_- = 0,$$

liquid boundary: $[\boldsymbol{n}.\boldsymbol{u}]_-^+ = 0$,

$[\boldsymbol{b}]_-^+ = 0$,

$\boldsymbol{b} = (\boldsymbol{n}.\boldsymbol{b})\,\boldsymbol{n}$,

free boundary: $\boldsymbol{b} = 0$,

all boundaries: $[\varphi_1]_-^+ = 0$,

$[g_1]_-^+ = 0$,

where

$$\boldsymbol{b} = \boldsymbol{n}.\bar{\mathbf{T}} - \boldsymbol{n}\,\mathrm{div}_s(\pi_0 \boldsymbol{u}) + \pi_0\,\mathrm{grad}_s \boldsymbol{u}.\boldsymbol{n}, \qquad (3.33)$$

$$g_1 = \boldsymbol{n}.\mathrm{grad}\,\varphi_1 + 4\pi G \varrho_0 \boldsymbol{n}.\boldsymbol{u}.$$

4. FREE OSCILLATIONS OF THE SNREI MODEL OF THE EARTH

The plan of this Section is as follows. We begin with derivation of the equations of motion and boundary conditions of free oscillations of a spherically symmetric Earth. The equations of motion are then separated into two parts-toroidal and spheroidal. Following the formulation by Alterman et al. [8] we modify the fundamental equations for both parts into forms which are most convenient for numerical calculations. This problem was discussed by many authors [8, 13, 85, 118, 134].

4.1. Equations of motion and boundary conditions

To study the free oscillations of a spherically symmetric, non-rotating, perfectly elastic, isotropic, gravitating model of the Earth (hereinafter referred to as the SNREI Earth model), it is advantageous to use spherical coordinates r, ϑ, φ with their origin in the centre of the Earth. Assume this model to be composed of solid and liquid concentric spherical shells, separated by internal boundaries which are concentric spherical surfaces. The internal boundaries are again solid and liquid. The model is bounded by an external spherical boundary — a free surface. In equilibrium unstrained configuration the Earth model does not rotate, $\boldsymbol{\Omega} = 0$, $\psi = 0$. This equilibrium configuration B is considered to be the reference system, i.e. the equations of motion, boundary conditions and other relations will be expressed relative to this configuration B.

Assume the physical properties of the individual spherical shells to be described by the density prior to deformation, $\varrho_0(r)$, and by Lamé's coefficients $\lambda(r)$ and $\mu(r)$ which are only functions of r, the radial distance from the centre of the Earth model. The liquid regions are characterized by a zero parameter μ. Assume the initial static stress $\mathbf{t}_0$ to be hydrostatic,

$$\mathbf{t}_0 = -p_0(r)\,\mathbf{I}, \qquad (4.1)$$

where $p_0(r)$ is the hydrostatic pressure prior to deformation. The quantity π_0, given by Eq. (2.41), is then equal to $-p_0$. The inital equilibrium configuration of the Earth model is described by Eqs (3.3) and (3.4) which, in this particular case, read

(4.2)
$$\operatorname{grad} p_0 = -\varrho_0 \operatorname{grad} \varphi_0,$$
$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -\varrho_0 \frac{\mathrm{d}\varphi_0}{\mathrm{d}r}.$$

The derived quantities, the gravitational potential $\varphi_0(r)$ of the equilibrium configuration, the hydrostatic pressure $p_0(r)$ and the gravitational acceleration prior to deformation,

(4.3)
$$\boldsymbol{g}_0(r, \vartheta, \varphi) = -g_0(r)\,\boldsymbol{e}_r, \quad g_0(r) = \frac{\mathrm{d}\varphi_0}{\mathrm{d}r} > 0\,,$$

are also functions of the radial distance r only. In this particular case, Poisson's equation (3.5) takes the form

(4.4)
$$\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\varphi_0}{\mathrm{d}r} = 4\pi G \varrho_0,$$
$$\frac{\mathrm{d}g_0}{\mathrm{d}r} + \frac{2}{r} g_0 = 4\pi G \varrho_0.$$

Boundary conditions (3.6) now reduce to

(4.5)
$$[p_0]_-^+ = 0\,, \quad [\varphi_0]_-^+ = 0\,, \quad [g_0]_-^+ = 0\,.$$

The elastic gravitational free oscillations of the Earth model will be described by the time-variable displacement field, $\boldsymbol{u}(\boldsymbol{r}, t)$, ε now having been put equal to unit. It is again assumed that the deformations of the model are small in the sense that the tensor of finite strains is approximated by the tensor of small strains. If the position of a mass particle before and after deformation is given by the vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$, with a view to (3.9) the displacement vector may be expressed as

(4.6)
$$\boldsymbol{u} = \boldsymbol{r}' - \boldsymbol{r}.$$

Note: Since spherical coordinates are orthogonal curvilinear coordinates, we shall represent tensors or vectors by their physical components (ref. to Section A.5), because these components are the same in orthogonal curvilinear coordinates for all kinds of tensor components. Moreover, we shall not denote physical components by indices in parentheses, but only by covariant indices. For example, the physical components of the displacement vector $\boldsymbol{u}$ are $u_r, u_\vartheta, u_\varphi$; for the sake of simplicity we shall denote them by u, v, w.

The resultant stress after deformation in the reference configuration B is given by the Piola—Kirchhoff stress tensor $\mathbf{T}$, see (2.10),

(4.7)
$$\mathbf{T}(r, t) = \mathbf{t}_0(r) + \bar{\mathbf{T}}(r, t),$$

where the increment of the Piola—Kirchhoff stress tensor $\bar{\mathbf{T}}$ is expressed in terms of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ by Eq. (2.20), which now reads

(4.8)
$$\bar{\mathbf{T}} = \bar{\mathbf{t}} + p_0[(\operatorname{grad} \boldsymbol{u})^{\mathrm{T}} - \operatorname{div} \boldsymbol{u}\mathbf{I}].$$

By substituting Eqs (4.1) and (4.2) into the equations of motion (3.21) and (3.23), these equations can be expressed in terms of the increment of the Piola—Kirchhoff stress tensor $\bar{\mathbf{T}}$ and of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$, respectively,

(4.9)
$$\varrho_0\left(\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \operatorname{grad} \varphi_1 + \operatorname{grad} \operatorname{grad} \varphi_0 \,.\, \boldsymbol{u}\right) = \operatorname{div} \bar{\mathbf{T}},$$

and

(4.10)
$$\varrho_0\left(\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \operatorname{grad} \varphi_1 + \operatorname{grad}(\boldsymbol{u} \,.\, \operatorname{grad} \varphi_0) - \operatorname{div} \boldsymbol{u} \operatorname{grad} \varphi_0\right) = \operatorname{div} \bar{\mathbf{t}}.$$

The last equation may be modified by using the increment of density ϱ_1, given by Eq. (3.13),

(4.11)
$$\varrho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \varrho_0 \operatorname{grad} \varphi_1 + \varrho_1 \operatorname{grad} \varphi_0 + \operatorname{grad}(\varrho_0 \boldsymbol{u} \,.\, \operatorname{grad} \varphi_0) = \operatorname{div} \bar{\mathbf{t}}.$$

It should be emphasized that Poisson's equation (3.18),

(4.12)
$$\nabla^2 \varphi_1 + 4\pi G \operatorname{div}(\varrho_0 \boldsymbol{u}) = 0.$$

should be added to the equation of motion (4.9) or (4.10).

The equations of motion, mentioned above, do not depend on the rheological model of the Earth. We are assuming a linearly elastic and isotropic Earth model. As already explained in Chapter 2, we consider the elastic coefficients of the Earth model in configuration B to be independent. From these two assumptions it then follows that the increment of Cauchy's stress tensor

(4.13)
$$\bar{\mathbf{t}} = \lambda \operatorname{div} \boldsymbol{u}\mathbf{I} + 2\mu\mathbf{E},$$

where $\mathbf{E}$ is the tensor of small strains,

(4.14)
$$\mathbf{E} = \tfrac{1}{2}[\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^{\mathrm{T}}].$$

By substituting (4.13) into (4.8) we would be able to express the increment of the Piola—Kirchhoff stress tensor. Using Eqs (A.129)—(A.133), we can express the r.h.s. of the equation of motion (4.10),

(4.15)
$$\operatorname{div} \bar{\mathbf{t}} = (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \operatorname{div} \boldsymbol{u} \operatorname{grad} \lambda - \\ - \mu \operatorname{rot} \operatorname{rot} \boldsymbol{u} + \operatorname{grad} \mu \,.\, [\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^{\mathrm{T}}].$$

Conditions (3.32) hold true without change for the vector of boundary conditions $\boldsymbol{b}$ and scalar g_1 which are, in this case, expressed by the relations

(4.16)
$$\boldsymbol{b} = \boldsymbol{n} . \bar{\mathbf{t}}, \qquad g_1 = \partial \varphi_1 / \partial r + 4\pi G \varrho_0 u .$$

4.2. Separation of the equations of motion

We have derived the equations of motion (4.9) and (4.10) for the Earth model being considered. If we substitute from (4.15) into (4.10), the equations of motion take the form

(4.17)
$$\varrho_0 \left[\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \operatorname{grad} \varphi_1 + \operatorname{grad}(\boldsymbol{u} . \operatorname{grad} \varphi_0) - \operatorname{div} \boldsymbol{u} \operatorname{grad} \varphi_0 \right] = (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} + \operatorname{div} \boldsymbol{u} \operatorname{grad} \lambda - \mu \operatorname{rot} \operatorname{rot} \boldsymbol{u} + \operatorname{grad} \mu . [\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^T] .$$

Considering that the physical parameters ϱ_0, λ, μ are functions of coordinate r only, we can express Eq. (4.17) for the r-component,

(4.18)
$$\varrho_0 \left[\frac{\partial^2 u}{\partial t^2} - g_0 \Delta + \frac{\partial}{\partial r} (\varphi_1 + g_0 u) \right] = (\lambda + 2\mu) \frac{\partial \Delta}{\partial r} + \Delta \frac{d\lambda}{dr} + \mu \left[\nabla^2 u - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \Delta) + \frac{2}{r} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) \right] + 2 \frac{d\mu}{dr} \frac{\partial u}{\partial r} ,$$

where we have put $\Delta = \operatorname{div} \boldsymbol{u}$ and used the relation

(4.19)
$$(\operatorname{rot} \operatorname{rot} \boldsymbol{u})_r = -\nabla^2 u + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \Delta) - \frac{2}{r} \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) .$$

We shall now apply operation rot to Eq. (4.17) and use Eq. (A.124):

(4.20)
$$\varrho_0 \left[\frac{\partial^2 \operatorname{rot} \boldsymbol{u}}{\partial t^2} - \operatorname{grad} \operatorname{div} \boldsymbol{u} \times \operatorname{grad} \varphi_0 \right] + \operatorname{grad} \varrho_0 \times \left[\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \operatorname{grad} \varphi_1 + \operatorname{grad}(\boldsymbol{u} . \operatorname{grad} \varphi_0) - \operatorname{div} \boldsymbol{u} \operatorname{grad} \varphi_0 \right] = \operatorname{grad}(\lambda + 2\mu) \times \operatorname{grad} \operatorname{div} \boldsymbol{u} + \operatorname{grad} \operatorname{div} \boldsymbol{u} \times \operatorname{grad} \lambda - \mu \operatorname{rot} \operatorname{rot} \operatorname{rot} \boldsymbol{u} - \operatorname{grad} \mu \times \operatorname{rot} \operatorname{rot} \boldsymbol{u} + \operatorname{rot} \{ \operatorname{grad} \mu . [\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^T] \} .$$

Let us express the last equation for the r-component only,

(4.21)
$$\varrho_0 \frac{\partial^2 \omega_r}{\partial t^2} = -\mu (\operatorname{rot} \operatorname{rot} \operatorname{rot} \boldsymbol{u})_r + \frac{d\mu}{dr} (\operatorname{grad} \operatorname{rot} \boldsymbol{u})_{rr} ,$$

where we have put $\omega_r = (\mathrm{rot}\, \boldsymbol{u})_r$. If we use the relation

$$(\mathrm{rot\,rot\,rot}\, \boldsymbol{u})_r = -\nabla^2 \omega_r - \frac{2}{r}\left(\frac{\partial \omega_r}{\partial r} + \frac{\omega_r}{r}\right), \tag{4.22}$$

we can express Eq. (4.21) in the following form

$$\varrho_0 \frac{\partial^2 \omega_r}{\partial t^2} = \mu \nabla^2 \omega_r + \frac{2\mu}{r}\left(\frac{\partial \omega_r}{\partial r} + \frac{\omega_r}{r}\right) + \frac{\mathrm{d}\mu}{\mathrm{d}r}\frac{\partial \omega_r}{\partial r}. \tag{4.23}$$

We now apply the differential operator div to the equation of motion (4.17),

$$\begin{aligned} \varrho_0 \Bigg[\frac{\partial^2 \mathrm{div}\, \boldsymbol{u}}{\partial t^2} &+ \nabla^2(\varphi_1 + \boldsymbol{u}\,.\,\mathrm{grad}\, \varphi_0) - \\ -\mathrm{div}\,(\mathrm{div}\, \boldsymbol{u}\, \mathrm{grad}\, \varphi_0) \Bigg] &+ \mathrm{grad}\, \varrho_0 \,.\, \Bigg[\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \mathrm{grad}\, \varphi_1 + \\ &+ \mathrm{grad}\,(\boldsymbol{u}\,.\,\mathrm{grad}\, \varphi_0) - \mathrm{div}\, \boldsymbol{u}\, \mathrm{grad}\, \varphi_0 \Bigg] = \\ = (\lambda + 2\mu) \nabla^2 \mathrm{div}\, \boldsymbol{u} &+ \mathrm{grad}\,(\lambda + 2\mu)\,.\,\mathrm{grad\,div}\, \boldsymbol{u} + \\ &+ \mathrm{div}\,(\mathrm{div}\, \boldsymbol{u}\, \mathrm{grad}\, \lambda) - \mathrm{grad}\, \mu\,.\,\mathrm{rot\,rot}\, \boldsymbol{u} + \\ + \mathrm{grad}\, \mu\,.\,[\mathrm{div\,grad}\, \boldsymbol{u} &+ \mathrm{grad\,div}\, \boldsymbol{u}] + \mathrm{tr}\,\{\mathrm{grad\,grad}\, \mu\,.\,[\mathrm{grad}\, \boldsymbol{u} + (\mathrm{grad}\, \boldsymbol{u})^{\mathrm{T}}]\}, \end{aligned} \tag{4.24}$$

where we have used the relation for vector $\boldsymbol{u}$ and the 2nd-order tensor $\mathbf{A}$,

$$\mathrm{div}\,(\boldsymbol{u}\,.\,\mathbf{A}) = \boldsymbol{u}\,.\,\mathrm{div}\, \mathbf{A}^{\mathrm{T}} + \mathrm{tr}\,(\mathrm{grad}\, \boldsymbol{u}\,.\,\mathbf{A}). \tag{4.25}$$

Equation (4.24) can be further modified to read

$$\begin{aligned} \varrho_0 \left[\frac{\partial^2 \Delta}{\partial t^2} + \nabla^2(\varphi_1 + g_0 u) - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_0 \Delta) \right] &+ \\ + \frac{\mathrm{d}\varrho_0}{\mathrm{d}r}\left[\frac{\partial^2 u}{\partial t^2} - g_0 \Delta + \frac{\partial}{\partial r}(\varphi_1 + g_0 u) \right] &= (\lambda + 2\mu) \nabla^2 \Delta + \\ + \frac{\partial \Delta}{\partial r}\frac{\mathrm{d}}{\mathrm{d}r}(\lambda + 2\mu) &+ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \Delta \frac{\mathrm{d}\lambda}{\mathrm{d}r}\right) + \\ + 2\frac{\mathrm{d}\mu}{\mathrm{d}r}\left[\nabla^2 u - \frac{2}{r}\Delta + \frac{2}{r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \right] &+ 2\frac{\mathrm{d}^2 \mu}{\mathrm{d}r^2}\frac{\partial u}{\partial r}. \end{aligned} \tag{4.26}$$

We have derived the equations of motion (4.18), (4.23) and (4.26) from the equations of motion (4.17). Equation (4.23) only contains the variable ω_r, whereas the remaining two equations contain the variables u and Δ. It folows that the general solution of the equations of motion of the free oscillations of the SNREI Earth model consists of two parts. From the part for which u and Δ are zero, whereas ω_r satisfies Eq. (4.23), and from the part for which ω_r is zero, whereas u and Δ satisfy Eq. (4.18) and Eq. (4.26). In the latter case, Poisson's equation (4.12) for the additional potential φ_1 is added to the equations of motion (4.18) and (4.26).

4.3. Toroidal free oscillations

Let us first study the oscillations for which u and Δ are equal to zero. Such oscillations are referred to as toroidal, because the radial component of the displacement vector is equal to zero (the horizontal components are non-zero), or also as torsional, because the volume dilatation Δ is equal to zero. Since no radial motions occur and the volume changes are zero, it follows from (3.13) that toroidal oscillations are not accompanied by changes of density, $\varrho_1 = 0$, nor by disturbances of the gravitational potential, $\varphi_1 = 0$.

Using the relation $u = 0$, the equation $\Delta = 0$ can be expressed in spherical coordinates, see (A.152), as follows:

$$\frac{\partial(v \sin \vartheta)}{\partial \vartheta} + \frac{\partial w}{\partial \varphi} = 0 . \tag{4.27}$$

This yields the components of the displacement vector,

$$v = \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \varphi}, \quad w = -\frac{\partial f}{\partial \vartheta}, \tag{4.28}$$

where f is any function of the coordinates and time. Since we are studying the free oscillations of a body of finite dimensions, the eigenfrequencies will be discrete values. We, therefore, resolve the displacement vector into an infinity sum of normal oscillations with discrete angular frequencies ω_n, i.e.

$$f(r, \vartheta, \varphi, t) = \sum_{n=0}^{\infty} F_n(r, \vartheta, \varphi) \, e^{i\omega_n t} . \tag{4.29}$$

For the SNREI Earth model, let us seek to determine the functions $F_n(r, \vartheta, \varphi)$ in a partly separated form,

$$F_n(r, \vartheta, \varphi) = W_n(r) \, S_n(\vartheta, \varphi) , \tag{4.30}$$

where $S_n(\vartheta, \varphi)$ is a spherical function of the nth degree which satisfies the differential equation

$$\nabla_\Omega^2 S_n(\vartheta, \varphi) = -n(n+1) S_n(\vartheta, \varphi) . \tag{4.31}$$

The symbol ∇_Ω^2 represents the angular part of Laplace's operator ∇^2. By using Eq. (A.154), we can express (4.31) as

$$\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta}\left(\sin \vartheta \frac{\partial S_n}{\partial \vartheta}\right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 S_n}{\partial \varphi^2} + n(n+1) S_n = 0 . \tag{4.32}$$

To summarize, we are seeking to determine function $f(r, \vartheta, \varphi, t)$ in the following form:

$$f(r, \vartheta, \varphi, t) = \sum_{n=0}^{\infty} W_n(r)\, S_n(\vartheta, \varphi)\, e^{i\omega_n t}. \tag{4.33}$$

It also holds that

$$S_n(\vartheta, \varphi) = \sum_{m=-n}^{n} Y_{nm}(\vartheta, \varphi), \tag{4.34}$$

where $Y_{nm}(\vartheta, \varphi)$ is a fully normalized spherical function of degree n and order m, expressed in terms of the associated Legendre function $P_{nm}(\cos\vartheta)$ by the formula [1, 12, 77]

$$Y_{nm}(\vartheta, \varphi) = e^{im\varphi} \sqrt{\left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]}\, P_{nm}(\cos\vartheta). \tag{4.35}$$

If we substitute expansion (4.33) into Eq. (4.28), we can express the components of the displacement vector for a single given normal toroidal oscillations with the angular frequency ω_n in the following norm:

$$\begin{aligned} u &= 0, \\ v &= \frac{W_n(r)}{\sin\vartheta} \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi} e^{i\omega_n t}, \\ w &= -W_n(r) \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta} e^{i\omega_n t}. \end{aligned} \tag{4.36}$$

Now, substitute (4.36) into the equation of motion (4.23) and use Eq. (4.32). For $n \neq 0$ this yields an ordinary 2nd-order differential equation for function $W_n(r)$,

$$\begin{aligned} &\mu\left(\frac{d^2 W_n}{dr^2} + \frac{2}{r}\frac{dW_n}{dr}\right) + \frac{d\mu}{dr}\left(\frac{dW_n}{dr} - \frac{W_n}{r}\right) + \\ &+ \left[\varrho_0 \omega_n^2 - \frac{n(n+1)\mu}{r^2}\right] W_n = 0. \end{aligned} \tag{4.37}$$

The components of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ on spherical surface r then read

$$\begin{aligned} \bar{t}_{rr} &= 0, \\ \bar{t}_{r\vartheta} &= 2\mu e_{r\vartheta} = \mu\left(\frac{dW_n}{dr} - \frac{W_n}{r}\right)\frac{1}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi} e^{i\omega_n t}, \\ \bar{t}_{r\varphi} &= 2\mu e_{r\varphi} = -\mu\left(\frac{dW_n}{dr} - \frac{W_n}{r}\right)\frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta} e^{i\omega_n t}. \end{aligned} \tag{4.38}$$

The boundary conditions for displacement vector $\boldsymbol{u}$ and for the vector of boundary conditions $\boldsymbol{b}$, which is given by Eq. (4.16), are expressed by relations (3.32).

The horizontal component of the displacement vector (the normal component is zero) and the components $\bar{t}_{r\vartheta}$ and $\bar{t}_{r\varphi}$ of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ must be continuous across a solid undeformed boundary. This yields the following conditions for function $W_n(r)$ at the solid boundary σ:

$$[W_n(r)]_-^+ = 0, \tag{4.39}$$

$$\left[\mu\left(\frac{\mathrm{d}W_n}{\mathrm{d}r} - \frac{W_n}{r}\right)\right]_-^+ = 0$$

At a liquid and free boundary, the horizontal components of the displacement vector may vary arbitrarily, and components $\bar{t}_{r\vartheta}$ and $\bar{t}_{r\varphi}$ are zero at these boundaries. This yields the following condition for function $W_n(r)$ at liquid and free boundaries:

$$\mu\left(\frac{\mathrm{d}W_n}{\mathrm{d}r} - \frac{W_n}{r}\right) = 0. \tag{4.40}$$

It is advantageous to transform the ordinary 2nd-order differential equation (4.37) for function $W_n(r)$ with boundary conditions (4.39) and (4.40) to a system of two ordinary differential equations of the 1st order with simpler boundary conditions. For this purpose we shall introduce new variables

$$y_1 = \tilde{W}_n,\ y_2 = \mu\left(\frac{\mathrm{d}W_n}{\mathrm{d}r} - \frac{W_n}{r}\right), \tag{4.41}$$

where $y_1(r)$ is the radial part of horizontal displacements and $y_2(r)$ the radial part of horizontal stresses for a given normal toroidal oscillation with frequency ω_n, as

$$u = 0, \tag{4.42}$$

$$v = \frac{y_1(r)}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial\varphi}\mathrm{e}^{\mathrm{i}\omega_n t},$$

$$w = -y_1(r)\frac{\partial S_n(\vartheta, \varphi)}{\partial\vartheta}\mathrm{e}^{\mathrm{i}\omega_n t},$$

$$\bar{t}_{rr} = 0, \tag{4.43}$$

$$\bar{t}_{r\vartheta} = \frac{y_2(r)}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial\varphi}\mathrm{e}^{\mathrm{i}\omega_n t},$$

$$\bar{t}_{r\varphi} = -y_2(r)\frac{\partial S_n(\vartheta, \varphi)}{\partial\vartheta}\mathrm{e}^{\mathrm{i}\omega_n t}.$$

The ordinary 2nd-order differential equation (4.37), after substituting (4.41), changes to an equivalent system of two ordinary differential equations of the 1st order,

$$\frac{dy_1}{dr} = \frac{1}{r} y_1 + \frac{1}{\mu} y_2 , \tag{4.44}$$

$$\frac{dy_2}{dr} = \left[\frac{\mu(n-1)(n+2)}{r^2} - \varrho_0 \omega_n^2 \right] y_1 - \frac{3}{r} y_2 .$$

At a solid boundary, functions $y_1(r)$ and $y_2(r)$ must be continuous, at liquid and free boundaries function $y_1(r)$ is arbitrary, but finite, and function $y_2(r)$ zero,

$$y_2(r) = 0 . \tag{4.45}$$

4.4. Spheroidal free oscillations

We shall now deal with the second type of free oscillations for which the r-component of the vector rot $\boldsymbol{u}$ is zero, $\omega_r = 0$, but the quantities u and $\Delta = \operatorname{div} \boldsymbol{u}$ are non-zero. Such oscillations are referred to as spheroidal. Since the volume dilatation is non-zero, we must also take disturbances of the gravitational potential into account.

In spherical coordinates, equation $\omega_r = 0$ reads (see A.153))

$$\frac{\partial (w \sin \vartheta)}{\partial \vartheta} - \frac{\partial v}{\partial \varphi} = 0 . \tag{4.46}$$

We may thus put

$$v = \frac{\partial f}{\partial \vartheta} , \quad w = \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \varphi} , \tag{4.47}$$

where f is a function of the coordinates and time. We shall again seek to determine this function in the form of (4.33),

$$f(r, \vartheta, \varphi, t) = \sum_{n=0}^{\infty} V_n(r) \, S_n(\vartheta, \varphi) \, e^{i\omega_n t} . \tag{4.48}$$

Similarly, we seek to determine the radial displacement component and the additional gravitational potential in the form of series:

$$u(r, \vartheta, \varphi, t) = \sum_{n=0}^{\infty} U_n(r) \, S_n(\vartheta, \varphi) \, e^{i\omega_n t} , \tag{4.49}$$

$$\varphi_1(r, \vartheta, \varphi, t) = \sum_{n=0}^{\infty} F_n(r) \, S_n(\vartheta, \varphi) \, e^{i\omega_n t} .$$

For a particular normal spheroidal oscillations with frequency ω_n the components of the displacement vector are

(4.50)
$$u = U_n(r)\, S_n(\vartheta, \varphi)\, e^{i\omega_n t},$$
$$v = V_n(r)\, \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\, e^{i\omega_n t},$$
$$w = \frac{V_n(r)}{\sin \vartheta}\, \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\, e^{i\omega_n t}.$$

and the additional gravitational potential

(4.51)
$$\varphi_1 = F_n(r)\, S_n(\vartheta, \varphi)\, e^{i\omega_n t}.$$

With the aid of Eqs (4.50) it is easy to derive the formula for the relative change of volume under strain,

(4.52)
$$\Delta = X_n(r)\, S_n(\vartheta, \varphi)\, e^{i\omega_n t},$$

where

(4.53)
$$X_n(r) = \frac{dU_n}{dr} + \frac{2U_n}{r} - \frac{n(n+1)\, V_n}{r}.$$

Poisson's equation (4.12) will reduce to an ordinary 2nd-order differential equation for function $F_n(r)$,

(4.54)
$$\frac{d^2F_n}{dr^2} + \frac{2}{r}\frac{dF_n}{dr} - \frac{n(n+1)\, F_n}{r^2} = -4\pi G\left(\varrho_0 X_n + U_n \frac{d\varrho_0}{dr}\right).$$

After substituting Eqs (4.50)—(4.53) into the equations of motion (4.18) and (4.26) and by using Eq. (4.32), we arrive at

(4.55)
$$\varrho_0 \omega_n^2 U_n + \varrho_0 g_0 X_n - \varrho_0 \frac{dF_n}{dr} - \varrho_0 \frac{d}{dr}(g_0 U_n) + \frac{d}{dr}\left(\lambda X_n + 2\mu \frac{dU_n}{dr}\right) + \mu\left[\frac{4}{r}\frac{dU_n}{dr} - \frac{4}{r^2} U_n + n(n+1)\left(-\frac{U_n}{r^2} - \frac{1}{r}\frac{dV_n}{dr} + \frac{3V_n}{r^2}\right)\right] = 0,$$

(4.56)
$$\varrho_0 \omega_n^2 r V_n - \varrho_0 F_n - \varrho_0 g_0 U_n + \lambda X_n + r\frac{d}{dr}\left[\mu\left(\frac{dV_n}{dr} - \frac{V_n}{r} + \frac{U_n}{r}\right)\right] + \frac{\mu}{r}\left[5U_n + 3r\frac{dV_n}{dr} - V_n - 2n(n+1)\, V_n\right] = 0.$$

Equations (4.54)—(4.56) represent a system of three ordinary differential equations of the 2nd order for functions U_n, V_n and F_n. These equations were first derived in [8].

The components of the increment of Cauchy's stress tensor $\bar{\mathbf{t}}$ on spherical surface r read

$$\bar{t}_{rr} = \lambda\Delta + 2\mu e_{rr} = \left(\lambda X_n + 2\mu \frac{\mathrm{d}U_n}{\mathrm{d}r}\right) S_n(\vartheta, \varphi)\, \mathrm{e}^{\mathrm{i}\omega_n t}, \tag{4.57}$$

$$\bar{t}_{r\vartheta} = 2\mu e_{r\vartheta} = \mu\left(\frac{\mathrm{d}V_n}{\mathrm{d}r} - \frac{V_n}{r} + \frac{U_n}{r}\right) \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\, \mathrm{e}^{\mathrm{i}\omega_n t},$$

$$\bar{t}_{r\varphi} = 2\mu e_{r\varphi} = \mu\left(\frac{\mathrm{d}V_n}{\mathrm{d}r} - \frac{V_n}{r} + \frac{U_n}{r}\right) \frac{1}{\sin\vartheta} \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\, \mathrm{e}^{\mathrm{i}\omega_n t}.$$

The boundary conditions for the displacement vector $\boldsymbol{u}$, for the vector of boundary conditions $\boldsymbol{b}$, given by Eq. (4.16)$_1$, and the additional gravitational potential φ_1 are expressed by Eqs (3.32). These yield the boundary conditions for functions $U_n(r)$, $V_n(r)$ and $F_n(r)$,

(4.58) solid boundary

$$[U_n(r)]_-^+ = 0,$$

$$[V_n(r)]_-^+ = 0,$$

$$\left[\lambda X_n + 2\mu \frac{\mathrm{d}U_n}{\mathrm{d}r}\right]_-^+ = 0,$$

$$\left[\mu\left(\frac{\mathrm{d}V_n}{\mathrm{d}r} - \frac{V_n}{r} + \frac{U_n}{r}\right)\right]_-^+ = 0,$$

$$[F_n(r)]_-^+ = 0,$$

$$\left[\frac{\mathrm{d}F_n}{\mathrm{d}r} + 4\pi G\varrho_0 U_n\right]_-^+ = 0,$$

(4.59) liquid boundary

$$[U_n(r)]_-^+ = 0,$$

$$\left[\lambda X_n + 2\mu \frac{\mathrm{d}U_n}{\mathrm{d}r}\right]_-^+ = 0,$$

$$\mu\left(\frac{\mathrm{d}V_n}{\mathrm{d}r} - \frac{V_n}{r} + \frac{U_n}{r}\right) = 0,$$

$$[F_n(r)]_-^+ = 0,$$

$$\left[\frac{\mathrm{d}F_n}{\mathrm{d}r} + 4\pi G\varrho_0 U_n\right]_-^+ = 0,$$

(4.60) free boundary
$$\lambda X_n + 2\mu \frac{dU_n}{dr} = 0,$$
$$\mu\left(\frac{dV_n}{dr} - \frac{V_n}{r} + \frac{U_n}{r}\right) = 0,$$
$$[F_n]_-^+ = 0,$$
$$\frac{dF_n}{dr} + \frac{n+1}{r} F_n + 4\pi G \varrho_0 U_n = 0.$$

Let us derive the last boundary condition $(4.60)_4$. Let φ_1^i denote the additional gravitational potential within the Earth model, φ_1^e the additional gravitational potential in free space outside the model. At the surface of the Earth model $(r = a)$ according to condition (3.32),

(4.61)
$$\varphi_1^i = \varphi_1^e,$$
$$\frac{\partial \varphi_1^i}{\partial r} + 4\pi G \varrho_0 u = \frac{\partial \varphi_1^e}{\partial r}.$$

The additional potential φ_1^e outside the Earth can be expressed as [12]

(4.62)
$$\varphi_1^e(r, \vartheta, \varphi) = \sum_{n=0}^{\infty} F_n^e(r)\, S_n(\vartheta, \varphi) =$$
$$= \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{n+1} S_n(\vartheta, \varphi) \quad \text{for } r \geq a.$$

Consequently, function $F_n^e(r)$ satisfies the equation

(4.63)
$$\frac{dF_n^e}{dr} = -\frac{n+1}{r} F_n^e.$$

Equation $(4.61)_2$ may then be altered to read

(4.64)
$$\frac{dF_n}{dr} + \frac{n+1}{r} F_n + 4\pi G \varrho_0 U_n = 0 \quad \text{for } r = a,$$

in which the quantities involved refer only to the internal regions of the Earth model.

It is again convenient to introduce new variables [8],

(4.65)
$$y_1 = U_n,$$
$$y_2 = \lambda X_n + 2\mu \frac{dU_n}{dr},$$

$X_n = \frac{dU}{dr} + \frac{2U}{r} - \frac{n(n+1)V}{r}$

$$y_3 = V_n,$$
$$y_4 = \mu\left(\frac{\mathrm{d}V_n}{\mathrm{d}r} - \frac{V_n}{r} + \frac{U_n}{r}\right),$$
$$y_5 = F_n,$$
$$y_6 = \frac{\mathrm{d}F_n}{\mathrm{d}r} + \frac{n+1}{r}F_n + 4\pi G\varrho_0 U_n,$$

where y_1 is the radial part of vertical displacement, y_2 the radial part of vertical stress, y_3 the radial part of horizontal displacements, y_4 the radial part of horizontal stresses, y_5 the radial part of the additional gravitational potential, and y_6 the radial part of the gradient of the additional gravitational potential for a single normal spheroidal oscillation with frequency ω_n, because it holds that

(4.66)
$$u = y_1(r)\, S_n(\vartheta, \varphi)\, \mathrm{e}^{\mathrm{i}\omega_n t},$$
$$v = y_3(r)\frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\mathrm{e}^{\mathrm{i}\omega_n t},$$
$$w = \frac{y_3(r)}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\mathrm{e}^{\mathrm{i}\omega_n t},$$

(4.67)
$$\bar{t}_{rr} = y_2(r)\, S_n(\vartheta, \varphi)\, \mathrm{e}^{\mathrm{i}\omega_n t},$$
$$\bar{t}_{r\vartheta} = y_4(r)\frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\mathrm{e}^{\mathrm{i}\omega_n t},$$
$$\bar{t}_{r\varphi} = \frac{y_4(r)}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\mathrm{e}^{\mathrm{i}\omega_n t},$$

(4.68)
$$\varphi_1 = y_5(r)\, S_n(\vartheta, \varphi)\, \mathrm{e}^{\mathrm{i}\omega_n t}.$$

After applying substitution (4.65), the system of three ordinary differential equations of the 2nd order (4.54)—(4.56) will convert to the following system of six ordinary differential equations of the 1st order,

(4.69)
$$\frac{\mathrm{d}y_1}{\mathrm{d}r} = -\frac{2\lambda}{\lambda + 2\mu}\frac{y_1}{r} + \frac{y_2}{\lambda + 2\mu} + \frac{N\lambda}{\lambda + 2\mu}\frac{y_3}{r},$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}r} = \left[-\varrho_0\omega_n^2 - \frac{4\varrho_0 g_0}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2}\right]y_1 - \frac{4\mu}{\lambda + 2\mu}\frac{y_2}{r} +$$
$$+ N\left[\frac{\varrho_0 g_0}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2}\right]y_3 + N\frac{y_4}{r} - \frac{(n+1)\varrho_0}{r}y_5 +$$
$$+ \varrho_0 y_6,$$
$$\frac{\mathrm{d}y_3}{\mathrm{d}r} = -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu},$$

$$\frac{\mathrm{d}y_4}{\mathrm{d}r} = \left[\frac{\varrho_0 g_0}{r} - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu) r^2}\right] y_1 - \frac{\lambda}{\lambda + 2\mu}\frac{y_2}{r} + \left\{-\varrho_0 \omega_n^2 + \frac{1}{r^2}\left[\frac{4\mu N(\lambda + \mu)}{\lambda + 2\mu} - 2\mu\right]\right\} y_3 - \frac{3}{r} y_4 - \frac{\varrho_0}{r} y_5 ,$$

$$\frac{\mathrm{d}y_5}{\mathrm{d}r} = -4\pi G \varrho_0 y_1 - \frac{n+1}{r} y_5 + y_6 ,$$

$$\frac{\mathrm{d}y_6}{\mathrm{d}r} = -4\pi G \varrho_0 (n+1)\frac{y_1}{r} + 4\pi G \varrho_0 N \frac{y_3}{r} + (n-1)\frac{y_6}{r} ,$$

where $N = n(n+1)$. We shall use this system of differential equations to calculate the spheroidal free oscillations in the solid regions of the Earth model. In the liquid regions,

(4.70) $$\mu = 0 , \quad y_2 = \lambda X_n , \quad y_4 = 0 .$$

Equations (4.69) then reduce to

(4.71) $$\frac{\mathrm{d}y_1}{\mathrm{d}r} = -\frac{2y_1}{r} + \frac{y_2}{\lambda} + N\frac{y_3}{r} ,$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}r} = -\left(\varrho_0 \omega_n^2 + \frac{4\varrho_0 g_0}{r}\right) y_1 + N\frac{\varrho_0 g_0}{r} y_3 - \frac{(n+1)\varrho_0}{r} y_5 + \varrho_0 y_6 ,$$

$$\frac{\mathrm{d}y_5}{\mathrm{d}r} = -4\pi G \varrho_0 y_1 - \frac{n+1}{r} y_5 + y_6 ,$$

$$\frac{\mathrm{d}y_6}{\mathrm{d}r} = -4\pi G \varrho_0 (n+1)\frac{y_1}{r} + 4\pi G \varrho_0 N \frac{y_3}{r} + (n-1)\frac{y_6}{r} ,$$

$$y_3 = \frac{1}{r\omega_n^2}\left(g_0 y_1 - \frac{y_2}{\varrho_0} + y_5\right) .$$

The boundary conditions are as follows:

Solid boundary: all functions y_i must be continuous and finite at the origin ($r = 0$);

Liquid boundary: y_1, y_2, y_5 and y_6 must be continuous, $y_4 = 0$, y_3 arbitrary and may even change at a jump;

Free boundary: $y_2 = y_4 = y_6 = 0$.

4.5. Radial free oscillations

A special type of spheroidal free oscillations are oscillations with $n = 0$. These oscillations are called radial, because only the radial component of the displacement vector is non-zero. As Eq. (4.50) indicates, the horizontal components of displacement are zero,

$$u = U(r)\,e^{i\omega t}, \; v = w = 0 . \tag{4.72}$$

Equations (4.52) and (4.53) yield the relation

$$\Delta = \left(\frac{dU}{dr} + \frac{2U}{r}\right) e^{i\omega t} . \tag{4.73}$$

With a view to (4.51), the additional gravitational potential

$$\varphi_1 = F(r)\,e^{i\omega t}, \tag{4.74}$$

where function $F(r)$ satisfies Eq. (4.54) which, in this particular case, takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right) + \frac{4\pi G}{r^2}\frac{d}{dr}(r^2\varrho_0 U) = 0 . \tag{4.75}$$

By integrating (4.75) with respect to r we arrive at

$$\frac{dF}{dr} + 4\pi G\varrho_0 U = 0 . \tag{4.76}$$

and this yields the relation for function y_6:

$$y_6 = \frac{1}{r}y_5 = \frac{1}{r}F . \tag{4.77}$$

The system of differential equations (4.69) reduces to a system of two differential equations,

$$\frac{dy_1}{dr} = -\frac{2\lambda}{\lambda + 2\mu}\frac{y_1}{r} + \frac{y_2}{\lambda + 2\mu}, \tag{4.78}$$

$$\frac{dy_2}{dr} = \left[-\varrho_0\omega^2 - \frac{4\varrho_0 g_0}{r} + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)r^2}\right]y_1 - \frac{4\mu}{\lambda + 2\mu}\frac{y_2}{r} .$$

The boundary conditions require that functions y_1 and y_2 be continuous at the solid and liquid boundaries. At the free boundary y_2 must equal zero.

The radial part of the gravitational potential can then be derived from the relation

$$\frac{dy_5}{dr} = -4\pi G \varrho_0 y_1 . \tag{4.79}$$

To conclude, let us express the displacement vector and the increment of Cauchy's stress tensor on a spherical surface in a uniform way for toroidal and spheroidal free oscillations. Let us resolve the gradient nabla operator to read

$$\nabla = \boldsymbol{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \nabla_s , \tag{4.80}$$

where ∇_s is the surface gradient nabla operator on the spherical surface give by the relation

$$\nabla_s = \boldsymbol{e}_\vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{\sin \vartheta} \boldsymbol{e}_\varphi \frac{\partial}{\partial \varphi} \tag{4.81}$$

and $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$, $\boldsymbol{e}_\varphi$ are unit vectors in the directions of r, ϑ, φ. The displacement vector for a single normal free oscillations with frequency ω_n may then be expressed as

$$\boldsymbol{u}(\boldsymbol{r}) = \{U_n(r) S_n(\vartheta, \varphi) \boldsymbol{e}_r + V_n(r) \nabla_s S_n(\vartheta, \varphi) - W_n(r) [\boldsymbol{e}_r \times \nabla_s S_n(\vartheta, \varphi))]\} e^{i\omega_n t} . \tag{4.82}$$

Functions $U_n(r)$, $V_n(r)$ and $W_n(r)$ satisfies the differential equations (4.54)—(4.56) and (4.37), respectively.

The increment of Cauchy's stress tensor on a spherical surface is

$$\boldsymbol{n} . \bar{\mathbf{t}} = \boldsymbol{e}_r . \bar{\mathbf{t}} = (\bar{t}_{rr}, \bar{t}_{r\vartheta}, \bar{t}_{r\varphi}) . \tag{4.83}$$

For a single normal free oscillation with frequency ω_n

$$\boldsymbol{n} . \bar{\mathbf{t}}(\boldsymbol{r}) = \{P_n(r) S_n(\vartheta, \varphi) \boldsymbol{e}_r + Q_n(r) \nabla_s S_n(\vartheta, \varphi) - R_n(r) [\boldsymbol{e}_r \times \nabla_s S_n(\vartheta, \varphi)]\} e^{i\omega_n t}, \tag{4.84}$$

where

$$P_n = \lambda \left(\frac{dU_n}{dr} + \frac{2U_n}{r} - \frac{n(n+1) V_n}{r} \right) + 2\mu \frac{dU_n}{dr}, \tag{4.85}$$

$$Q_n = \mu \left(\frac{dV_n}{dr} - \frac{V_n}{r} + \frac{U_n}{r} \right),$$

$$R_n = \mu \left(\frac{dW_n}{dr} - \frac{W_n}{r} \right).$$

5. FREE OSCILLATIONS OF A HOMOGENEOUS MODEL OF THE EARTH

We shall now study the free oscillations of an Earth model which is homogeneous and isotropic, i.e. we shall assume that the density ϱ_0 and Lamé's constants λ and μ are constant and independent of the coordinates. This problem is important especially in two cases: a) in the matrix solution of the free oscillations of the Earth, b) in defining the initial values of numerical integration of derived ordinary differential equations of free oscillations.

The free oscillations of uniform sphere were first considered by Lamb [90] and the additional effect of gravity was included by Love [95]. Several authors [118, 134] have dealt with this problem, especially with regard to defining the initial values of numerical integration. However, the solutions given are incomplete with respect to the matrix solution of free oscillations. Application of Thomson—Haskell method to the calculatiion for the torsional oscillations was made by Gilbert and McDonald [72] and to the Rayleigh waves in spherical medium was made by Bhattacharya [22, 23, 24], Gaulon et al. [66], Teng [136]. We shall now deal with the detailed solution of this problem.

5.1. Toroidal free oscillations

This type of oscillation is described by the equation of motion (4.37) which, in the case of a homogeneous Earth model, takes the form

$$\frac{d^2W_n}{dr^2} + \frac{2}{r}\frac{dW_n}{dr} + \left[\frac{\varrho_0\omega_n^2}{\mu} - \frac{n(n+1)}{r^2}\right]W_n = 0. \tag{5.1}$$

If we define the $\mathscr{L}^2$ operator as

$$\mathscr{L}^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{n(n+1)}{r^2}, \tag{5.2}$$

Eq. (5.1) can be expressed in a more compact form,

$$(\mathscr{L}^2 + k^2)\,W_n = 0, \tag{5.3}$$

where

$$k^2 = \varrho_0\omega_n^2/\mu, \quad x = kr. \tag{5.4}$$

Equations (5.1) and (5.4) are Bessel's differential equations for spherical wave functions. Their general solution is given by the linear combination either of Bessel's spherical functions of the 1st and 2nd kind,

$$j_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)}\cdot J_{n+\frac{1}{2}}(x), \quad y_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)}\cdot Y_{n+\frac{1}{2}}(x), \tag{5.5}$$

or of Hankel's spherical functions of the 1st and 2nd kind,

$$h_n^{(1)}(x) = j_n(x) + \mathrm{i}y_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} \, . \, H_{n+\frac{1}{2}}^{(1)}(x) \,, \tag{5.6}$$

$$h_n^{(2)}(x) = j_n(x) - \mathrm{i}y_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} \, . \, H_{n+\frac{1}{2}}^{(2)}(x) \,,$$

where $J_n(x)$ and $Y_n(x)$ are Bessel's functions of the 1st and 2nd kind, and $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are Hankel's functions of the 1st and 2nd kind. Only Bessel's spherical function of the 1st kind, $j_n(x)$, satisfies the condition of regularity at the origin. To save space, let us put

$$f_n(x) = \begin{cases} a_1 j_n(x) + a_2 y_n(x) \\ b_1 h_n^{(1)}(x) + b_2 h_n^{(2)}(x) \end{cases}, \tag{5.7}$$

where a_i, b_i, $i = 1, 2$, are constants. We shall use the following recurrent formulas [1, 12] in the computations that follow:

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x) \,, \tag{5.8}$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) \frac{\mathrm{d}f_n(x)}{\mathrm{d}x} \,,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [x^{n+1} f_n(x)] = x^{n+1} f_{n-1}(x) \,,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} [x^{-n} f_n(x)] = -x^{-n} f_{n+1}(x) \,.$$

Let us go back to Eq. (5.1) and (5.3). Their solution is

$$y_1(r) = W_n(r) = f_n(x) \,, \tag{5.9}$$

where x is given by (5.4) and $f_n(x)$ by (5.7). Using recurrent formulas (5.8) it is easy to to derive

$$y_2(r) = \mu \left(\frac{\mathrm{d}W_n}{\mathrm{d}r} - \frac{W_n}{r} \right) = \frac{\mu}{r} [(n-1) f_n(x) - x f_{n+1}(x)] \,. \tag{5.10}$$

5.2. Spheroidal free oscillations

Let us again consider a homogeneous Earth model. We shall first prove that for this model the acceleration of gravitation within the Earth model is a linear function of distance from the Earth's centre. Poisson's equation (4.4) applies,

$$\frac{dg_0}{dr} + \frac{2g_0}{r} = 4\pi G \varrho_0 , \tag{5.11}$$

in which ϱ_0 is now constant. The solution of this equation reads

$$g_0 = \gamma r, \ \gamma = \tfrac{4}{3}\pi G \varrho_0 . \tag{5.12}$$

For the homogeneous Earth model, the equations of motion (4.17), (4.20) and (4.24) take the following form:

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \operatorname{grad} \varphi_1 + \gamma \operatorname{grad}(ru) + \boldsymbol{g}_0 \Delta = \frac{\lambda + 2\mu}{\varrho_0} \operatorname{grad} \Delta - \frac{\mu}{\varrho_0} \operatorname{rot} \operatorname{rot} \boldsymbol{u} , \tag{5.13}$$

$$\frac{\partial^2 \operatorname{rot} \boldsymbol{u}}{\partial t^2} - \boldsymbol{g}_0 \times \operatorname{grad} \Delta = -\frac{\mu}{\varrho_0} \operatorname{rot} \operatorname{rot} \operatorname{rot} \boldsymbol{u} , \tag{5.14}$$

$$\frac{\partial^2 \Delta}{\partial t^2} - 3\gamma\Delta + \gamma \nabla^2 (ru) - \gamma \Delta - \gamma r \frac{\partial \Delta}{\partial r} = \frac{\lambda + 2\mu}{\varrho_0} \nabla^2 \Delta . \tag{5.15}$$

We shall now derive the relation for volume dilatation Δ. The components of vector rot $\boldsymbol{u}$ for spheroidal free oscillations can be expressed, using (A.153) and (4.50), as

$$\begin{aligned} \omega_r &= 0 , \\ \omega_\vartheta &= -H_n(r) \frac{1}{\sin \vartheta} \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi} e^{i\omega_n t} , \\ \omega_\varphi &= H_n(r) \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta} e^{i\omega_n t} , \end{aligned} \tag{5.16}$$

where

$$H_n(r) = \frac{dV_n}{dr} + \frac{V_n}{r} - \frac{U_n}{r} . \tag{5.17}$$

Using these relations, it is easy to derive the formulas

$$\begin{aligned} (\operatorname{rot} \operatorname{rot} \operatorname{rot} \boldsymbol{u})_\vartheta &= \mathscr{L}^2 H_n \frac{1}{\sin \vartheta} \frac{\partial S_n}{\partial \varphi} e^{i\omega_n t} , \\ (\operatorname{rot} \operatorname{rot} \operatorname{rot} \boldsymbol{u})_\varphi &= -\mathscr{L}^2 H_n \frac{\partial S_n}{\partial \vartheta} e^{i\omega_n t} , \end{aligned} \tag{5.18}$$

in which the $\mathscr{L}^2$-operator is defined by Eq. (5.2). It also holds that

$$(\boldsymbol{g}_0 \times \operatorname{grad} \Delta)_\vartheta = \gamma X_n(r) \frac{1}{\sin \vartheta} \frac{\partial S_n}{\partial \varphi} e^{i\omega_n t} , \tag{5.19}$$

$$(\boldsymbol{g}_0 \times \operatorname{grad} \Delta)_\varphi = -\gamma X_n(r) \frac{\partial S_n}{\partial \vartheta} e^{i\omega_n t},$$

where function $X_n(r)$ is defined by Eq. (4.53). Using Eqs (5.18) and (5.19) we can now express the ϑ- and φ-component of Eq. (5.14). In both cases we arrive at the equation

$$\frac{\mu}{\varrho_0} \mathscr{L}^2 H_n + \omega_n^2 H_n = \gamma X_n. \tag{5.20}$$

If we also make use of Eqs (A.122), (4.52) and (4.53), Eq. (5.15) becomes

$$\frac{\lambda + 2\mu}{\varrho_0} \mathscr{L}^2 X_n + (\omega_n^2 + 4\gamma) X_n = \gamma n(n+1) H_n. \tag{5.21}$$

By applying the operator $(\mu/\varrho_0)\mathscr{L}^2 + \omega_n^2$ to Eq. (5.21) and then using Eq. (5.20), we arrive at

$$\left(\frac{\mu}{\varrho_0} \mathscr{L}^2 + \omega_n^2\right)\left[\frac{\lambda + 2\mu}{\varrho_0} \mathscr{L}^2 X_n + (\omega_n^2 + 4\gamma) X_n\right] = \gamma^2 n(n+1) X_n. \tag{5.22}$$

An analogous equation holds for function H_n; indeed, if we apply the operator $[(\lambda + 2\mu)/\varrho_0] \mathscr{L}^2 + \omega_n^2 + 4\gamma$ to Eq. (5.20) and use Eq. (5.21), we obtain

$$\left(\frac{\mu}{\varrho_0} \mathscr{L}^2 + \omega_n^2\right)\left[\frac{\lambda + 2\mu}{\varrho_0} \mathscr{L}^2 H_n + (\omega_n^2 + 4\gamma) H_n\right] = \gamma^2 n(n+1) H_n. \tag{5.23}$$

We shall modify Eq. (5.22) to read

$$(\mathscr{L}^2 + k_1^2)(\mathscr{L}^2 + k_2^2) X_n = 0, \tag{5.24}$$

where

$$k_{1,2}^2 = \tfrac{1}{2}\left\{\frac{\omega_n^2 + 4\gamma}{\alpha^2} + \frac{\omega_n^2}{\beta^2} \pm \sqrt{\left[\left(\frac{\omega_n^2 + 4\gamma}{\alpha^2} - \frac{\omega_n^2}{\beta^2}\right)^2 + \frac{4n(n+1)\gamma^2}{\alpha^2 \beta^2}\right]}\right\} \tag{5.25}$$

and α, β are the velocities of longitudinal and transverse seismic waves. The general solution of Eq. (5.24) is given by Bessel's or Hankel's spherical functions of the 1st and 2nd kind which are denoted uniformly as $f_n(kr)$ — see (5.7),

$$X_n(r) = a_1 f_n(k_1 r) + a_2 f_n(k_2 r), \tag{5.26}$$

a_1 and a_2 being constants. Equation (5.20) now yields

$$H_n(r) = -\frac{a_1}{f_1} f_n(k_1 r) - \frac{a_2}{f_2} f_n(k_2 r), \tag{5.27}$$

where

$$f_{1,2} = \frac{1}{\gamma}(\beta^2 k_{1,2}^2 - \omega_n^2). \tag{5.28}$$

We have thus obtained the relations for functions $X_n(r)$ and $H_n(r)$. If we substitute (5.26) into (4.53) and (5.27) into (5.17), we obtain a system of two ordinary 1st-order differential equations,

$$\frac{dU_n}{dr} + \frac{2U_n}{r} - \frac{n(n+1)V_n}{r} = af_n(kr), \tag{5.29}$$

$$\frac{dV_n}{dr} + \frac{V_n}{r} - \frac{U_n}{r} = -\frac{a}{f}f_n(kr).$$

We first solve the homogeneous system by substituting $r = e^t$. The solution of the homogeneous system is

$$U_n^h(r) = b_1 r^{n-1} + b_2 r^{-n-2}, \tag{5.30}$$

$$V_n^h(r) = \frac{b_1}{n} r^{n-1} - \frac{b_2}{n+1} r^{-n-2}.$$

Using the method of variations of constants, $b_1 = b_1(r)$, $b_2 = b_2(r)$, we now solve system (5.29). By applying recurrent formulas (5.8), we obtain the particular solution:

$$rU_n(r) = nhf_n(x) - fxf_{n+1}(x), \tag{5.31}$$

$$rV_n(r) = hf_n(x) + xf_{n+1}(x),$$

where

$$x = kr, \ h = f - (n+1). \tag{5.32}$$

The general solution of the system of differential equations (5.29) is

$$ry_1(r) \equiv rU_n(r) = b_1 r^n + b_2 r^{-n-1} + nhf_n(x) - fxf_{n+1}(x), \tag{5.33}$$

$$ry_3(r) \equiv rV_n(r) = \frac{b_1}{n} r^n + \frac{b_2}{n+1} r^{-n-1} + hf_n(x) + xf_{n+1}(x),$$

b_1 and b_2 being constants. From this we determine the multiplicative constant of function $X_n(r)$,

$$X_n(r) = -k^2 ff_n(kr). \tag{5.34}$$

Function $y_5(r) = F_n(r)$ can be determined, for example, from Eq. (4.56):

$$y_5(r) = -b_1\left(\gamma - \frac{\omega_n^2}{n}\right)r^n - b_2\left(\gamma + \frac{\omega_n^2}{n+1}\right)r^{-n-1} - 3\gamma ff_n(kr). \tag{5.35}$$

The remaining functions $y_i(r)$ can be determined from the definition relations (4.65):

$$r^2 y_2(r) = 2\mu b_1 (n-1) r^n - 2\mu b_2 (n+2) r^{-n-1} - \\ - (\lambda + 2\mu) f x^2 f_n(kr) + 2\mu \{n(n-1) h f_n(kr) + [2f + n(n+1)] x f_{n+1}(kr)\}, \tag{5.36}$$

$$r^2 y_4(r) = 2\mu \frac{b_1}{n}(n-1) r^n + 2\mu b_2 \frac{n+2}{n+1} r^{-n-1} + \\ + \mu [x^2 f_n(kr) + 2(n-1) h f_n(kr) - 2(f+1) x f_{n+1}(kr)], \tag{5.37}$$

$$r y_6(r) = -(2n+1) b_1 \left(\gamma - \frac{\omega_n^2}{n}\right) r^n + 3\gamma b_1 r^n + \\ + 3\gamma b_2 r^{-n-1} - 3(2n+1)\gamma f f_n(kr) + 3n\gamma h f_n(kr). \tag{5.38}$$

If $\mu = 0$, one of the solutions (5.25) will vanish, and we are left with

$$k^2 = \frac{1}{\alpha^2}\left[\omega_n^2 + 4\gamma - \frac{n(n+1)\gamma^2}{\omega_n^2}\right], \quad f = -\frac{\omega_n^2}{\gamma}, \quad h = f - (n+1). \tag{5.39}$$

6. EQUATIONS OF MOTION OF SPHEROIDAL FREE OSCILLATIONS AT THE EARTH'S CENTRE

The equations of motion of spheroidal free oscillations (4.69) have a singular point at the origin of the coordinate system, which is placed in the Earth's centre of gravity. In integrating the equations of motion numerically, particularly if oscillations of low orders are involved, this singularity has to be eliminated. This problem can be solved in two ways. One way is defining the initial values of numerical integration by analytical solution of the equations of motion for a homogeneous medium (see Chapter 5). The second is to use the expansions of the equations of motion into a power series in r in the neighbourhood of the origin ($r = 0$). Although this problem has already been discussed in the literature [36], a detailed derivation has not been published yet. Since most of the more recent Earth models assume a solid inner core, we shall begin with the expansion of the equations of motion of spheroidal free oscillations for a solid medium.

Let us solve the system of equations of motion (4.69) for spheroidal oscillations of a solid elastic body in the neighbourhood of the origin by expansion into the series

$$y_i(r) = \sum_{m=0}^{\infty} A_{i,m} r^{m+k_i}, \qquad i = 1, 2, \ldots, 6. \tag{6.1}$$

where m, k_i are integral variables and $A_{i,m}$ are the coefficients we are seeking to determine. By differentiating (6.1) with respect to r, we arrive at

$$\frac{\mathrm{d}y_i(r)}{\mathrm{d}r} = \sum_{m=0}^{\infty} (m + k_i) A_{i,m} r^{m+k_i-1}. \tag{6.2}$$

However, Eqs (4.69) immediately yield the relation for the integral constants k_i,

$$k_2 = k - 1,\ k_3 = k,\ k_4 = k - 1,\ k_5 = k + 1,\ k_6 = k, \tag{6.3}$$

where $k = k_1$ is the only unknown integral constant. Equations (6.1) and (6.3) then yield the relation

$$y_i(r) = \sum_{m=0}^{\infty} A_{i,m} r^{m+k+j} = \sum_{m=0}^{\infty} A_{i,m} r^{s+j}, \tag{6.4}$$

for $i = 1, 2, \ldots, 6$, in which we have put $s = k + m$ to save space. We can determine the constant j in the following manner: $j = -1$ for $i = 2, 4$; $j = 0$ for $i = 1, 3, 6$; $j = 1$ for $i = 5$.

Let us now expand the gravitational acceleration g_0 into a series in the neighbourhood of the origin. Let

$$g_0 = \sum_{k=0}^{\infty} a_k r^k. \tag{6.5}$$

However, $a_0 = 0$ since $g_0 = 0$ for $r = 0$. By substituting into Poisson's equation (4.4) and comparing the coefficifents of the powers r^k, we find that $a_1 = \gamma$, $a_2 = a_3 = \ldots = 0$. Therefore,

$$g_0 = \gamma r,\quad \gamma = \tfrac{4}{3}\pi G \varrho_0, \tag{6.6}$$

where ϱ_0 is the density in the neighbourhood of the Earth's centre. Now substitute (6.6) into the equations of motion (4.69),

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}r} &= \frac{c_{11}}{r} y_1 + c_{12} y_2 + \frac{c_{13}}{r} y_3, \\
\frac{\mathrm{d}y_2}{\mathrm{d}r} &= \left(c_{21}^0 + \frac{c_{21}^1}{r^2}\right) y_1 + \frac{c_{22}}{r} y_2 + \left(c_{23}^0 + \frac{c_{23}^1}{r^2}\right) y_3 + \frac{c_{24}}{r} y_4 + \\
&\quad + \frac{c_{25}}{r} y_5 + c_{26} y_6, \\
\frac{\mathrm{d}y_3}{\mathrm{d}r} &= \frac{c_{31}}{r} y_1 + \frac{c_{33}}{r} y_3 + c_{34} y_4, \\
\frac{\mathrm{d}y_4}{\mathrm{d}r} &= \left(c_{41}^0 + \frac{c_{41}^1}{r^2}\right) y_1 + \frac{c_{42}}{r} y_2 + \left(c_{43}^0 + \frac{c_{43}^1}{r^2}\right) y_3 + \frac{c_{44}}{r} y_4 + \frac{c_{45}}{r} y_5, \\
\frac{\mathrm{d}y_5}{\mathrm{d}r} &= c_{51} y_1 + \frac{c_{55}}{r} y_5 + c_{56} y_6,
\end{aligned} \tag{6.7}$$

$$\frac{dy_6}{dr} = \frac{c_{61}}{r} y_1 + \frac{c_{63}}{r} y_3 + \frac{c_{66}}{r} y_6 ,$$

where

(6.8)
$$\begin{aligned}
&c_{11} = -2\lambda b,\ c_{12} = b,\ c_{13} = N\lambda b,\\
&c_{21}^0 = -\varrho_0 \omega_n^2 - 4\varrho_0 \gamma,\ c_{21}^1 = 2d,\\
&c_{22} = -4\mu b,\ c_{23}^0 = N\varrho_0\gamma,\ c_{23}^1 = -Nd,\\
&c_{24} = N,\ c_{25} = -(n+1)\varrho_0,\ c_{26} = \varrho_0,\\
&c_{31} = -1,\ c_{33} = 1,\ c_{34} = 1/\mu,\\
&c_{41}^0 = \varrho_0\gamma,\ c_{41}^1 = -d,\ c_{42} = -\lambda b,\\
&c_{43}^0 = -\varrho_0\omega_n^2,\ c_{43}^1 = 4N\mu(\lambda+\mu)b - 2\mu,\\
&c_{44} = -3,\ c_{45} = \varrho_0,\ c_{51} = -3\gamma,\\
&c_{55} = -(n+1),\ c_{56} = 1,\ c_{61} = -3\gamma(n+1),\\
&c_{63} = 3\gamma N,\ c_{66} = n-1.
\end{aligned}$$

In Eqs (6.8) we have put

(6.9)
$$\begin{aligned}
&b = 1/(\lambda + 2\mu),\ d = 2\mu(3\lambda + 2\mu)b,\\
&\gamma = \tfrac{4}{3}\pi G\varrho_0,\ N = n(n+1).
\end{aligned}$$

Let us substitute expansions (6.1) and (6.2) together with conditions (6.3) into Eqs (6.7). If we compare the coefficients of powers of r, we arrive at the following system of equations:

(6.10)
$$\begin{aligned}
&(c_{11} - s)A_{1,m} + c_{12}A_{2,m} + c_{13}A_{3,m} = 0,\\
&c_{31}A_{1,m} + (c_{33} - s)A_{3,m} + c_{34}A_{4,m} = 0, \qquad \text{for } m = 0, 1, 2, \ldots,
\end{aligned}$$

(6.11)
$$\begin{aligned}
&c_{21}^1 A_{1,m} + (c_{22} - s + 1)A_{2,m} + c_{23}^1 A_{3,m} + c_{24}A_{4,m} = 0,\\
&c_{41}^1 A_{1,m} + c_{42}A_{2,m} + c_{43}^1 A_{3,m} +\\
&\qquad + (c_{44} - s + 1)A_{4,m} = 0, \qquad \text{for } m = 0, 1,
\end{aligned}$$

(6.12)
$$\begin{aligned}
&c_{21}^0 A_{1,m-2} + c_{21}^1 A_{1,m} + (c_{22} - s + 1)A_{2,m} + c_{23}^0 A_{3,m-2} +\\
&+ c_{23}^1 A_{3,m} + c_{24}A_{4,m} + c_{25}A_{5,m-2} + c_{26}A_{6,m-2} = 0,\\
&c_{41}^0 A_{1,m-2} + c_{41}^1 A_{1,m} + c_{42}A_{2,m} + c_{43}^0 A_{3,m-2} + c_{43}^1 A_{3,m} +\\
&+ (c_{44} - s + 1)A_{4,m} + c_{45}A_{5,m-2} = 0, \qquad \text{for } m = 2, 3, \ldots,
\end{aligned}$$

(6.13)
$$\begin{aligned}
&c_{51}A_{1,m} + (c_{55} - s - 1)A_{5,m} + c_{56}A_{6,m} = 0,\\
&c_{61}A_{1,m} + c_{63}A_{3,m} + (c_{66} - s)A_{6,m} = 0, \qquad \text{for } m = 0, 1, 2, \ldots.
\end{aligned}$$

We can calculate the coefficients $A_{2,m}$, $A_{4,m}$, $A_{5,m}$ and $A_{6,m}$ from system (6.10) and (6.13) with the aid of the coefficients $A_{1,m}$ and $A_{3,m}$:

$$(6.14)\qquad A_{2,m} = \frac{s - c_{11}}{c_{12}} A_{1,m} - \frac{c_{13}}{c_{12}} A_{3,m},$$

$$A_{4,m} = -\frac{c_{31}}{c_{34}} A_{1,m} + \frac{s - c_{33}}{c_{34}} A_{3,m},$$

$$A_{5,m} = -3\gamma \frac{(s+2) A_{1,m} - N A_{3,m}}{(s+2+n)(s+1-n)},$$

$$A_{6,m} = -3\gamma \frac{(n+1) A_{1,m} - N A_{3,m}}{s+1-n}, \qquad \text{for } m = 0, 1, 2, \ldots .$$

If we substitute from (6.14) into (6.11) and (6.12), we arrive at the following system of equations:

$$(6.15)\qquad Q_{11}(s) A_{1,m} + Q_{12}(s) A_{3,m} = 0.$$

$$Q_{21}(s) A_{1,m} + Q_{22}(s) A_{3,m} = 0, \qquad \text{for } m = 0, 1,$$

$$(6.16)\qquad Q_{11}(s) A_{1,m} + Q_{12}(s) A_{3,m} + c_{21}^0 A_{1,m-2} + c_{23}^0 A_{3,m-2} + c_{25} A_{5,m-2} + c_{26} A_{6,m-2} = 0,$$

$$Q_{21}(s) A_{1,m} + Q_{22}(s) A_{3,m} + c_{41}^0 A_{1,m-2} + c_{43}^0 A_{3,m-2} + c_{45} A_{5,m-2} = 0, \qquad \text{for } m = 2, 3, \ldots .$$

In the preceding system we put

$$(6.17)\qquad Q_{11}(s) = c_{21}^1 + \frac{(c_{22} - s + 1)(s - c_{11})}{c_{12}} - \frac{c_{31} c_{24}}{c_{34}},$$

$$Q_{22}(s) = c_{43}^1 + \frac{(c_{44} - s + 1)(s - c_{33})}{c_{34}} - \frac{c_{13} c_{42}}{c_{12}},$$

$$Q_{12}(s) = c_{23}^1 - \frac{c_{13}}{c_{12}}(c_{22} - s + 1) + \frac{c_{24}}{c_{34}}(s - c_{33}),$$

$$Q_{21}(s) = c_{41}^1 + \frac{c_{42}}{c_{12}}(s - c_{11}) - \frac{c_{31}}{c_{34}}(c_{44} - s + 1),$$

If we substitute from (6.8) into (6.17), we obtain

$$(6.18)\qquad Q_{11}(s) = (\lambda + 2\mu)(1 - s)(s + 2) + N\mu,$$

$$Q_{22}(s) = -\mu s(s + 1) + N(\lambda + 2\mu),$$

$$Q_{12}(s) = N[s(\lambda + \mu) - \lambda - 3\mu],$$

$$Q_{21}(s) = -s(\lambda + \mu) - 2(\lambda + 2\mu).$$

The determinant of the system of equations (6.15) is

$$D_S(s) = Q_{11}(s)\,Q_{22}(s) - Q_{12}(s)\,Q_{21}(s). \tag{6.19}$$

By making use of Eqs (6.18), we obtain

$$D_S(s) = \mu(\lambda + 2\mu)[s^4 + 2s^3 - (2N + 1)s^2 - 2(N + 1)s + N(N - 2)]. \tag{6.20}$$

The determinant of (6.20) can then be expressed as

$$D_S(s) = \mu(\lambda + 2\mu)\,(s - n - 1)\,(s - n + 1)\,(s + n)\,(s + n + 2). \tag{6.21}$$

For the system of equations (6.15) to have a non-trivial solution, the determinant of the system must be equal to zero,

$$D_S(s) = 0, \qquad \text{for } m = 0,\, 1, \tag{6.22}$$

which is an algebraic equation of the 4th degree in the unknown parameters s or k. Its solution are the values $s \equiv k + m = n + 1,\ n - 1,\ -n,\ -n - 2$ for $m = 0, 1$. We shall eliminate the values $-n$, $-n - 2$, because functions $y_i(r)$ would then diverge at the origin of the coordinate system. Equation (6.4) yields

$$y_i(r) = A_i^- r^{n-1+j} + A_i^+ r^{n+1+j} + \sum_{m=2}^{\infty} (A_{i,m}^- r^{m+n-1+j} + A_{i,m}^+ r^{m+n+1+j}) \tag{6.23}$$

for $i = 1, 2, \ldots, 6$, where A_i represents the coefficients $A_{i,m}$ for $m = 0, 1$, the suffices $-$ and $+$ of the coefficients A_i and $A_{i,m}$ indicate whether the coefficient refers to parameter $s = n - 1$, or to $s = n + 1$.

Before attempting to determine the coefficifents $A_i^\pm$ and $A_{i,m}^\pm$, we shall determine the auxiliary quantities $Q_{ij}(s)$, $i, j = 1, 2$, for the values $s = n - 1$ and $s = n + 1$. Substituting these values into (6.18) yields

$$\begin{aligned} Q_{11}(n-1) &= -(n+1)[\lambda(n-2) + \mu(n-4)], \\ Q_{22}(n-1) &= n[\lambda(n+1) + \mu(n+3)], \\ Q_{12}(n-1) &= n(n+1)[\lambda(n-2) + \mu(n-4)], \\ Q_{21}(n-1) &= -[\lambda(n+1) + \mu(n+3)], \end{aligned} \tag{6.24}$$

$$\begin{aligned} Q_{11}(n+1) &= -\mu p_2, \\ Q_{22}(n+1) &= -\mu p_2/n, \\ Q_{12}(n+1) &= n(n+1)z_1, \\ Q_{21}(n+1) &= (n+1)z_1, \end{aligned} \tag{6.25}$$

where

$$\begin{aligned} p_2 &= n(n+5) + n(n+3)(\lambda/\mu), \\ z_1 &= \lambda n + \mu(n-2). \end{aligned} \tag{6.26}$$

The solution of Eq. (6.15) for $s = n - 1$ and $s = n + 1$ reads

(6.27) $$A_3^- = \frac{1}{n} A_1^- , \quad A_3^+ = \frac{\mu p_2}{N z_1} A_1^+ .$$

We have adopted A_1^- and A_1^+ as the independent coefficients. Coefficients $A_2^\pm$ and $A_4^\pm$ can be obtained from Eqs (6.14),

(6.28) $$A_2^- = 2(n-1)\mu A_1^- , \quad A_4^- = \frac{2(n-1)\mu}{n} A_1^- ,$$

(6.29) $$A_2^+ = \frac{2\mu}{z_1}[\lambda(n^2 - n - 3) + \mu(n^2 - n - 2)] A_1^+ , \quad A_4^+ = \frac{\mu p_1}{N z_1} A_1^+ ,$$

where

(6.30) $$p_1 = 2n^2(n+2)\lambda + 2n(n^2 + 2n - 1)\mu .$$

The coefficients A_5^- and A_6^- will be determined from the system of equations (6.13) for $s = n - 1$. Since the second equation in this system is satisfied identically for this value of parameter s, we shall adopt another independent coefficient, e.g. A_6^-; consequently,

(6.31) $$A_5^- = -\frac{3\gamma}{2n+1} A_1^- + \frac{1}{2n+1} A_6^- .$$

Coefficients A_5^+ and A_6^+ can be determined from Eqs (6.14) for $s = n + 1$:

(6.32) $$A_5^+ = \frac{3\gamma\mu}{z_1} A_1^+ , \quad A_6^+ = \frac{3\gamma}{z_1}[n\lambda + (3n+1)\mu] A_1^+ .$$

Let us now solve the system of equations (6.16) for $m = 2$ and $k = n - 1$ ($s = n + 1$), i.e. determine the coefficients $A_{i,2}^-$, $i = 1, 2, \ldots, 6$. For the given parameters m and k, this system contains two linearly independent equations, because

(6.33) $$\begin{aligned} c_{21}^0 A_1^- + c_{23}^0 A_3^- + c_{25} A_5^- + c_{26} A_6^- &= \\ = n(c_{41}^0 A_1^- + c_{43}^0 A_3^- + c_{45} A_5^-) &= \\ = \left\{ n\varrho_0 \left[-\frac{\omega_n^2}{n} + 2\gamma \frac{n-1}{2n+1} \right] A_1^- + \frac{1}{2n+1} A_6^- \right\} &= z_2 \end{aligned}$$

and the determinant of the system is equal to zero. For example, let us take the coefficient $A_{1,2}^-$ to be independent. Equation $(6.16)_2$ then yields

(6.34) $$A_{3,2}^- = \frac{\mu p_2}{N z_1} A_{1,2}^- - \frac{z_2}{N z_1} .$$

The remaining coefficients can be calculated from (6.14) for $m = 2$ and $s = n + 1$,

$$(6.35)\qquad \begin{aligned} A^-_{2,2} &= \frac{2\mu}{z_1}[\lambda(n^2 - n - 3) + \mu(n^2 - n - 2)]\, A^-_{1,2} + \frac{\lambda z_2}{z_1}, \\ A^-_{4,2} &= \frac{\mu p_1}{N z_1} A^-_{1,2} - \frac{\mu n z_2}{N z_1}, \\ A^-_{5,2} &= \frac{3\gamma\mu}{z_1} A^-_{1,2} - \frac{3\gamma}{2(2n+3)} \frac{z_2}{z_1} \\ A^-_{6,2} &= \frac{3\gamma}{z_1}[n\lambda + (3n+1)\mu]\, A^-_{1,2} - \frac{3\gamma z_2}{2 z_1}. \end{aligned}$$

The resultant coefficients of the powers of r^{n+1} will read

$$(6.36)\qquad B_i = A_i^+ + A^-_{i,2} \qquad \text{for } i = 1, 2, \ldots, 6.$$

For example, let as take the coefficient B_1 to be independent. Equations (6.29), (6.32), (6.34) and (6.35) then yield

$$(6.37)\qquad \begin{aligned} B_2 &= \frac{2\mu}{z_1}[\lambda(n^2 - n - 3) + \mu(n^2 - n - 2)]\, B_1 + \frac{\lambda z_2}{z_1}, \\ B_3 &= \frac{\mu p_2}{N z_1} B_1 - \frac{z_2}{N z_1}, \\ B_4 &= \frac{\mu p_1}{N z_1} B_1 - \frac{\mu n z_2}{N z_1}, \\ B_5 &= \frac{3\gamma\mu}{z_1} B_1 - \frac{3\gamma}{2(2n+3)} \frac{z_2}{z_1}, \\ B_6 &= \frac{3\gamma}{z_1}[n\lambda + (3n+1)\mu]\, B_1 - \frac{3\gamma z_2}{2 z_1}. \end{aligned}$$

It is convenient to adopt B_4 as the independent coefficifent. Consequently,

$$(6.38)\qquad \begin{aligned} B_3 &= \frac{p_2}{p_1} B_4 + \frac{\varrho_0}{p_1}\left\{-\frac{n}{2n+1} A_6^- + \left[\omega_n^2 - 2\gamma\frac{n(n-1)}{2n+1}\right] A_1^-\right\}, \\ B_2 &= -q_1 B_3 + q_2 B_4, \\ B_1 &= -n B_3 + \frac{1}{\mu} B_4, \\ B_5 &= -\frac{3\gamma}{2(2n+3)}[(n+3) B_1 - n(n+1) B_3], \\ B_6 &= (2n+3) B_5 + 3\gamma B_1, \end{aligned}$$

where

(6.39)
$$q_1 = 2n(n+2)\lambda + 2n(n+1)\mu,$$
$$q_2 = 2(n+1) + (n+3)(\lambda/\mu).$$

To summarize: The expansions into a series of functions $y_i(r)$, $i = 1, 2, \ldots, 6$, in the neighbourhood of the origin of the Earth model for spheroidal free oscillations take the form

(6.40)
$$\begin{aligned} y_1 &= A_1 r^{n-1} + B_1 r^{n+1} + \ldots, \\ y_2 &= A_2 r^{n-2} + B_2 r^n + \ldots, \\ y_3 &= A_3 r^{n-1} + B_3 r^{n+1} + \ldots, \\ y_4 &= A_4 r^{n-2} + B_4 r^n + \ldots, \\ y_5 &= A_5 r^n + B_5 r^{n+2} + \ldots, \\ y_6 &= A_6 r^{n-1} + B_6 r^{n+1} + \ldots. \end{aligned}$$

Among the coefficients $A_i = A_i^-$, which are given by Eqs $(6.27)_1$, (6.28) and (6.31), there are two independent coefficients, A_1 and A_6,

(6.41)
$$\begin{aligned} A_2 &= 2(n-1)\mu A_1, \\ A_3 &= \frac{1}{n} A_1, \\ A_4 &= \frac{2(n-1)}{n}\mu A_1, \\ A_5 &= -\frac{3\gamma}{2n+1} A_1 + \frac{1}{2n+1} A_6. \end{aligned}$$

Among the cofefficients B_i, which are given by Eqs (6.38), there is one independent coefficient, B_4,

(6.42)
$$\begin{aligned} B_3 &= \frac{p_2}{p_1} B_4 + \frac{\varrho_0}{p_1}\left\{-\frac{n}{2n+1} A_6 + \left[\omega_n^2 - 2\gamma\frac{n(n-1)}{2n+1}\right] A_1\right\}, \\ B_2 &= -q_1 B_3 + q_2 B_4, \\ B_1 &= -nB_3 + \frac{1}{\mu} B_4, \\ B_5 &= -\frac{3\gamma}{2(2n+3)}[(n+3)B_1 - n(n+1)B_3], \\ B_6 &= (2n+3)B_5 + 3\gamma B_1, \end{aligned}$$

where

(6.43)
$$\begin{aligned} p_1 &= 2n^2(n+2)\lambda + 2n(n^2+2n-1)\mu, \\ p_2 &= n(n+5) + n(n+3)(\lambda/\mu), \\ q_1 &= 2n(n+2)\lambda + 2n(n+1)\mu, \\ q_2 &= 2(n+1) + (n+3)(\lambda/\mu). \end{aligned}$$

It would seem that functions $y_2(r)$ and $y_4(r)$ in Eqs (6.40) are singular at the origin for $n = 1$. However, the relations for A_2 and A_4 clearly show that these coefficients are zero for $n = 1$. Consequently, expansion (6.40) holds for $n \geqq 1$.

Since three arbitrary constants A_1, A_6 and B_4 occur in the solution (6.40) to equations (6.7), Eqs (6.7) have to be integrated three times with three boundary conditions to determine the eigenfunctions $y_i(r)$. It is then convenient to adopt the following values for constants A_1, A_6 and B_4,

(6.44) $$A_1 = (1,0,0)\,, \quad A_6 = (0,1,0)\,, \quad B_4 = (0,0,1)$$

at a small distance, $r = r_1$, from the origin. The initial values of functions $y_i(r)$ are

$$y^{(1)} = \begin{bmatrix} 1 \\ 2(n-1)\mu/r_1 \\ 1/n \\ 2(n-1)\mu/nr_1 \\ -3\gamma r_1/(2n+1) \\ 0 \end{bmatrix}, \quad y^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ r_1/(2n+1) \\ 1 \end{bmatrix}, \quad y^{(3)} = \begin{bmatrix} B_1 r_1^2 \\ B_2 r_1 \\ B_3 r_1^2 \\ r_1 \\ B_5 r_1^3 \\ B_6 r_1^2 \end{bmatrix},$$

where B_i are given by Eqs (6.42) for $B_4 = 1$.

Solution (6.40) with the constants given by Eqs (6.41) and (6.42) was valid in the solid regions of the Earth model. We shall now specify this solution for the liquid regions for which Eq. (4.70) holds true. $A_2 = A_4 = 0$ follows from the condition that $\mu = 0$. Since $y_4 = 0$, also $B_4 = 0$. The expansions into series of eigenfunctions in the neighbourhood of the origin for a liquid medium then read

(6.46) $$\begin{aligned} y_1 &= A_1 r^{n-1} + \dots, & y_2 &= B_2 r^n + \dots, \\ y_5 &= A_5 r^n + \dots, & y_6 &= A_6 r^{n-1} + \dots, \end{aligned}$$

where

(6.47) $$\begin{aligned} B_2 &= \left\{\varrho_0\left[-\frac{\omega_n^2}{n} + 2\gamma\frac{n-1}{2n+1}\right]A_1 + \frac{1}{2n+1}A_6\right\}, \\ A_5 &= -\frac{3\gamma}{2n+1}A_1 + \frac{1}{2n+1}A_6 \end{aligned}$$

and A_1 and A_6 are arbitrary constants. It is easy to prove that expansions (6.46) satisfy the differential equations (4.71).

For completeness, let us give the series expansion of the eigenfunctions for radial free oscillations in the neighbourhood of the origin. Since the first part of solution (6.10) with the coefficients A_i is singular at the origin for $n = 0$, we shall only use the second part with the coefficients B_i for $n = 0$,

(6.48) $$y_1 = B_1 r + \dots, \; y_2 = B_2 + \dots, \; y_5 = B_5 r^2 + \dots.$$

The relations for coefficients B_2 and B_5 follow from (6.37), (6.49) $$B_2 = (3\lambda + 2\mu)\, B_1\,, \quad B_5 = -\tfrac{3}{2}\gamma B_1\,,$$

B_1 being an arbitrary constant.

7. SOLUTION OF THE SECULAR EQUATION BY MEANS OF THE VARIATION PRINCIPLE

Here we follow the formulation by Verreault [142] and Backus, Gilbert [15], who have pointed out that a generalization of variational principle for torsional [142] and spheroidal [15] modes by including a boundary term provides a rapidly convergent iterative scheme for numerical calculation of the eigen frequencies.

Let us, once again, study the SNREI Earth model. Assume a certain angular frequency ω, which is not necessarily an eigenfrequency of the Earth model, a vector field $\mathbf{u}(\mathbf{r})$ and scalar field $\varphi_1(\mathbf{r})$ which, within the Earth model, satisfy the equations of motion (4.10) and Poisson's equation (4.12),

(7.1) $$\omega^2 \varrho_0 \mathbf{u} = -\operatorname{div} \bar{\mathbf{t}} + \varrho_0 \operatorname{grad} \varphi_1 + \operatorname{grad}(\mathbf{u} \,.\, \operatorname{grad} \varphi_0) - \operatorname{div} \mathbf{u} \operatorname{grad} \varphi_0\,,$$

(7.2) $$\operatorname{div}\left(\frac{1}{4\pi G} \operatorname{grad} \varphi_1 + \varrho_0 \mathbf{u}\right) = 0\,.$$

Assume that the characteristic field $\mathbf{u}(\mathbf{r})$ and $\varphi_1(\mathbf{r})$ satisfy the boundary conditions at the internal boundaries, surface σ, but not necessarily at the outher boundary — surface S. The problem is to change the angular frequency ω so that the boundary condition is satisfied also at the external boundary, and that our knowledge of functions $\mathbf{u}(\mathbf{r})$ and $\varphi_1(\mathbf{r})$ can be exploited.

Let us multiply Eq. (7.1) scalarly by the vector $\mathbf{u}$ and integrate is over the volume of the model V. By using Gauss' theorem, we obtain

(7.3) $$\omega^2 \int_V \varrho_0 u^2 \, dV = \int_V \{\operatorname{tr}(\operatorname{grad} \mathbf{u} \,.\, \bar{\mathbf{t}}) + \varrho_0[\mathbf{u} \,.\, \operatorname{grad} \varphi_1 + $$
$$+ \mathbf{u} \,.\, \operatorname{grad}(\mathbf{u} \,.\, \operatorname{grad} \varphi_0) - \operatorname{div} \mathbf{u} (\mathbf{u} \,.\, \operatorname{grad} \varphi_0)]\} \, dV - \int_S (\mathbf{u} \,.\, \bar{\mathbf{t}} \,.\, \mathbf{n}) \, dS\,,$$

where we have made use of Eq. (4.25) and denoted the vector of the normal external to surface S by $\mathbf{n}$. With a view to the assumptions made, we have used

(7.4) $$\int_\sigma [\mathbf{u} \,.\, \bar{\mathbf{t}}]_-^+ \,.\, \mathbf{n} \, dS = 0\,.$$

Let us also assume that the additional gravitational potential φ_1 satisfies Laplace's equation outside the volume V,

(7.5) $$\nabla^2 \varphi_1 = 0 \qquad \text{outside } V,$$

and that φ_1 is a continuous function on surface S. Now multiply Eq. (7.2) by function φ_1 and integrate over the whole of space E. By using Gauss' theorem we obtain

$$(7.6)\qquad \int_V \varrho_0 \boldsymbol{u}\,.\,\mathrm{grad}\,\varphi_1\,\mathrm{d}V + \frac{1}{4\pi G}\int_E |\mathrm{grad}\,\varphi_1|^2\,\mathrm{d}V + + \int_S \varphi_1 \boldsymbol{n}\,.\left[\frac{\mathrm{grad}\,\varphi_1}{4\pi G} + \varrho_0 \boldsymbol{u}\right]_-^+ \mathrm{d}S = 0\,.$$

Adding Eqs (7.3) and (7.6) yields

$$(7.7)\qquad \omega^2 \mathfrak{T} = \mathfrak{K} - \mathfrak{B}\,,$$

where

$$(7.8)\qquad \mathfrak{T} = \int_V \varrho_0 u^2\,\mathrm{d}V\,,$$

$$(7.9)\qquad \mathfrak{K} = \int_V \{\mathrm{tr}\,(\mathrm{grad}\,\boldsymbol{u}\,.\,\boldsymbol{\hat{t}}) + \varrho_0[2\,\boldsymbol{u}\,.\,\mathrm{grad}\,\varphi_1 + + \boldsymbol{u}\,.\,\mathrm{grad}\,(\boldsymbol{u}\,.\,\mathrm{grad}\,\varphi_0) - \mathrm{div}\,\boldsymbol{u}(\boldsymbol{u}\,.\,\mathrm{grad}\,\varphi_0)]\}\,\mathrm{d}V + \frac{1}{4\pi G}\int_E |\mathrm{grad}\,\varphi_1|^2\,\mathrm{d}V\,,$$

$$(7.10)\qquad \mathfrak{B} = \int_S \boldsymbol{n}\,.\left\{\boldsymbol{u}\,.\,\boldsymbol{\bar{t}} - \varphi_1\left[\frac{\mathrm{grad}\,\varphi_1}{4\pi G} + \varrho_0 \boldsymbol{u}\right]_-^+\right\}\mathrm{d}S\,.$$

The expression $\mathfrak{T}$ represents double the kinetic energy of elastic oscillations with the angular frequency ω. The expression $\mathfrak{K}$ is equal to double the total potential energy which is composed of the elastic energy of oscillations, the work performed against hydrostatic pressure, gravitational energy and the energy used to change the density by ϱ_1.

If $\boldsymbol{u}(\boldsymbol{r})$ and $\varphi_1(\boldsymbol{r})$ are eigenfunctions of the oscillations, the boundary conditions on surface S imply that $\mathfrak{B} = 0$, and (7.8) will reduce to the functional of the action of the system,

$$(7.11)\qquad \omega^2 \mathfrak{T} = \mathfrak{K}.$$

If the Earth model is spherically symmetrical, the displacement vector and the increment of Cauchy's stress tensor on a spherical surface are given by Eqs (4.82) and (4.84). By substituting these relations into (7.8) and (7.10), we obtain

$$(7.12)\qquad \mathfrak{T} = \int_0^a \varrho_0(r)\{U_n^2(r) + n(n+1)[V_n^2(r) + W_n^2(r)]\}\,r^2\,\mathrm{d}r\,,$$

$$(7.13)\qquad \mathfrak{B} = a^2\left\{U_n(a)\,P_n(a) + n(n+1)[V_n(a)\,Q_n(a) + + W_n(a)\,R_n(a)] + \frac{1}{4\pi G}\,F_n(a)\,G_n(a)\right\},$$

where $r = a$ is the radius of the surface S of the Earth model. Functions U_n, V_n and F_n are given by Eqs (4.65), function W_n by Eq. (4.41), functions P_n, Q_n and R_n by Eqs (4.85), and function $G_n = y_6$ by (4.65). In Eqs (7.12) and (7.13) in which we have omitted the factor $2n + 1$ on the r.h.s., we have made use of the relations [1]

(7.14)
$$\int_0^{2\pi}\int_0^{\pi} S_n(\vartheta, \varphi)\, S_n^*(\vartheta, \varphi) \sin\vartheta \,\mathrm{d}\vartheta\,\mathrm{d}\varphi = 2n + 1\,,$$

$$\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial S_n}{\partial\vartheta}\frac{\partial S_n^*}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial S_n}{\partial\varphi}\frac{\partial S_n^*}{\partial\varphi}\right]\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = $$
$$= n(n+1)(2n+1)\,.$$

If ω_t, $\boldsymbol{u}_t$ and φ_{1t} are close to some free oscillation, we obtain a better estimate of the eigenfrequency ω_b due to the validity of Hamilton's variation principle (7.11), with the aid of the relation

(7.15)
$$\omega_b^2 = \omega_t^2 + \mathfrak{B}/\mathfrak{T} + \varepsilon$$

where ε is a term of the order of magnitude of $|\boldsymbol{u}_t - \boldsymbol{u}_b|^2$ and $|\varphi_{1t} - \varphi_{1b}|^2$. The terms $\mathfrak{B}$ and $\mathfrak{T}$ are determined with the aid of functions $\boldsymbol{u}_t$ and φ_{1t}.

If we integrate the equations of motion from a particular $r = r_1$ to $r = a$ for some frequency ω_t, close to the eigenfrequency ω_n, Eq. (7.15) without the term ε can then be used to improve the estimate of the eigenfrequency ω_n.

8. CALCULATION OF THE PERIODS OF THE FREE OSCILLATIONS OF THE EARTH

We integrated the systems of ordinary differential equations of the 1st order, describing the free oscillations of the SNREI Earth model, numerically on a computer. We used the one-step Runge—Kutta method of the 4th order [81]. It is important that the integration take place from the Earth's centre to the surface. If we integrate in the opposite sense, the numerical error of the integration will increase exponentially with depth [3, 35, 100, 103, 112, 132—134].

8.1. Toroidal free oscillations

According to (4.42), the components of the displacement vector for toroidal free oscillations with the frequency ω take the form

(8.1)
$$u = 0\,,$$
$$v = \frac{y_1(r)}{\sin\vartheta}\frac{\partial S_n(\vartheta, \varphi)}{\partial\varphi}\,\mathrm{e}^{\mathrm{i}\omega t},$$

$$w = -y_1(r)\,\frac{\partial S_n(\vartheta,\varphi)}{\partial \vartheta}\,e^{i\omega t}.$$

The components of the increment of Cauchy's stress tensor on a spherical surface, in virtue of (4.43), read

(8.2)
$$\bar{t}_{rr} = 0\,,$$
$$\bar{t}_{r\vartheta} = \frac{y_2(r)}{\sin\vartheta}\,\frac{\partial S_n(\vartheta,\varphi)}{\partial\varphi}\,e^{i\omega t},$$
$$\bar{t}_{r\varphi} = -y_2(r)\,\frac{\partial S_n(\vartheta,\varphi)}{\partial\vartheta}\,e^{i\omega t}.$$

The scalar functions $y_1(r)$ and $y_2(r)$ satisfy the system of two ordinary 1st-order differential equations (4.44),

(8.3)
$$\frac{d}{dr}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r^{-1}, & \mu^{-1} \\ \mu(n-1)(n+2)r^{-2} - \varrho_0\omega^2, & -3r^{-1} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with the boundary conditions

(8.4)
$$y_2(a) = y_2(b) = 0$$

at the Earth's surface ($r = a$) and at the core-mantle boundary ($r = b$).

The system of toroidal equations (8.3) is integrated from the core-mantle boundary to the Earth's surface. The initial value for the integration of function $y_2(r)$ is given by condition $(8.4)_2$, and of function $y_1(r)$ by a selected initial value, i.e.

(8.5)
$$y_1(b) = 1, \quad y_2(b) = 0\,.$$

The integration is caried out from the core-mantle boundary to the Earth's surface where condition $(8.4)_1$ must be satisfed. The secular equation for toroidal free oscillations then reads

(8.6)
$$D_T(\omega, n) = y_2(a; \omega, n) = 0\,.$$

At the beginning of the integration we select a series of free oscillations n and an estimate of the eigenfrequency ω_t. If the selected initial frequency does not render the secular function $y_2(a)$ zero at the surface of the Earth model, the angular frequency is improved to value ω_b by means of the variation method with the boundary terms described in Chapter 7,

(8.7)
$$\omega_b^2 = \omega_t^2 + \mathfrak{B}/\mathfrak{I}\,,$$

in which, for toroidal oscillations,

(8.8)
$$\mathfrak{B} = a^2 y_1(a)\,y_2(a)\,,$$
$$\mathfrak{I} = \int_b^a \varrho_0(r)\,y_1^2(r)\,r^2\,dr$$

functions $y_1(r)$ and $y_2(r)$ being computed for the tested frequency ω_t. This procedure is applied iteratively until the eigenfrequency of the toroidal oscillation is reached. After determining the eigenfrequency of the normal mode, the last values of functions $y_1(r)$ and $y_2(r)$ correspond to the eigenfunctions of differential operator (8.3) with the eigenvalue ω.

For higher toroidal modes of a higher order n, the integration from the core-mantle boundary may lead to an overflow of the value's magnitude during numerical computation. In this case, we begin to integrate at a higher level in the mantle ($r = r_1 > b$). Moreover, we assume that the material is homogeneous inside the sphere of radius r_1. This assumption is easy to satisfy because we can always take r_1 small enough to avoid the properties within the sphere, radius r_1, affecting the characteristic functions on the Earth's surface. The appropriate initial values of functions $y_1(r)$ and $y_2(r)$ are then given by the solution of the toroidal free oscillation of the homogeneous sphere, radius $r = r_1$ (see Chapter 5),

(8.9)
$$y_1(r) = j_n(kr),$$
$$y_2(r) = \frac{\mu}{r}[(n-1)j_n(kr) - (kr)j_{n+1}(kr)],$$
$$k^2 = \varrho_0\omega^2/\mu,$$

where $j_n(kr)$ is a spherical Bessel function of the 1st kind which is convergent in the neighbourhood of the origin. In Eqs (8.9) for calculating the initial values of functions y_1 and y_2, only the ratio

(8.10)
$$z_n(x) = xj_{n+1}(x)/j_n(x)$$

occurs. The initial values are then

(8.11)
$$y_1(r) = 1\,, \quad y_2(r) = \frac{\mu}{r}[(n-1) - z_n(kr)]\,.$$

The recurrent formula $(5.8)_1$ immediately yields the recurrent relation for function $z_n(x)$,

(8.12)
$$z_{n-1}(x) = \frac{x^2}{(2n+1) - z_n(x)}\,.$$

The limiting value of function $z_n(x)$ for $n \to \infty$,

(8.13)
$$\lim_{n\to\infty} z_n(x) = \frac{x^2}{2n+3}$$

is derived in Supplement C. If we start with a sufficiently large n in the initial value of $z_n(x)$ according to (8.13), the recurrent formula (8.12) can be used to calculate function $z_n(x)$ of a lower degree.

The spheroidal free oscillations of the Earth perturb the gravitational field of the Earth. We therefore consider coupled elastic and gravitational oscillations which simultaneously satisfy the elastic equations of motion and Poisson's equation. The components of the displacement vector for spheroidal free oscillations with frequency ω take the form of (4.66),

$$u = y_1(r)\, S_n(\vartheta, \varphi)\, e^{i\omega t}, \tag{8.14}$$
$$v = y_3(r) \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\, e^{i\omega t},$$
$$w = \frac{y_3(r)}{\sin \vartheta} \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\, e^{i\omega t}.$$

The additional gravitational potential

$$\varphi_1 = y_5(r)\, S_n(\vartheta, \varphi)\, e^{i\omega t}, \tag{8.15}$$

The increment of Cauchy's stress tensor on the spherical surface, according to (4.67), is

$$\bar{t}_{rr} = y_2(r)\, S_n(\vartheta, \varphi)\, e^{i\omega t}, \tag{8.16}$$
$$\bar{t}_{r\vartheta} = y_4(r) \frac{\partial S_n(\vartheta, \varphi)}{\partial \vartheta}\, e^{i\omega t},$$
$$\bar{t}_{r\varphi} = \frac{y_4(r)}{\sin \vartheta} \frac{\partial S_n(\vartheta, \varphi)}{\partial \varphi}\, e^{i\omega t}.$$

The scalar functions $y_i(r)$, $i = 1, 2, \ldots, 6$, satisfy the system of six ordinary differential equations of the 1st order (4.69), which we shall express in matrix form,

$$\frac{d\boldsymbol{y}}{dr} = \mathbf{A}^s \boldsymbol{y}, \tag{8.17}$$

where $\boldsymbol{y}^T = (y_1, y_2, \ldots, y_6)$ and the elements of matrix $A^s_{6\times 6}$ are

$$\begin{aligned}
&A^s_{11} = -2\lambda b r^{-1}, \; A^s_{12} = b, \; A^s_{13} = N\lambda b r^{-1}, \\
&A^s_{21} = -\varrho_0 \omega^2 + 4(d r^{-2} - \varrho_0 g_0 r^{-1}), \\
&A^s_{22} = -4\mu b r^{-1}, \; A^s_{23} = N(\varrho_0 g_0 r^{-1} - 2 d r^{-2}), \\
&A^s_{24} = N r^{-1}, \; A^s_{25} = -(n+1)\varrho_0 r^{-1}, \; A^s_{26} = \varrho_0, \\
&A^s_{31} = -r^{-1}, \; A^s_{33} = r^{-1}, \; A^s_{34} = \mu^{-1}, \\
&A^s_{41} = \varrho_0 g_0 r^{-1} - 2 d r^{-2}, \; A^s_{42} = -\lambda b r^{-1}, \\
&A^s_{43} = -\varrho_0 \omega^2 + [4N\mu(\lambda + \mu) b - 2\mu] r^{-2}, \\
&A^s_{44} = -3 r^{-1}, \; A^s_{45} = \varrho_0 r^{-1}, \; A^s_{51} = -3\gamma,
\end{aligned} \tag{8.18}$$

$$A^s_{55} = -(n+1)r^{-1}, \; A^s_{56} = 1 .$$
$$A^s_{61} = -3\gamma(n+1)r^{-1}, \; A^s_{63} = 3\gamma N r^{-1},$$
$$A^s_{66} = (n-1)r^{-1},$$

where

(8.19) $$b = 1/(\lambda + 2\mu), \; d = \mu(3\lambda + 2\mu)b, \quad \gamma = \tfrac{4}{3}\pi G\varrho_0, \; N = n(n+1).$$

The other elements of matrix $\mathbf{A}^s$ are zero. The boundary conditions at the Earth's surface are

(8.20) $$y_2(a) = y_4(a) = y_6(a) = 0 .$$

The system of six ordinary 1st-order differential equations (8.17) has six independent solutions. However, we are only interested in the solutions which are regular at the origin. Therefore, we shall start by integrating three regular solutions in the solid inner core and integrate up to the Earth's surface. The secular equation for spheroidal eigenoscillations then reads

(8.21) $$D_s(\omega, n) \equiv \begin{vmatrix} y_{21}(a), & y_{22}(a), & y_{23}(a) \\ y_{41}(a), & y_{42}(a), & y_{43}(a) \\ y_{61}(a), & y_{62}(a), & y_{63}(a) \end{vmatrix} = 0 .$$

If the outer core of the Earth is modelled by a liquid layer with a zero modulus of torsion, the system of six differential equations will reduce to a system of four differential equations (4.71), because no tangential stresses can exist in this medium. In Eqs (8.17) $b = 1/\lambda$, $d = 0$ and

(8.22) $$y_3 = \frac{1}{r\omega^2}\left(g_0 y_1 - \frac{y_2}{\varrho_0} + y_5\right),$$
$$\frac{dy_1}{dr} = -\frac{2y_1}{r} + \frac{y_2}{\lambda} + N\frac{y_3}{r},$$
$$\frac{dy_2}{dr} = -\left(\varrho_0\omega^2 + \frac{4\varrho_0 g_0}{r}\right)y_1 + N\frac{\varrho_0 g_0}{r}y_3 - \frac{(n+1)\varrho_0}{r}y_5 + \varrho_0 y_6,$$
$$\frac{dy_5}{dr} = -4\pi G\varrho_0 y_1 - \frac{n+1}{r}y_5 + y_6,$$
$$\frac{dy_6}{dr} = -4\pi G\varrho_0(n+1)\frac{y_1}{r} + 4\pi G\varrho_0 N\frac{y_3}{r} + (n-1)\frac{y_6}{r}.$$

In integrating the differential equations of free oscillations from the inner core to the Earth's surface, we must extend the solution from one region of the Earth into the other in such a way that the conditions at the boundary are satisfied. Let us, therefore, describe the method of connecting up the solutions in the individual regions.

The solution in the solid inner core can be expressed as the superposition of three regular solutions,

$$y_i^s = K_1^s y_{i1}^s + K_2^s y_{i2}^s + K_3^s y_{i3}^s, \qquad i = 1, 2, \dots, 6, \tag{8.23}$$

where the superscript s indicates quantities relating to the solid inner core, and K_1^s, K_2^s and K_3^s are integration constants.

In the liquid outer core,

$$y_i^l = K_1^l y_{i1}^l + K_2^l y_{i2}^l + K_3^l y_{i3}^l, \qquad i = 1, 2, 5, 6, \tag{8.24}$$

where the superscript l indicates quantities relating to the liquid outer core. Finally, in the mantle,

$$y_i^m = K_1^m y_{i1}^m + K_2^m y_{i2}^m + K_3^m y_{i3}^m, \qquad i = 1, 2, \dots, 6, \tag{8.25}$$

where the superscript m indicates quantities relating to the solid Earth's mantle.

At the inner-outer core boundary the tangential component of stress y_{4i} will vanish and, consequently, one of the three integration constants in the inner core may be eliminated, for example,

$$K_3^s = -\frac{y_{41}^s}{y_{43}^s} K_1^s - \frac{y_{42}^s}{y_{43}^s} K_2^s. \tag{8.26}$$

By substituting (8.26) into (8.23) and comparing with Eq. (8.24) we find that

$$K_1^l y_{i1}^l + K_2^l y_{i2}^l = K_1^s \left(y_{i1}^s - \frac{y_{41}^s}{y_{43}^s} y_{i3}^s \right) + \\ + K_2^s \left(y_{i2}^s - \frac{y_{42}^s}{y_{43}^s} y_{i3}^s \right), \qquad i = 1, 2, 5, 6. \tag{8.27}$$

At the inner-outer core boundary the conditions of continuity of functions y_1, y_2, y_5, y_6 are guaranteed by the folowing relations:

$$y_{i1}^l = y_{i1}^s - \frac{y_{41}^s}{y_{43}^s} y_{i3}^s, \\ y_{i2}^l = y_{i2}^s - \frac{y_{42}^s}{y_{43}^s} y_{i3}^s, \qquad i = 1, 2, 5, 6, \\ K_1^s = K_1^l, \; K_2^s = K_2^l. \tag{8.28}$$

Once the integration reaches the outer core — mantle boundary, both solutions can be extended directly from the core to the mantle,

$$y_{ij}^m = y_{ij}^l, \qquad i = 1, 2, 5, 6, \; j = 1, 2, \tag{8.29}$$

$$y^m_{31} = y^m_{41} = y^m_{32} = y^m_{42} = 0\,,$$
$$K^l_j = K^m_j\,, \qquad j = 1, 2.$$

The third system of boundary conditions at the core-mantle boundary follows from the fact that the tangential displacement y_3 does not necessarily have to be continuous at this boundary. One may, therefore, take the values

(8.30) $$y^m_{33} = 1\,, \;\; y^m_{i3} = 0\,, \qquad i = 1, 2, 4, 5, 6.$$

We shall determine the integration constants K^m_1, K^m_2 and K^m_3 by satisfying the boundary conditions at the Earth's surface — see below.

For spheroidal free oscillations we again integrate the differential equations for a particular order n, chosen in advance, and we vary the angular frequency of the oscillations until the secular function $D_s(\omega, n)$ is zero. Using the variation method with the boundary term, described in Chapter 7, we can obtain a better estimate ω_b from the estimate of the eigenfrequency ω_t with the aid of the relation

(8.31) $$\omega_b^2 = \omega_t^2 + \mathfrak{B}/\mathfrak{T}\,,$$

in which, for spheroidal oscillations,

(8.32) $$\mathfrak{B} = a^2\Big[y_1(a)\,y_2(a) + n(n+1)\,y_3(a)\,y_4(a) + \\ + \frac{1}{4\pi G}\,y_5(a)\,y_6(a)\Big]\,,$$
$$\mathfrak{T} = \int_0^a \varrho_0(r)\,[y_1^2(r) + n(n+1)\,y_3^2(r)]\,r^2\,\mathrm{d}r$$

The system of functions y_i, $i = 1, 2, \ldots, 6$, used in these relations, must be a good approximation of the eigenfunctions of the oscillations.

As we have seen, the numerical integration produces three independent regular solutions y_{ij}, $i = 1, 2, \ldots, 6$, $j = 1, 2, 3$. It remains to determine the integration constants K^m_1, K^m_2 and K^m_3 in such a way that the eigenfunctions of the oscillations are suitably approximated. For this purpose, we select coefficifents K^m_j, $j = 1, 2, 3$, in the same way as if ω_t were the eigenfrequency. If ω_t is the eigenfrequency of the oscillation, the matrix

(8.33) $$\begin{bmatrix} y_{21}(a), & y_{22}(a), & y_{23}(a) \\ y_{41}(a), & y_{42}(a), & y_{43}(a) \\ y_{61}(a), & y_{62}(a), & y_{63}(a) \end{bmatrix}$$

is then singular and the matrix adjoint to it, $Q_{ij}(a)$, i.e. the transposed matrix of cofactors, has rank 1. The eigenvectors are then

(8.34) $$y_i(r) = \sum_{j=1}^{3} Q_{kj}(a)\,y_{ij}(r)\,, \qquad i = 1, 2, \ldots, 6;\; k = 1, 2, \text{ or } 3.$$

If ω_t is not the eigenfrequency, we define function $y_i(r)$ in the same way but, for reasons of numerical stability, we select the index k to maximize the largest element of matrix Q_{kj}.

The initial values of numerical integration may be defined in two ways. For spheroidal oscillations of higher orders and higher modes, it is convenient to define function $y_i(r)$ on the surface of an oscillating sphere, radius r_1. Should the magnitude of the number represented in the computer overflow during the computation, we define the initial values at a higher level. In Chapter 5 we derived the analytical solution of the equations of motion for a homogeneous body. To define the initial values, we shall use three of the six solutions, regular at the origin. The first two sets of initial values are

(8.35)
$$\begin{aligned} ry_1(r) &= nhj_n(x) - fxj_{n+1}(x), \\ r^2y_2(r) &= -(\lambda + 2\mu)fx^2j_n(x) + 2\mu\{n(n-1)hj_n(x) + \\ &\quad + [2f + n(n+1)]xj_{n+1}(x)\}, \\ ry_3(r) &= hj_n(x) + xj_{n+1}(x), \\ r^2y_4(r) &= \mu[x^2j_n(x) + 2(n-1)hj_n(x) - 2(f+1)xj_{n+1}(x)], \\ y_5(r) &= -3\gamma fj_n(x), \\ ry_6(r) &= 3n\gamma hj_n(x) + (2n+1)y_5(r), \end{aligned}$$

where $x = kr$ and $j_n(x)$ is a spherical Bessel function of the 1st kind,

(8.36)
$$k^2 = \frac{1}{2}\left\{\frac{\omega^2 + 4\gamma}{\alpha^2} + \frac{\omega^2}{\beta^2} \pm \sqrt{\left[\left(\frac{\omega^2 + 4\gamma}{\alpha^2} - \frac{\omega^2}{\beta^2}\right)^2 + \frac{4n(n+1)\gamma^2}{\alpha^2\beta^2}\right]}\right\},$$

α, β being the velocities of longitudinal and transverse seismic waves,

(8.37)
$$\gamma = \tfrac{4}{3}\pi G\varrho_0, \; f = \frac{1}{\gamma}(\beta^2k^2 - \omega^2), \; h = f - (n+1).$$

In Eqs (8.35), which define the initial values, there again only occurs the ratio $z_n(x) = xj_{n+1}(x)/j_n(x)$, for which recurrent formula (8.12) and the limiting value (8.13) hold true.

The third set of initial values regular at the origin

(8.38)
$$\begin{aligned} ry_1(r) &= nr^n, \\ r^2y_2(r) &= 2\mu n(n-1)r^n, \\ ry_3(r) &= r^n, \\ r^2y_4(r) &= 2\mu(n-1)r^n, \\ y_5(r) &= (\omega^2 - n\gamma)r^n, \end{aligned}$$

$$ry_6(r) = (2n + 1)y_5(r) + 3n\gamma r^n .$$

If $\mu=0$, one of the solutions (8.35) will vanish and we shall be left with

(8.39) $$k^2 = \frac{1}{\alpha^2}\left[\omega^2 + 4\gamma - \frac{n(n+1)\gamma^2}{\omega^2}\right], \; f = -\frac{\omega^2}{\gamma}, \; h = f - (n+1).$$

For spheroidal oscillations of low orders, for which the integration is carried out from the vicinity of the Earth's centre, the second way of defining the initial values is convenient, i.e. by expanding function $y_i(r)$ into a power series in r in the neighbourhood of the origin. According to the results of Chapter 6, at a small distance $r = r_1$ from the Earth's centre

(8.40) $$y^{(1)} = \begin{bmatrix} 1 \\ 2(n-1)\mu/r_1 \\ 1/n \\ 2(n-1)\mu/nr_1 \\ -3\gamma r_1/(2n+1) \\ 0 \end{bmatrix}, \quad y^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ r_1/(2n+1) \\ 1 \end{bmatrix},$$

$$y^{(3)} = \begin{bmatrix} B_1 r_1^2 \\ B_2 r_1 \\ B_3 r_1^2 \\ r_1 \\ B_5 r_1^3 \\ B_6 r_1^2 \end{bmatrix},$$

where

(8.41) $$\begin{aligned} B_3 &= p_2/p_1 , \\ B_2 &= -q_1 B_3 + q_2 , \\ B_1 &= -nB_3 + 1/\mu , \\ B_5 &= -3\gamma[(n+3)B_1 - n(n+1)B_3]/2(2n+3) , \\ B_6 &= (2n+3)B_5 + 3\gamma B_1 \end{aligned}$$

and

(8.42) $$\begin{aligned} p_1 &= 2n^2(n+2)\lambda + 2n(n^2+2n-1)\mu , \\ p_2 &= n(n+5) + n(n+3)(\lambda/\mu) , \\ q_1 &= 2n(n+2)\lambda + 2n(n+1)\mu , \\ q_2 &= 2(n+1) + (n+3)(\lambda/\mu) . \end{aligned}$$

If $\mu = 0$, the initial values of functions $y_i(r)$, $i = 1, 2, 5, 6$, are

(8.43) $$y^{(1)} = \begin{bmatrix} 0 \\ \varrho_0 r/(2n+1) \\ r/(2n+1) \\ 1 \end{bmatrix}, \quad y^{(2)} = \begin{bmatrix} 1 \\ \varrho_0 r\left[-\dfrac{\omega^2}{n} + \dfrac{2(n-1)\gamma}{2n+1}\right] \\ -3\gamma r/(2n+1) \\ 0 \end{bmatrix}.$$

8.3. Radial free oscillations

If we are considering spheroidal oscillations of zero order, $n = 0$, the displacement vector and the increment of Cauchy's stress tensor on the spherical surface only have radial components:

(8.44) $$u = y_1(r)\,e^{i\omega t}, \quad v = w = 0, \qquad \bar{t}_{rr} = y_2(r)\,e^{i\omega t}, \quad \bar{t}_{r\vartheta} = \bar{t}_{r\varphi} = 0.$$

Functions $y_1(r)$ and $y_2(r)$ satisfy the system of two ordinary 1st-order differential equations,

(8.45) $$\mathrm{d}\boldsymbol{y}/\mathrm{d}r = \mathbf{A}^r\boldsymbol{y},$$

where $\boldsymbol{y}^{\mathrm{T}} = (y_1, y_2)$ and the elements of matrix $A^r_{2\times 2}$ are

(8.46) $$A^r_{11} = -2\lambda b r^{-1}, \quad A^r_{12} = b,$$
$$A^r_{21} = -\varrho_0\omega^2 + 4(dr^{-2} - \varrho_0 g_0 r^{-1}), \quad A^r_{22} = -4\mu b r^{-1},$$

quantities b and d being defined by (8.19). The boundary condition at the Earth's surface reads

(8.47) $$y_2(a) = 0.$$

The initial conditions of integration for function $y_i(r)$ may, for example, be defined from the expansion of functions $y_i(r)$ into power series in r in the neighbourhood of the origin,

(8.48) $$y_1(r) = r, \quad y_2(r) = 3\lambda + 2\mu.$$

The integration is carried out from a small distance $r = r_1$ from the Earth's centre to its surface, using the same system of differential equations in the solid and liquid parts of the Earth; consequently, there is no problem with extending functions $y_i(r)$ from the solid to the liquid region and vice versa. The secular function for radial oscillations is the surface value of function $y_2(r)$.

The radial part of the additional gravitational potential can then be derived by solving the 1st-order differential equation

(8.49) $$dy_5/dr = -4\pi G\varrho_0 y_1 .$$

The estimate of eigenfrequency ω_t can again be improved using the variation method, described in Chapter 7. The improved estimate of the eigenfrequency ω_b is

(8.50) $$\omega_b^2 = \omega_t^2 + \mathfrak{B}/\mathfrak{I},$$

in which, for radial oscillations,

(8.51) $$\mathfrak{B} = a^2\left[y_1(a)\,y_2(a) + \frac{1}{4\pi Ga}y_5^2(a)\right],$$

$$\mathfrak{I} = \int_0^a \varrho_0(r)\,y_1^2(r)\,r^2\,dr$$

with functions $y_i(r)$ computed for the tested frequency ω_t.

8.4. Periods of the free oscillations of the Earth

Drawing on the theory we derived, we have written programs for computing the free oscillations of the SNREI Earth model. We tested them on Earth model 1066A, the physical parameters of which are given in [70] and tabulated by subroutine M1066A. This SNREI Earth model consists of the solid inner core, liquid outer core and solid mantle and crust.

The normal modes of this Earth model are described by the type of oscillation (toroidal, spheroidal and radial), by the angular number n, radial number k (also by the mode number), the period or angular frequency of oscillation and by the eigenfunctions of the oscillations. We shall denote the periods of the toroidal, spheroidal and radial characteristic mode, which is given by the spherical function

(8.52) $$S_n(\vartheta, \varphi) = \sum_{m=-n}^{n} Y_{nm}(\vartheta, \varphi),$$

by the symbols

(8.53) $${}_kT_n,\ {}_kS_n \text{ and } {}_kS_0,$$

respectively.

The geometric configuration of the oscillations can be illustrated in relation to nodal surfaces. Nodal surfaces are defined as surfaces on which the displacement vector is zero. There are three types of nodal surfaces: a) concentric spherical surfaces within the Earth; b) concentric conical surfaces with apexes in the Earth's centre, which intersect the Earth's surface in parallel circles. The number of circles is $n - |m|$; c) planes which pass through both poles (one of them is the origin of the earthquake) and which intersect the Earth's surface in meridional circles. The number of circles is $2m$.

The simplest spheroidal oscillations are oscillations with $n = 0$, radial oscillations. The fundamental mode, ${}_0S_0$, represents alternate densification and rarification. Higher modes, ${}_1S_0$, ${}_2S_0$, ... have one, two, ... nodal surfaces within the Earth. The fundamental spheroidal mode, ${}_0S_2$, is referred to as the "football mode". In this case, the Earth is alternately flattened and elongated. Fundamental mode ${}_0S_3$ represents an alternate change to pear shape. In mode ${}_0T_2$ two hemispheres shear oscillate in opposite phase with a nodal plane between them. Higher fundamental toroidal modes divide the Earth's surface into 3, 4, ... zones with opposite motions. Some types of free oscillations cannot be observed. For example, mode ${}_0S_1$ represents the translation of the whole surface and centre of gravity of the Earth, mode ${}_0T_1$ represents the variation of the Earth's rotational velocity.

In Table 1, some of the theoretical periods T_{1066A} of free oscillations of Earth model 1066A are compared with the observed periods T_{obs}. These were taken from [70]; they are the average values, mostly observed during the large earthquakes in Chile (May 22, 1960), Alaska (Mar. 28, 1964) and Columbia (July 31, 1970). Moreover, the periods of the oscillations of the lowest orders were averaged in the sense of the diagonal summation rule [67]. According to this rule, the period of a spherically symmetric and non-rotating Earth model is equal to the average period of the observed spectral multiplet. Note that the void positions in the T_{obs} column indicate that these modes have as yet not been recorded.

The theoretical periods T_{1066A} of the free oscillations of Earth model 1066A were computed using the numerical method described in the preceding sections. Let us briefly summarize our experience with this method. The advantage of this method is the relative simplicity and illustrativeness of the operations it consists of. The accuracy of the method is comparatively high. Practical numerical computations have shown that the eigenperiods and eigenfunctions can be determined with a relative error not exceeding 0.01—0.05 %. Another advantage is the minimal demand on the inherent computer store. In the ICL 4-72 computer we used an inherent store of up to 110 k bytes.

The main disadvantage of the method described is the required computer time, particularly for computing the spheroidal modes. The computer time required to compute the periods and eigenfunctions of one spheroidal mode is 100—200 ETU (6—12 mins) with an ICL 4-72 computer. The time required to compute the periods and eigenfunctions of one toroidal mode is shorter, about 20—30 ETU.

From Table 1 we can draw two conclusions: a) The maximum differences between the observed T_{obs} and theoretical periods T_{1066A} are 0.5 %. This indicates

Table 1. Comparison of observed eigenperiods T_{obs} and theoretical values T_{1066A}, computed for model 1066A

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
0 T 2		2631.8	0 T 47		174.0
0 T 3	1706.0	1703.2	0 T 48		170.7
0 T 4	1305.5	1304.2	0 T 49		167.5
0 T 5	1076.0	1076.0	0 T 50		164.5
0 T 6	925.8	925.9			
0 T 7	819.3	818.3			
0 T 8	736.9	736.7	1 T 2	756.6	756.5
0 T 9	671.8	672.0	1 T 3	695.2	694.0
0 T 10	619.0	619.3	1 T 4	630.0	630.0
0 T 11	574.6	575.3	1 T 5		570.6
0 T 12	536.8	537.8	1 T 6	519.1	518.7
0 T 13	505.0	505.4	1 T 7	475.2	474.8
0 T 14	477.5	477.1	1 T 8	438.5	438.0
0 T 15	451.8	452.1	1 T 9	407.7	407.3
0 T 16	429.2	429.8	1 T 10	381.2	381.3
0 T 17	410.2	409.7	1 T 11	359.1	359.0
0 T 18	391.0	391.6	1 T 12	339.5	339.5
0 T 19	374.8	375.1	1 T 13	322.1	322.4
0 T 20	359.6	359.9	1 T 14	307.0	307.1
0 T 21	346.1	346.1	1 T 15	293.3	293.3
0 T 22	333.2	333.2	1 T 16	280.6	280.8
0 T 23	321.2	321.4	1 T 17	269.7	269.5
0 T 24	310.4	310.4	1 T 18	259.1	259.2
0 T 25	300.2	300.1	1 T 19	249.6	249.6
0 T 26	290.3	290.5	1 T 20	240.8	240.9
0 T 27	281.4	281.5	1 T 21	232.5	232.8
0 T 28	272.9	273.0	1 T 22	225.2	225.4
0 T 29	264.9	265.0	1 T 23	218.4	218.4
0 T 30	257.3	257.5	1 T 24	211.9	212.0
0 T 31	250.3	250.5	1 T 25	205.9	205.9
0 T 32	244.3	243.8	1 T 26	200.3	200.3
0 T 33	237.4	237.4	1 T 27	194.9	195.0
0 T 34	231.3	231.4	1 T 28	190.1	190.1
0 T 35	224.9	225.6	1 T 29	185.3	185.4
0 T 36	220.7	220.2	1 T 30	180.8	181.0
0 T 37	213.9	215.0	1 T 31	176.8	176.8
0 T 38	209.8	210.0	1 T 32	173.0	172.9
0 T 39	204.3	205.3	1 T 33	169.2	169.2
0 T 40	200.0	200.8	1 T 34	165.7	165.7
0 T 41	195.9	196.5	1 T 35	162.3	162.3
0 T 42	191.3	192.3	1 T 36	159.1	159.1
0 T 43	187.4	188.3	1 T 37	156.0	156.0
0 T 44	183.8	184.5	1 T 38	153.1	153.1
0 T 45	180.3	180.8	1 T 39	150.3	150.3
0 T 46	176.9	177.3	1 T 40	147.7	147.7

Table 1 (*continued*)

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
			2 T 15		238.3
1 T 41	145.1	145.1	2 T 16		228.6
1 T 42	142.7	142.6	2 T 17	219.9	219.9
1 T 43	140.2	140.2	2 T 18	211.9	212.0
1 T 44	138.0	138.0	2 T 19	204.6	204.7
1 T 45	135.6	135.8	2 T 20		198.0
1 T 46	133.6	133.6	2 T 21	191.9	191.8
1 T 47	131.6	131.6	2 T 22	186.2	186.1
1 T 48	129.6	129.6	2 T 23		180.7
1 T 49	127.7	127.7	2 T 24		175.7
1 T 50	125.9	125.8	2 T 25	170.9	170.9
1 T 51	124.1	124.0	2 T 26	166.5	166.5
1 T 52	122.3	122.3	2 T 27	162.3	162.3
1 T 53		120.6	2 T 28	158.4	158.4
1 T 54	119.0	119.0	2 T 29	154.7	154.7
1 T 55		117.4	2 T 30		151.1
1 T 56		115.8	2 T 31	147.7	147.7
1 T 57	114.4	114.3	2 T 32	144.6	144.5
1 T 58	112.9	112.8	2 T 33	141.5	141.5
1 T 59	111.4	111.4	2 T 34	138.6	138.6
1 T 60	110.2	110.0	2 T 35	135.7	135.8
1 T 61		108.7	2 T 36	133.1	133.1
1 T 62	107.4	107.4	2 T 37	130.5	130.6
1 T 63		106.1	2 T 38	128.2	128.2
1 T 64	104.9	104.8	2 T 39	125.7	125.8
1 T 65		103.6	2 T 40	123.6	123.6
1 T 66	102.6	102.4	2 T 41	121.3	121.4
1 T 67		101.2	2 T 42	119.3	119.4
1 T 68		100.1	2 T 43		117.5
1 T 69		99.0	2 T 44	115.5	115.5
1 T 70		97.9	2 T 45	113.6	113.7
			2 T 46		111.9
			2 T 47	110.2	110.2
2 T 2	447.3	447.6	2 T 48		108.6
2 T 3		435.2	2 T 49	107.0	107.0
2 T 4	419.4	419.8	2 T 50		105.5
2 T 5	401.8	402.2	2 T 51	104.0	104.0
2 T 6		383.1	2 T 52	102.6	102.6
2 T 7	363.7	363.2	2 T 53		101.3
2 T 8	343.3	343.3	2 T 54	99.9	99.9
2 T 9		324.1	2 T 55	98.6	98.7
2 T 10		348.4	2 T 56	97.4	97.4
2 T 11		289.3	2 T 57		96.2
2 T 12		274.3	2 T 58	95.1	95.1
2 T 13		260.9	2 T 59		94.0
2 T 14		249.0	2 T 60		92.9

Table 1 (*continued*)

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
2 T 61	91.9	91.8	3 T 41	108.9	108.9
2 T 62		90.8	3 T 42	107.0	107.1
2 T 63		89.8	3 T 43		105.4
2 T 64		88.8	3 T 44		103.7
2 T 65		87.9	3 T 45		102.1
			3 T 46	100.6	100.5
			3 T 47	99.1	99.1
3 T 2		308.9	3 T 48		97.6
3 T 3		304.6	3 T 49		96.2
3 T 4		299.1	3 T 50		94.9
3 T 5		292.6	3 T 51	93.7	93.5
3 T 6		285.2	3 T 52		92.3
3 T 7		277.1	3 T 53	91.2	91.0
3 T 8		268.5	3 T 54	89.9	89.8
3 T 9	259.3	259.5	3 T 55		88.7
3 T 10		250.3	3 T 56	87.7	87.5
3 T 11	240.5	241.0	3 T 57	86.4	86.4
3 T 12		231.8	3 T 58	85.3	85.4
3 T 13		222.7	3 T 59	84.4	84.4
3 T 14		214.0	3 T 60		83.3
3 T 15		205.7	3 T 61	82.4	82.3
3 T 16		198.0	3 T 62	81.4	81.4
3 T 17	190.0	190.9	3 T 63		80.5
3 T 18	184.1	184.4	3 T 64		79.6
3 T 19	178.2	178.4	3 T 65	78.7	78.7
3 T 20	172.7	172.9	3 T 66		77.8
3 T 21	167.7	167.8	3 T 67		77.0
3 T 22		163.1	3 T 68	76.2	76.2
3 T 23	158.5	158.7	3 T 69	75.4	75.4
3 T 24	154.6	154.6	3 T 70		74.6
3 T 25	150.7	150.8	3 T 71		73.9
3 T 26	147.2	147.1	3 T 72	73.2	73.2
3 T 27	143.7	143.7	3 T 73	72.4	72.4
3 T 28	140.4	140.4	3 T 74		71.7
3 T 29	137.2	137.2	3 T 75		71.1
3 T 30	134.2	134.2			
3 T 31	131.4	131.5			
3 T 32	128.7	128.7	4 T 2		231.0
3 T 33	126.2	126.1	4 T 3		229.2
3 T 34	123.8	123.6	4 T 4		227.0
3 T 35		121.3	4 T 5		224.2
3 T 36		119.0	4 T 6		220.9
3 T 37	116.9	116.8	4 T 7	216.8	217.3
3 T 38	114.7	114.7	4 T 8		213.4
3 T 39		112.7	4 T 9		209.2
3 T 40		110.7	4 T 10		204.7

Table 1 (*continued*)

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
4 T 11	199.7	200.0	4 T 57		80.0
4 T 12		195.3	4 T 58		79.0
4 T 13		190.4	4 T 59		78.1
4 T 14	184.9	185.4	4 T 60		77.2
4 T 15		180.4	4 T 61		76.3
4 T 16	174.7	175.3	4 T 62	75.5	75.5
4 T 17	169.7	170.3	4 T 63	74.7	74.6
4 T 18	164.7	165.4	4 T 64	73.8	73.8
4 T 19	160.1	160.5	4 T 65	72.9	73.0
4 T 20	155.6	155.9	4 T 66	72.3	72.2
4 T 21	151.2	151.5	4 T 67	71.1	71.5
4 T 22	147.5	147.3			
4 T 23	143.7	143.4			
4 T 24		139.7	0 S 0	1227.6	1230.5
4 T 25	136.3	136.2	1 S 0	613.6	613.8
4 T 26		133.0	2 S 0	398.6	398.5
4 T 27	130.0	129.9	3 S 0	305.8	305.8
4 T 28		127.0	4 S 0	243.7	243.7
4 T 29		124.3	5 S 0	204.6	204.9
4 T 30		121.7	6 S 0	174.3	174.3
4 T 31		119.2	7 S 0	151.9	152.0
4 T 32		116.8	8 S 0	134.6	134.6
4 T 33		114.6	9 S 0		
4 T 34		112.4	10 S 0	110.7	110.4
4 T 35		110.4	11 S 0	101.3	101.1
4 T 36		108.4	12 S 0	93.5	93.5
4 T 37		106.5			
4 T 38		104.7			
4 T 39		102.9	0 S 2	3233.3	3232.4
4 T 40	101.3	101.3	0 S 3	2133.6	2135.1
4 T 41	99.7	99.7	0 S 4	1547.3	1545.9
4 T 42		98.1	0 S 5	1190.1	1190.4
4 T 43		96.6	0 S 6	963.2	963.7
4 T 44		95.2	0 S 7	811.5	812.4
4 T 45	93.8	93.8	0 S 8	707.6	708.0
4 T 46	92.3	92.4	0 S 9	634.0	634.1
4 T 47	91.1	91.1	0 S 10	580.1	579.6
4 T 48	89.8	89.9	0 S 11	537.0	537.4
4 T 49		88.6	0 S 12	502.3	502.9
4 T 50	87.5	87.5	0 S 13	473.2	473.7
4 T 51		86.3	0 S 14	448.3	448.5
4 T 52		85.2	0 S 15	426.2	426.5
4 T 53		84.1	0 S 16	406.8	407.1
4 T 54	83.0	83.0	0 S 17	389.3	389.8
4 T 55		82.0	0 S 18	374.0	374.3
4 T 56		81.0	0 S 19	360.2	360.3

Table 1 (*continued*)

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
0 S 20	347.7	347.6	0 S 66	141.2	141.1
0 S 21	336.0	336.0	0 S 67		139.3
0 S 22	325.3	325.3	0 S 68		137.5
0 S 23	315.4	315.5	0 S 69		135.7
0 S 24	306.3	306.3	0 S 70		134.0
0 S 25	297.7	297.8	0 S 71		132.3
0 S 26	289.7	289.8	0 S 72		130.7
0 S 27	282.3	282.4	0 S 73		129.1
0 S 28	275.2	275.3	0 S 74		127.6
0 S 29	268.5	268.6	0 S 75		126.1
0 S 30	262.2	262.3	0 S 76		124.6
0 S 31	256.1	256.3	0 S 77		123.1
0 S 32	250.3	250.5	0 S 78		121.7
0 S 33	245.0	245.1	0 S 79		120.3
0 S 34	239.7	239.8	0 S 80		118.9
0 S 35	234.7	234.8			
0 S 36	229.9	230.0			
0 S 37	225.2	225.3	1 S 2	1470.9	1470.3
0 S 38	220.8	220.9	1 S 3	1060.8	1063.5
0 S 39	216.5	216.6	1 S 4	852.7	852.1
0 S 40	212.4	212.5	1 S 5	730.6	729.4
0 S 41	208.4	208.6	1 S 6	657.6	657.0
0 S 42	204.6	204.7	1 S 7	603.9	604.3
0 S 43	200.9	201.0	1 S 8	556.0	556.1
0 S 44	197.4	197.5	1 S 9	510.0	509.7
0 S 45	194.0	194.0	1 S 10	465.5	466.0
0 S 46	190.6	190.7	1 S 11		426.4
0 S 47	187.4	187.4	1 S 12		367.7
0 S 48	184.3	184.3	1 S 13		
0 S 49	181.3	181.3	1 S 14	337.0	336.5
0 S 50	178.3	178.3	1 S 15	316.1	315.5
0 S 51	175.4	175.5	1 S 16	299.5	299.4
0 S 52	172.6	172.7	1 S 17	286.0	286.1
0 S 53	170.1	170.0	1 S 18	274.8	274.3
0 S 54	167.4	167.4	1 S 19	263.6	263.5
0 S 55	164.8	164.9	1 S 20	254.0	253.7
0 S 56	162.4	162.4	1 S 21	244.9	244.6
0 S 57	160.2	160.0	1 S 22	236.2	236.2
0 S 58	157.7	157.7	1 S 23	228.4	228.4
0 S 59	155.6	155.4	1 S 24	221.0	221.1
0 S 60	153.4	153.3	1 S 25	214.4	214.2
0 S 61	151.2	151.1	1 S 26	207.7	207.9
0 S 62	149.2	149.0	1 S 27	201.7	201.9
0 S 63	147.1	147.0	1 S 28	196.3	196.2
0 S 64	145.0	145.0	1 S 29	190.9	191.0
0 S 65	143.0	143.0	1 S 30	185.9	185.9

Table 1 (*continued*)

Mode	T_{obs} (s)	T_{1066A} (s)	Mode	T_{obs} (s)	T_{1066A} (s)
1 S 31			1 S 47	131.5	131.4
1 S 32	176.7	176.8	1 S 48	129.2	129.3
1 S 33	172.3	172.6	1 S 49	127.1	127.3
1 S 34	168.3	168.6	1 S 50	125.4	125.4
1 S 35	164.6	164.8	1 S 51		
1 S 36	161.4	161.3	1 S 52	121.7	121.7
1 S 37	157.7	157.8	1 S 53	120.1	120.0
1 S 38	154.7	154.6	1 S 54	118.5	118.4
1 S 39	151.5	151.5	1 S 55	116.8	116.7
1 S 40	148.6	148.6	1 S 56	115.3	115.2
1 S 41	145.8	145.8	1 S 57		
1 S 42	143.2	143.1	1 S 58	112.3	112.2
1 S 43	140.6	140.6	1 S 59	110.9	110.8
1 S 44	138.3	138.1	1 S 60		
1 S 45			1 S 61	108.1	108.1
1 S 46					

that Earth model 1066A reflects the distribution of physical parameters within the Earth relatively well; b) With a view to the objective of this study, this model is also suitable for comparing the periods of free oscillations which we have computed, using the relevant programs, with the periods of the free oscillations of the same Earth model, published in [70]. The comparison indicates very good agreement. This test at least partly verified the correct function of the programs.

9. CONCLUSION

Let us summarize the most important results of this study. Using general curvilinear coordinates, introduced in Supplement A, we briefly derived the fundamental relations of continuum mechanics. On principle, we differentiate between descriptions in Lagrange and Euler coordinates. The relations we derived are used to derive the equations of motion and boundary conditions of elastic oscillations of a body pre-stressed by finite static stresses. We assume that the free oscillations will cause small deviations from equilibrium position, such that the tensor of finite deformations can be approximated by the tensor of small deformations. The expression of the boundary conditions at the liquid boundary in Langrange's description of oscillation is relatively complicated. The case of the free oscillations of the general Earth model, which we then dealt with, is special from the point of view of this theory.

The first approximation of the real Earth is a spherically symmetric, non-rotating, isotropic, linearly elastic (SNREI) model. The wave field of free oscillations of this model resolves precisely into two types of oscillations, viz. toroidal and spheroidal. The toroidal free oscillations are characterized by the radial component of the displacement vector and the volume dilatation being zero. Consequently, these oscillations are not concommitent with changes of density, nor with perturbations of the gravitational potential. The toroidal equations of motion consist of a system of two ordinary differential equations of the 1st order. Whereas the energy of the toroidal oscillation is restricted to the solid elastic regions of the Earth model, spheroidal oscillations may "propagate" even through a liquid. These oscillations are characterized by a zero radial component of the rotation of the displacement vector, but the other quantities are, in general, non-zero. The spheroidal equations of motion consist of a system of six ordinary 1st-order differential equations. Radial oscillations ($n = 0$) are a special case of spheroidal oscillations.

In defining the initial values of numerical integration of the equations of motion and in the matrix solution of free oscillations, the eigenfunctions for the homogeneous model have to be known. We have proved that, for this particular model, the eigenfunctions of the oscillations can be expressed by a combination of spherical Bessel functions. In defining the initial values of numerical integration of systems of equations of motion in the neighbourhood of the model's centre, we used a different method, i.e. the expansion of the eigenfunctions into a power series in r in the neighbourhood of the origin.

Another important problem is determining the roots of the secular function. For the SNREI Earth model, we have derived a relation for computing an improved value of the eigenfrequency, using the variation method with a boundary term, with the aid of the tested frequency and the eigenfunctions computed for this tested frequency. The first three sections of Chapter 8 describe the method of solving the system of ordinary differential equations numerically for the free oscillations of the SNREI Earth model. This then involves the description of program functions, inclusive of instructions for using them, as written for the purpose of solving the problems on hand numerically. In Section 4 of Chapter 8, we present some of the eigenperiods of model 1066A and compare them with observed eigenperiods.

SUPPLEMENT A. TENSOR ANALYSIS

A.1. Introduction

To facilitate the understanding of the principal part of this study, we shall briefly deal with tensor analysis in this supplement. Tensor analysis is a natural expansion of vector analysis. As in the case of vectors, we shall formulate the tensor calculus for an arbitrary coordinate system. We shall introduce tensor with the aid of invariant properties of coordinate transformation. Since physical laws are invariant with respect to a particular coordinate system, the introduction of tensors via their invariant properties will provide a natural and powerful tool for formulating physical laws. There exist a large number of books and monographs of various sophistication on the subject. We recommend [65, 82, 89, 104, 125]. An account particularly suitable to continuum physics can be found in [56—59].

A.2. Curvilinear coordinates

Assume the position of an arbitrary point P in three-dimensional space to be determined by its Cartesian coordinates y^1, y^2, y^3.

Consider the transformation of these coordinates,

$$x^k = x^k(y^1, y^2, y^3), \qquad k = 1, 2, 3, \tag{A.1}$$

under the assumption that functions x^k are defined and continuously differentiable at least up to the first order in a particular region of point $P(y^1, y^2, y^3)$. Also assume that the Jacobian of the transformation,

$$J \equiv \det\left|\frac{\partial y^k}{\partial x^m}\right| = \begin{vmatrix} \partial y^1/\partial x^1 & \partial y^1/\partial x^2 & \partial y^1/\partial x^3 \\ \partial y^2/\partial x^1 & \partial y^2/\partial x^2 & \partial y^2/\partial x^3 \\ \partial y^3/\partial x^1 & \partial y^3/\partial x^2 & \partial y^3/\partial x^3 \end{vmatrix} \tag{A.2}$$

differs from zero in the region being considered. From the implicit function theorem it follows that transformation (A.1) has a uniquely inverse transformation

$$y^k = y^k(x^1, x^2, x^3), \qquad k = 1, 2, 3. \tag{A.3}$$

Under these assumptions the coordinates x^k are uniquely assigned to coordinates y^k and vice versa. Coordinates x^k determine the position of point P in space uniquely and, therefore, they are referred to as the *curvilinear coordinates* of the point (Fig. A1).

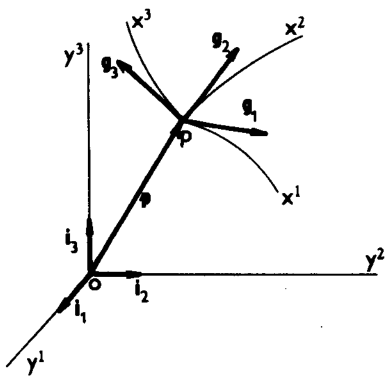

Fig. A1. Curvilinear coordinates.

A set of points in E_3 one of whose curvilinear coordinates is constant, is called a *coordinate surface*. Three different coordinate surfaces may pass through each point in E_3. The line of intersection of two mutually corresponding coordinate surfaces is called the *coordinate line*, i.e. a set of points in E_3 whose two curvilinear coordinates are constant. Once again, three different coordinate lines may pass through each point in E_3.

In Cartesian coordinates, the position vector $\boldsymbol{p}$ of point P is given by the relation

(A.4) $$\boldsymbol{p} = y^k \boldsymbol{i}_k ,$$

where $\boldsymbol{i}_k$ are unit basis vectors in Cartesian coordinates. In Eq. (A.4) and throughout the text as a whole we shall use Einstein's summation rule, i.e. we sum from one to three over all repeated indices which occur in diagonal position. No summation is carried out over the underscore indices.

We shall introduce the *base vectors* $\boldsymbol{g}_k(x^1, x^2, x^3)$ as follows:

(A.5) $$\boldsymbol{g}_k(\boldsymbol{x}) \equiv \frac{\partial \boldsymbol{p}}{\partial x^k} = \frac{\partial y^m}{\partial x^k} \boldsymbol{i}_m .$$

If we multiply (A.5) by $\partial x^k / \partial y^n$, we obtain

(A.6) $$\boldsymbol{i}_n = \frac{\partial x^k}{\partial y^n} \boldsymbol{g}_k .$$

Equation (A.5) implies that the vectors $\boldsymbol{g}_k$ are tangential to coordinate lines x^k like the vectors $\boldsymbol{i}_k$, which are located on the Cartesian axes y^k.

The infinitesimal vector at point P can be expressed as

(A.7) $$\mathrm{d}\boldsymbol{p} = \frac{\partial \boldsymbol{p}}{\partial x^k} \mathrm{d}x^k = \boldsymbol{g}_k \, \mathrm{d}x^k .$$

The square of the distance between two infinitesimally distant points is

(A.8) $$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{p} . \mathrm{d}\boldsymbol{p} = g_{kl}(\boldsymbol{x}) \, \mathrm{d}x^k \, \mathrm{d}x^l ,$$

where $g_{kl}(\boldsymbol{x})$ is the *covariant metric tensor* defined by the relation

(A.9) $$g_{kl}(\boldsymbol{x}) = \boldsymbol{g}_k \cdot \boldsymbol{g}_l = \frac{\partial y^m}{\partial x^k} \frac{\partial y^n}{\partial x^l} \delta_{mn} ,$$

where δ_{mn} is *Kronecker's delta symbol*, equal to unity if the indices are the same and to zero if the indices are different. If the metric tensor is known, the length of the vector and the angle between two vectors can be determined. Note that in general curvilinear coordinates $g_{kl} \neq 0$ for $k \neq l$. Therefore, vector $\boldsymbol{g}_k$ need not be orthogonal to vector $\boldsymbol{g}_l$. We shall refer to the coordinates as *orthogonal* if $g_{kl} = 0$ everywhere when $k \neq l$. Nor is g_{kk} necessarily equal to unity and, therefore, vectors $\boldsymbol{g}_k$ are *not* necessarily unit vectors. Equation (A.9) further implies that the covariant metric tensor is symmetric, $g_{kl} = g_{lk}$.

The *reciprocal base vectors* $\boldsymbol{g}^k(x)$ is determined by a system of nine equations

$$\boldsymbol{g}^k \cdot \boldsymbol{g}_l = \delta^k_l , \tag{A.10}$$

where δ^k_l is Kronecker's delta symbol. The solution to system (A.10) reads

$$\boldsymbol{g}^k = g^{kl}(\boldsymbol{x})\, \boldsymbol{g}_l , \tag{A.11}$$

where

$$g^{kl}(\boldsymbol{x}) = \frac{\text{alg. cofactor } g_{kl}}{g}, \quad g = \det(g_{kl}) . \tag{A.12}$$

From Eqs (A.9), (A.10) and (A.11) it is easy to derive the formulas

$$g^{kl} = \boldsymbol{g}^k \cdot \boldsymbol{g}^l , \quad g^k_l = \boldsymbol{g}^k \cdot \boldsymbol{g}_l = g^{km} g_{ml} = \delta^k_l . \tag{A.13}$$

Tensor g^{kl} is called the *contravariant metric tensor.* One can see that it is symmetric, $g^{kl} = g^{lk}$. Tensor g^k_l is a mixed metric tensor with the components $g^k_l = \delta^k_l$, where δ^k_l is Kronecker's delta symbol.

A.3. Tensors

Definition 1: We shall say that *tensor* **A** is defined in three-dimensional space if 3^{p+q} numbers $A^{k_1 \dots k_p}$ are assigned to every coordinate system, so that the coordinate transformation $x^{k'} = x^{k'}(x^1, x^2, x^3)$ transforms these numbers according to the relations

$$A^{k'_1 \dots k'_p}{}_{l'_1 \dots l'_q}(\boldsymbol{x}') = G^{k'_1 \dots k'_p l_1 \dots l_q}_{k_1 \dots k_p l'_1 \dots l'_q} A^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\boldsymbol{x}) , \tag{A.14}$$

where

$$G^{k'_1 \dots k'_p l_1 \dots l_q}_{k_1 \dots k_p l'_1 \dots l'_q} = \frac{\partial x^{k'_1}}{\partial x^{k_1}} \cdots \frac{\partial x^{k'_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial x^{l'_1}} \cdots \frac{\partial x^{l_q}}{\partial x^{l'_q}} . \tag{A.15}$$

We shall say that tensor **A** is p-times *contravariant* and q-times *covariant.* The total number of indices $p + q$ is the *rank* (*degree*) of the tensor, and the numbers $A^{k_1 \dots k_p}{}_{l_1 \dots l_q}$ are referred to as the *coordinates* of the tensor.

Example 1 (scalar): If we assign the same number A to every coordinate system, the number determines a zero-order tensor ($p = q = 0$), which is called a *scalar*,

$$A'(\boldsymbol{x}') = A(\boldsymbol{x}) . \tag{A.16}$$

Example 2 (vector): In changing the coordinates, the contravariant ($p = 1$,

$q = 0$), or covariant ($p = 0$, $q = 1$) coordinates of a *vector* are transformed according to the formulas

(A.17) $$A^{k'}(\boldsymbol{x}') = A^k(\boldsymbol{x})\,\partial x^{k'}/\partial x^k,$$

or

(A.18) $$A_{k'}(\boldsymbol{x}') = A_k(x)\,\partial x^k/\partial x^{k'},$$

respectively.

An example of a contravariant vector is the differential vector $\mathrm{d}x^k$,

(A.19) $$\mathrm{d}x^{k'} = (\partial x^{k'}/\partial x^k)\,\mathrm{d}x^k,$$

which agrees with (A.17) with $A^k = \mathrm{d}x^k$.

Similarly, the partial derivatives of a scalar is a covariant vector,

(A.20) $$\partial \Phi/\partial x^{k'} = (\partial \Phi/\partial x^k)\,\partial x^k/\partial x^{k'},$$

which agrees with (A.18) with $A_k = \partial \Phi/\partial x^k$.

Example 3 (2nd-order tensor): In changing the coordinates, the contravariant ($p = 2$, $q = 0$), covariant ($p = 0$, $q = 2$) and mixed ($p = 1$, $q = 1$) coordinates of a *2nd-order tensor* are transformed according to the formulas

(A.21) $$A^{k'l'}(\boldsymbol{x}') = A^{kl}(\boldsymbol{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^{l'}/\partial x^l),$$

(A.22) $$A_{k'l'}(\boldsymbol{x}') = A_{kl}(\boldsymbol{x})\,(\partial x^k/\partial x^{k'})\,(\partial x^l/\partial x^{l'}),$$

(A.23) $$A^{k'}{}_{l'}(\boldsymbol{x}') = A^k{}_l(\boldsymbol{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^{l'}/\partial x^l).$$

An example of a covariant or contravariant 2nd-order tensor is the metric tensor g_{kl} or g^{kl}, respectively, since

(A.24) $$g_{k'l'}(\boldsymbol{x}') = \boldsymbol{g}_{k'}(\boldsymbol{x}')\,.\,\boldsymbol{g}_{l'}(\boldsymbol{x}') = \boldsymbol{g}_k(\boldsymbol{x})\,.\,\boldsymbol{g}_l(\boldsymbol{x})\,(\partial x^k/\partial x^{k'})\,(\partial x^l/\partial x^{l'}).$$

The same applies to quantities g^{kl}. The quantities $g^k{}_l = \delta^k_l$ are the coordinates of a mixed 2nd-order tensor, since

(A.25) $$\delta^k_l(\boldsymbol{x})\,(\partial x^{k'}/\partial x^k)\,(\partial x^l/\partial x^{l'}) = \\ = (\partial x^{k'}/\partial x^k)\,(\partial x^k/\partial x^{l'}) = \partial x^{k'}/\partial x^{l'} = \delta^{k'}_{l'}(\boldsymbol{x}').$$

Lemma 1 (index law): Let $A^{k_1\ldots k_p}{}_{l_1\ldots l_q}$ be any p-times contravariant and q-times covariant tensor and let $s \geq q$, $t \geq p$. If the multiplication

(A.26) $$A^{k_1\ldots k_p}{}_{l_1\ldots l_q} X^{l_1\ldots l_q\ldots l_s}{}_{k_1\ldots k_p\ldots k_t} = B^{l_{q+1}\ldots l_s}{}_{k_{p+1}\ldots k_t}$$

produces an arbitrary (s—q)-times contravariant and (t—p)-times covariant tensor **B**, the quantity **X** is an s-times contravariant and t-times covariant tensor.

Proof: Assume Eq. (A.26) to hold in some coordinate system, i.e.

(A.27) $$A^{k'_1\ldots k'_p}{}_{l'_1\ldots l'_q} X^{l'_1\ldots l'_q\ldots l'_s}{}_{k'_1\ldots k'_p\ldots k'_t} = B^{l'_{q+1}\ldots l'_s}{}_{k'_{p+1}\ldots k'_t}.$$

Since **A** and **B** are tensors, the following transformation relations apply to them,

$$A^{k'_1\dots k'_p}{}_{l'_1\dots l'_q} = G^{k'_1\dots k'_p l_1\dots l_q}_{k_1\dots k_p l'_1\dots l'_q} A^{k_1\dots k_p}{}_{l_1\dots l_q}$$

and the same applies to tensor **B**. By substituting these transformations into (A.27) we obtain

(A.28) $$G^{k'_1\dots k'_p l_1\dots l_q}_{k_1\dots k_p l'_1\dots l'_q} A^{k_1\dots k_p}{}_{l_1\dots l_q} X^{l'_1\dots l'_q\dots l'_s}{}_{k'_1\dots k'_p\dots k'_t} = G^{l'_{q+1}\dots l'_s k_{p+1}\dots k_t}_{l_{q+1}\dots l_s k'_{p+1}\dots k'_t} B^{l_{q+1}\dots l_s}{}_{k_{p+1}\dots k_t}\,.$$

If we multiply (A.26) by $G^{l'_{q+1}\dots l'_s k_{p+1}\dots k_t}_{l_{q+1}\dots l_s k'_{p+1}\dots k'_t}$ and subtract the result from (A.28), we arrive at

(A.29) $$G^{k'_1\dots k'_p l_1\dots l_q}_{k_1\dots k_p l'_1\dots l'_q} A^{k_1\dots k_p}{}_{l_1\dots l_q}\left(X^{l'_1\dots l'_s}{}_{k'_1\dots k'_t} - G^{k_1\dots k_t l'_1\dots l'_s}_{k'_1\dots k'_t l_1\dots l_s} X^{l_1\dots l_s}{}_{k_1\dots k_t}\right) = 0\,,$$

where we have made use of the following properties of the quantities $G^{k'_1\dots k'_p l_1\dots l_q}_{k_1\dots k_p l'_1\dots l'_q}$ defined by Eq. (A.15),

(A.30) $$G^{k'_1\dots k'_p}_{k_1\dots k_p} G^{l_1\dots l_q}_{l'_1\dots l'_q} = G^{k'_1\dots k'_p l_1\dots l_q}_{k_1\dots k_p l'_1\dots l'_q}\,,$$

$$G^{k_1\dots k_p}_{k'_1\dots k'_p} G^{k'_1\dots k'_p}_{l_1\dots l_p} = \delta^{k_1}_{l_1}\delta^{k_2}_{l_2}\dots\delta^{k_p}_{l_p}\,.$$

Since the factor preceding the parentheses in Eq. (A.29) is an arbitrary tensor, the necessary and sufficient condition for (A.29) to be satisfied is that the expression in the parentheses should be zero. It then follows that

$$X^{l'_1\dots l'_s}{}_{k'_1\dots k'_t} = G^{k_1\dots k_t l'_1\dots l'_s}_{k'_1\dots k'_t l_1\dots l_s} X^{l_1\dots l_s}{}_{k_1\dots k_t}\,,$$

Q.E.D.

Definition 2 (transposed tensor): A tensor which is created by the permutation of two superscripts or two subscripts, is referred to as a tensor *transposed* with respect to these indices.

Example 4: The contravariant, covariant and mixed components of a transposed 2nd-order tensor are $(\mathbf{A}^T)^{kl} = A^{lk}$, $(\mathbf{A}^T)_{kl} = A_{lk}$, $(\mathbf{A}^T)^k{}_l = A_l{}^k$, $(\mathbf{A}^T)_l{}^k = A^k{}_l$.

Definition 3 (symmetric tensor): We shall refer to a tensor as *symmetric* with respect to the superscripts or subscripts, provided its coordinates remain unchanged under any permutation of these indices, e.g. tensor $A^n{}_{klm}$ is symmetric with respect to the first two subscripts provided $A^n{}_{klm} = A^n{}_{lkm}$.

Example 5: Metric tensors g_{kl}, g^{kl} and $g^k{}_l$ are symmetric tensors because $g_{kl} = \mathbf{g}_k \cdot \mathbf{g}_l = \mathbf{g}_l \cdot \mathbf{g}_k = g_{lk}$, and similarly for tensors g^{kl} and $g^k{}_l$. This implies the symmetry of Kronecker's delta symbol: $\delta^k{}_l = \delta_l{}^k$.

A.4. Tensor algebra

Definition 4 (equality of tensors): We say two tensors are *equal* if they are p-times contravariant and q-times covariant and if their coordinates are equal at least

in one coordinate system. Their coordinates are then equal in any coordinate system. Their coordinates are then equal in any coordinate system.
Definition 5 (addition of tensors): If two tensors are of the same order and type, the *sum* or difference of these tensors is a tensor of the same order and type, e.g.

(A.31) $$C^{kl}{}_m = A^{kl}{}_m + B^{kl}{}_m .$$

Definition 6 (outer product of tensors): The *outer product* of two tensors is obtained by simple multiplication of the tensor components, e.g.

(A.32) $$C^{kl}{}_m = A^{kl} B_m .$$

Lemma 1 implies that this operation yields a tensor whose order is equal to the sum of the orders of the factors.
Example 6 (dyadic product): The outer product of two vectors is called the *dyadic product,*

(A.33)
$$C^{kl} = A^k B^l \quad \text{contravariant component,}$$
$$C_{kl} = A_k B_l \quad \text{covariant component,}$$
$$C^k{}_l = A^k B_l \quad \text{mixed component.}$$

Definition 7 (tensor contraction): The algebraic operation in which we put the covariant and contravariant indices of a tensor equal to each other and add with respect to these identical indices is reffered to as tensor *contraction*, e.g.

(A.34) $$A^k{}_{kl}, \; A^k{}_{lk} .$$

Lemma 1 implies that the order of the contracted tensor is lower by two than the order of the original tensor. The type of contracted tensor is determined by the number of free indices. It is easy to prove that no tensor quantity is obtain, if this procedure is applied to two indices of the same type, i.e. either to both covariant or to both contravariant indices.
Definition 8 (raising and lowering the indices): The algebraic operation in which we assign the quantity $A_{k_1 \ldots k_p l_1 \ldots l_q}$ to every p-times contravariant and q-times covariant tensor $A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}$ by the relation

(A.35) $$A_{k_1 \ldots k_p l_1 \ldots l_q} = g_{m_1 k_1} \cdots g_{m_p k_p} A^{m_1 \ldots m_p}{}_{l_1 \ldots l_q} ,$$

where g_{kl} are the components of the covariant metric tensor, is referred to as *lowering* the indices of tensor **A**. Lemma 1 implies that the quantity $A_{k_1 \ldots k_p l_1 \ldots l_q}$ is a $(p + q)$-times covariant tensor.

Similarly, the algebraic operation in which we construct to tensor $A^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}$ a new $(p + q)$-times contravariant tensor

(A.36) $$A^{k_1 \ldots k_p l_1 \ldots l_q} = g^{m_1 l_1} \ldots g^{m_q l_q} A^{k_1 \ldots k_p}{}_{m_1 \ldots m_q} ,$$

where g^{kl} are the components of the contravariant metric tensor, is referred to as *raising* the indices of tensor **A**.

If we raise or lower only some of the indices of a tensor, we again obtain a tensor quantity. With tensors of higher orders than the first we use a gap (sometimes a dot) to indicate the original position of the indices we have lowered or raised. By raising or lowering indicates of a given tensor we obtain so-called *associated tensors*.

Example 7: By lowering the contravariant coordinates v^k of vector $\boldsymbol{v}$, we obtain its covariant coordinates and vice versa,

$$v_k = g_{kl}v^l, \quad v^k = g^{kl}v_l. \tag{A.37}$$

Example 8: Raising the indices of a 2nd-order tensor can be expressed as

$$A^k{}_l = g^{mk}A_{ml}, \quad A_l{}^k = g^{km}A_{lm}. \tag{A.38}$$

In general, tensors $A^k{}_l$ and $A_l{}^k$ are not equal. Only if tensor **A** is symmetric, $A^k{}_l = A_l{}^k$ and the relative positioning of the indices is unimportant.

Similarly, lowering the indicies can be expressed as

$$A^k{}_l = g_{lm}A^{km}, \quad A_l{}^k = g_{lm}A^{mk}. \tag{A.39}$$

The following relations also hold:

$$A^{kl} = g^{km}A_m{}^l = g^{lm}A^k{}_m = g^{km}g^{ln}A_{mn}, \tag{A.40}$$

$$A_{kl} = g_{ml}A_k{}^m = g_{km}A^m{}_l = g_{km}g_{ln}A^{mn}, \tag{A.41}$$

$$A^k{}_l = g^{km}g_{ln}A_m{}^n. \tag{A.42}$$

The associated tensors A^{kl}, $A^k{}_l$, $A_l{}^k$, A_{kl} characterize one and the same 2nd-order tensor **A**.

Definition 9 (inner product of tensors): We define the *inner (scalar) product* of tensors by contraction of the outer product of two tensors. The inner (scalar) product of vectors and tensors will be denoted by a dot.

Example 9 (scalar product of vectors): By contracting the dyadic product of two vectors, we define the *inner (scalar) product of two vectors*:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_k v^k = u^k v_k. \tag{A.43}$$

Lemma 2: The scalar product of vectors is an *invariant*, i.e. it is independent of the coordinate system.

Proof: With a view to Eqs (A.17) and (A.18)

$$u^{k'}v_{k'}(\boldsymbol{x}') = u^k(\boldsymbol{x})\, v_k(\boldsymbol{x})\, (\partial x^{k'}/\partial x^k)(\partial x^k/\partial x^{k'}) = u^k\, v_k(\boldsymbol{x}). \tag{A.44}$$

Example 10 (scalar product of a vector and 2nd-order tensor): By contracting the outer product of a vector $\boldsymbol{v}$ and 2nd-order tensor **A**, we define the *left-hand* and *right-hand scalar product* of a vector and 2nd-order tensor, (A.45) $$(\boldsymbol{v}.\mathbf{A})^k = A_l^{\ k} v^l = A^{lk} v_l,$$

and

(A.46) $$(\mathbf{A}.\boldsymbol{v})^k = A^k{}_l v^l = A^{kl} v_l,$$

respectively. By lowering the indices we obtain the covariant components of these vectors,

(A.47) $$(\boldsymbol{v}.\mathbf{A})_k = A_{lk} v^l = A^l{}_k v_l,$$

and

(A.48) $$(\mathbf{A}.\boldsymbol{v})_k = A_{kl} v^l = A_k{}^l v_l,$$

respectively.

Lemma 3: Assume φ to be a scalar $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ to be vectors and $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ to be 2nd-order tensors. It then holds that

(A.49) $$(\mathbf{A}_1 + \mathbf{A}_2).\boldsymbol{v} = \mathbf{A}_1.\boldsymbol{v} + \mathbf{A}_2.\boldsymbol{v},$$

(A.50) $$\mathbf{A}.(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \mathbf{A}.\boldsymbol{v}_1 + \mathbf{A}.\boldsymbol{v}_2,$$

(A.51) $$(\mathbf{A}.\varphi\boldsymbol{v}) = \varphi(\mathbf{A}.\boldsymbol{v}),$$

(A.52) $$\mathbf{A}.\boldsymbol{v} = \boldsymbol{v}.\mathbf{A}^{\mathrm{T}},$$

(A.53) $$(\boldsymbol{v}_1\boldsymbol{v}_2).\boldsymbol{v}_3 = \boldsymbol{v}_1(\boldsymbol{v}_2.\boldsymbol{v}_3),$$

(A.54) $$\boldsymbol{v}_1.(\boldsymbol{v}_2\boldsymbol{v}_3) = (\boldsymbol{v}_1.\boldsymbol{v}_2)\boldsymbol{v}_3,$$

where $\mathbf{A}^{\mathrm{T}}$ is the tensor transposed to tensor $\mathbf{A}$ and $\boldsymbol{v}_1\boldsymbol{v}_2$ is the dyadic product of vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Proof: Equations (A.49)—(A.52) follow immediately from the definition of the scalar product of a vector and 2nd-order tensor. Let us prove Eq. (A.53), e.g. for the contravariant component,

$$[(\boldsymbol{v}_1\boldsymbol{v}_2).\boldsymbol{v}_3]^k = (\boldsymbol{v}_1\boldsymbol{v}_2)^k{}_l\, v_3^l = v_1^k(\boldsymbol{v}_2)_l v_3^l = [\boldsymbol{v}_1(\boldsymbol{v}_2.\boldsymbol{v}_3)]^k.$$

Equation (A.54) can be proved in very much the same way.

Example 11 (scalar product of 2nd-order tensors): By contracting the outer product of two 2nd-order tensors $\mathbf{A}$ and $\mathbf{B}$, we define the *scalar product* of these tensors:

(A.55) $$(\mathbf{A}.\mathbf{B})^k{}_l = A^k{}_m B^m{}_l.$$

By lowering and raising the indices we obtain the second mixed component of this tensor:

(A.56) $$(\mathbf{A}.\mathbf{B})_l{}^k = A_l{}^m B_m{}^k.$$

Lemma 4: Let φ be a scalar, $\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ vectors, $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{B}, \mathbf{B}_1, \mathbf{B}_2$ 2nd-order tensors. It then holds that

(A.57) $$(\mathbf{A}_1 + \mathbf{A}_2) . \mathbf{B} = \mathbf{A}_1 . \mathbf{B} + \mathbf{A}_2 . \mathbf{B},$$

(A.58) $$\mathbf{A} . (\mathbf{B}_1 + \mathbf{B}_2) = \mathbf{A} . \mathbf{B}_1 + \mathbf{A} . \mathbf{B}_2,$$

(A.59) $$(\varphi \mathbf{A}) . \mathbf{B} = \varphi(\mathbf{A} . \mathbf{B}),$$

(A.60) $$\mathbf{A} . (\varphi \mathbf{B}) = \varphi(\mathbf{A} . \mathbf{B}),$$

(A.61) $$(\mathbf{A}_1 . \mathbf{A}_2) . \mathbf{B} = \mathbf{A}_1 . (\mathbf{A}_2 . \mathbf{B}),$$

(A.62) $$(\mathbf{A} . \mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} . \mathbf{A}^{\mathrm{T}},$$

(A.63) $$(\mathbf{A} . \mathbf{B}) . \boldsymbol{v} = \mathbf{A} . (\mathbf{B} . \boldsymbol{v}),$$

(A.64) $$\boldsymbol{v} . (\mathbf{A} . \mathbf{B}) = (\boldsymbol{v} . \mathbf{A}) . \mathbf{B},$$

(A.65) $$(\boldsymbol{v}_1 \boldsymbol{v}_2) . (\boldsymbol{v}_3 \boldsymbol{v}_4) = (\boldsymbol{v}_2 . \boldsymbol{v}_3) \boldsymbol{v}_1 \boldsymbol{v}_4 .$$

Proof: Equations (A.57)—(A.60) follow immediately from the definition of the scalar product of two 2nd-order tensors. Therefore, let us only prove the remaining Eqs (A.61)—(A.65):

$$[(\mathbf{A}_1 . \mathbf{A}_2) . \mathbf{B}]^k{}_l = (\mathbf{A}_1 . \mathbf{A}_2)^k{}_m B^m{}_l =$$
$$= (\mathbf{A}_1)^k{}_n (\mathbf{A}_2)^n{}_m \mathbf{B}^m{}_l = (\mathbf{A}_1)^k{}_n (\mathbf{A}_2 . \mathbf{B})^n{}_l = [\mathbf{A}_1 . (\mathbf{A}_2 . \mathbf{B})]^k{}_l,$$

$$[(\mathbf{A} . \mathbf{B})^{\mathrm{T}}]^k{}_l = (\mathbf{A} . \mathbf{B})_l{}^k = A_l{}^m B_m{}^k = (\mathbf{B}^{\mathrm{T}})^k{}_m (\mathbf{A}^{\mathrm{T}})^m{}_l = [(\mathbf{B}^{\mathrm{T}} . \mathbf{A}^{\mathrm{T}})]^k{}_l,$$

$$[(\mathbf{A} . \mathbf{B}) . \boldsymbol{v}]^k = (\mathbf{A} . \mathbf{B})^k{}_l v^l = A^k{}_m B^m{}_l v^l = A^k{}_m (\mathbf{B} . \boldsymbol{v})^m = [\mathbf{A} . (\mathbf{B} . \boldsymbol{v})]^k,$$

$$[\boldsymbol{v} . (\mathbf{A} . \mathbf{B})]^k = (\mathbf{A} . \mathbf{B})_l{}^k v^l = A_l{}^m B_m{}^k v^l = (\boldsymbol{v} . \mathbf{A})^m B_m{}^k = [(\boldsymbol{v} . \mathbf{A}) . \mathbf{B}]^k,$$

$$[(\boldsymbol{v}_1 \boldsymbol{v}_2) . (\boldsymbol{v}_3 \boldsymbol{v}_4)]^k{}_l = (\boldsymbol{v}_1 \boldsymbol{v}_2)^k{}_m (\boldsymbol{v}_3 \boldsymbol{v}_4)^m{}_l = (\boldsymbol{v}_1)^k (\boldsymbol{v}_2)_m (\boldsymbol{v}_3)^m (\boldsymbol{v}_4)_l =$$
$$= [(\boldsymbol{v}_2 . \boldsymbol{v}_3)(\boldsymbol{v}_1 \boldsymbol{v}_4)]^k{}_l .$$

The proofs for the other components are similar.

A.5. Physical components

We have so far represented a vector by its contravariant or covariant components

(A.66) $$\boldsymbol{v} = v^k \boldsymbol{g}_k = v_k \boldsymbol{g}^k,$$

where $\boldsymbol{g}_k$ is the vector tangential to the kth coordinate line at point x^k. Since vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are not generally unit vectors, components v^k and v_k do not have the same physical dimension as vector $\boldsymbol{v}$. Let us assign the unit vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}^k$ to vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$. However, the square of the length of a vector is given by the scalar product of this vector with itself; in virtue of Eqs (A.9) and (A.13)

(A.67) $$\boldsymbol{g}_k \cdot \boldsymbol{g}_k = g_{\underline{kk}}, \quad \boldsymbol{g}^k \cdot \boldsymbol{g}^k = g^{\underline{kk}},$$

where no summation is carried out over the underscore indices. The unit vectors to vector $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are then defined by the relations

(A.68) $$\boldsymbol{e}_k = \boldsymbol{g}_k/(g_{\underline{kk}})^{1/2}, \quad \boldsymbol{e}^k = \boldsymbol{g}^k/(g^{\underline{kk}})^{1/2}.$$

With a view to (A.66), vector $\boldsymbol{v}$ can be resolved into these unit vectors as

(A.69) $$\boldsymbol{v} = v^{(k)}\boldsymbol{e}_k = v_{(k)}\boldsymbol{e}^k,$$

where the quantities $v^{(k)}$ and $v_{(k)}$ are the *physical components* of vector $\boldsymbol{v}$. We use the term physical because these components have the same physical dimension as vector $\boldsymbol{v}$. By substituting Eqs (A.66) and (A.68) into (A.69), we obtain the formula for the physical components of vector $\boldsymbol{v}$,

(A.70) $$v^{(k)} = v^k(g_{\underline{kk}})^{1/2}, \quad v_{(k)} = v_k(g^{\underline{kk}})^{1/2}.$$

By substituting Eq. (A.37) into (A.70), we can derive the relation between the two kinds of physical components:

(A.71) $$v_{(k)} = \sum_l g_{kl}(g^{\underline{kk}}/g_{ll})^{1/2}v^{(l)}.$$

If the curvilinear coordinates are orthogonal, $g_{kl} = g^{kl} = 0$ for $k \neq l$,

(A.72) $$v_{(k)} = v^{(k)},$$

i.e. the difference between the two kinds of physical components of the vector vanishes.

This definition of the physical components can also be extended to tensors of higher orders with the aid of their relations with vectors. Let us demonstrate the resolution of the stress tensor $\mathbf{t}$ into physical components. For the time being, let us not assume that the stress tensor $\mathbf{t}$ is a symmetric 2nd-order tensor. The projection of the stress tensor $\mathbf{t}$ onto the unit external normal $\boldsymbol{n}$ defines the stress vector $\boldsymbol{t}$,

(A.73) $$\boldsymbol{t} = \mathbf{t} \, . \, \boldsymbol{n},$$

i.e. the components expressed with the aid of Eqs (A.46) and (A.48) read

(A.74) $$t^k = t^k{}_l n^l = t^{kl} n_l, \quad t_k = t_k{}^l n_l = t_{kl} n^l.$$

If we express vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ in terms of physical components (A.70), we obtain the relations

(A.75) $$t^{(k)} = t^{(k)}{}_{(l)} n^{(l)} = t^{(k)(l)} n_{(l)},$$
$$t_{(k)} = t_{(k)}{}^{(l)} n_{(l)} = t_{(k)(l)} n^{(l)},$$

where the quantities

(A.76)
$$t^{(k)}{}_{(l)} = t^k{}_l(g_{\underline{kk}}/g_{\underline{ll}})^{1/2},$$
$$t^{(k)(l)} = t^{kl}(g_{\underline{kk}}g_{\underline{ll}})^{1/2},$$
$$t_{(k)}{}^{(l)} = t_k{}^l(g_{\underline{ll}}/g_{\underline{kk}})^{1/2},$$
$$t_{(k)(l)} = t_{kl}(g_{\underline{kk}}g_{\underline{ll}})^{-1/2},$$

are the physical components of the stress tensor $\mathbf{t}$. Let it be emphasized that the quantities $t^{(k)}{}_{(l)}$, $t^{(k)(l)}$, $t_{(k)}{}^{(l)}$ and $t_{(k)(l)}$ are not tensor components. The relation between the right-hand and left-hand physical component can be derived with the aid of (A.42) and (A.76):

(A.77)
$$t_{(k)}{}^{(l)} = \sum_{m,n} (g_{nn}g_{\underline{ll}}/g_{\underline{kk}}g_{mm})^{1/2} g_{km}g^{ln}t^{(m)}{}_{(n)}.$$

If the curvilinear coordinates are orthogonal, $g_{kl} = g^{kl} = 0$,

(A.78)
$$t^{(k)}{}_{(l)} = t^{(k)(l)} = t_k{}^{(l)} = t_{(k)(l)},$$

i.e. the physical components of the stress tensor $\mathbf{t}$ in orthogonal curvilinear components are the same for all types of tensor coordinates.

A.6. Covariant derivative

As compared to Cartesian coordinates, the greatest difficulty in the system of curvilinear coordinates is that the basis vectors $\boldsymbol{g}_k$ and $\boldsymbol{g}^k$ are functions of the curvilinear coordinates x^k, so that in differentiating and integrating these vectors do not behave like constants. Therefore, let us first derive the formulas for differentiating these vectors with respect to the curvilinear coordinates. We shall put

(A.79)
$$\frac{\partial \boldsymbol{g}_k}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\frac{\partial \boldsymbol{p}}{\partial x^k}\right) = \frac{\partial^2 y^m}{\partial x^l \partial x^k}\boldsymbol{i}_m,$$

because the Cartesian unit vectors $\boldsymbol{i}_m$ do not depend on coordinates. After substituting for $\boldsymbol{i}_m$ from (A.6),

(A.80)
$$\frac{\partial \boldsymbol{g}_k}{\partial x^l} = \begin{Bmatrix} m \\ k\ \ l \end{Bmatrix} \boldsymbol{g}_m,$$

where the quantities

(A.81)
$$\begin{Bmatrix} m \\ k\ \ l \end{Bmatrix} = \frac{\partial^2 y^n}{\partial x^k \partial x^l}\frac{\partial x^m}{\partial y^n}$$

are referred to as *Christoffel's symbols of the 2nd kind. Christoffel's symbols of the 1st kind* are defined by the relations

(A.82) $$[kl,m] = g_{mn}\begin{Bmatrix} n \\ k\ l \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} m \\ k\ l \end{Bmatrix} = g^{mn}[kl,n].$$

By using (A.9) it is no difficult to prove that

(A.83) $$[kl, m] = \frac{1}{2}\left(\frac{\partial g_{km}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right).$$

Let it be emphasized that Christoffel's symbols are *not* tensors. However, they are symmetric with respect to indices k, l,

(A.84) $$[kl,m] = [lk,m], \quad \begin{Bmatrix} m \\ k\ l \end{Bmatrix} = \begin{Bmatrix} m \\ l\ k \end{Bmatrix}.$$

By making use of (A.11), we obtain a similar result to (A.79),

(A.85) $$\frac{\partial \boldsymbol{g}^k}{\partial x^l} = -\begin{Bmatrix} k \\ l\ m \end{Bmatrix}\boldsymbol{g}^m.$$

We can now calculate the partial derivatives of vector $\boldsymbol{v}$,

(A.86) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = \frac{\partial}{\partial x^k}(v^m \boldsymbol{g}_m) = \frac{\partial v^m}{\partial x^k}\boldsymbol{g}_m + v^m\frac{\partial \boldsymbol{g}_m}{\partial x^k} = \left(\frac{\partial v^m}{\partial x^k} + \begin{Bmatrix} m \\ k\ l \end{Bmatrix} v^l\right)\boldsymbol{g}_m,$$

which can be abbreviated to

(A.87) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = v^m{}_{;k}\boldsymbol{g}_m,$$

where the expression

(A.88) $$v^m{}_{;k} = \frac{\partial v^m}{\partial x^k} + \begin{Bmatrix} m \\ k\ l \end{Bmatrix} v^l$$

is the *covariant partial derivative* of the contravariant vector v^m.

A covariant partial derivative, or the partial derivative of any tensor is denoted by adding a semi-colon after the last tensor index, or a comma, and a further index appropriate to the coordinate with respect to which the covariant partial derivative, or partial derivative, respectively, is being performed.

By differentiating the expression $\boldsymbol{v} = v_m \boldsymbol{g}^m$, we obtain the covariant partial derivative of the covariant vector v_m,

(A.89) $$\frac{\partial \boldsymbol{v}}{\partial x^k} = v_{m;k}\boldsymbol{g}^m,$$

where

(A.90) $$v_{m;k} = v_{m,k} - \begin{Bmatrix} l \\ m\ k \end{Bmatrix} v_l.$$

The reason for introducing, besides ordinary partial derivatives, also covariant partial derivatives, is that applying the covariant derivative to any tensor increases the order of the tensor by one covariant index, whereas a partial derivative of a tensor is not, in general, a tensor quantity.

Since Christoffel's symbols are identically equal to zero in Cartesian coordinates, covariant partial derivatives in this coordinate system reduce to "ordinary" partial derivatives.

The covariant partial derivative of a scalar is identical with an "ordinary" partial derivative, because a scalar is a covariant tensor of order zero. Covariant partial derivatives of higher-order tensors are defined in a similar fashion as the covariant derivatives of vectors, e.g. the covariant partial derivative of a 2nd-order tensor,

(A.91)
$$A^{kl}{}_{;m} = A^{kl}{}_{,m} + \begin{Bmatrix} k \\ m\ n \end{Bmatrix} A^{nl} + \begin{Bmatrix} l \\ m\ n \end{Bmatrix} A^{kn},$$
$$A^{k}{}_{l;m} = A^{k}{}_{l,m} - \begin{Bmatrix} n \\ l\ m \end{Bmatrix} A^{k}{}_{n} + \begin{Bmatrix} k \\ m\ n \end{Bmatrix} A^{n}{}_{l},$$
$$A_{kl;m} = A_{kl,m} - \begin{Bmatrix} n \\ k\ m \end{Bmatrix} A_{nl} - \begin{Bmatrix} n \\ l\ m \end{Bmatrix} A_{kn},$$

is a tensor of the 3rd order.

Lemma 5 (Ricci): The covariant partial derivatives of any metric tensor are zero,

(A.92)
$$g_{kl;m} = g^{kl}{}_{;m} = g^{k}{}_{l;m} = g_{;k} = 0\,,$$

where $g = \det(g_{kl})$.

Proof: With a view to $(A.91)_3$,

(A.93)
$$g_{kl;m} = g_{kl,m} - \begin{Bmatrix} n \\ k\ m \end{Bmatrix} g_{nl} - \begin{Bmatrix} n \\ l\ m \end{Bmatrix} g_{kn}\,.$$

By using Eqs $(A.82)_2$ and (A.83) we can prove that the r.h.s. of Eq. (A.93) is equal to zero, $g_{kl;m} = 0$. The other relations of (A.92) can be proved in very much the same way.

Equation $(A.92)_4$ yields the following useful relation:

(A.94)
$$(\log\sqrt{g})_{,k} = \begin{Bmatrix} m \\ m\ k \end{Bmatrix}, \qquad g \equiv \det(g_{kl})\,.$$

Lemma 5 implies that metric tensors under covariant differentiation behave like constants, consequently, whether we raise or lower the index before covariant differentiation or after it is unimportant. It is easy to prove, for example, that

(A.95)
$$A^{k}{}_{;l} = (g^{km} A_m)_{;l} = g^{km} A_{m;l}\,.$$

It is also easy to prove that the product rule of differentiation holds for the covariant partial differentiation, e.g.,

(A.96) $$(A^k B_{lm})_{;n} = A^k{}_{;n} B_{lm} + A^k B_{lm;n} .$$

Sometimes, by means of the covariant partial derivative, we also introduce the contravariant partial derivative as

(A.97) $$A^k{}_{l;}{}^m = A^k{}_{l;n} g^{nm} .$$

A.7. Invariant differential operators

The invariant differential operators *gradient* (grad) of scalar Φ, *divergence* (div) and *rotation* (rot) of vector $\boldsymbol{v}$, are defined by the relations

(A.98) $$\operatorname{grad} \Phi = \Phi_{,k} \boldsymbol{g}^k ,$$

(A.99) $$\operatorname{div} \boldsymbol{v} = v^k{}_{;k} ,$$

(A.100) $$\operatorname{rot} \boldsymbol{v} = \varepsilon^{klm} v_{m;l} \boldsymbol{g}_k ,$$

(A.101) $$\varepsilon^{klm} = e^{klm}/\sqrt{g} , \qquad \varepsilon_{klm} = e_{klm}\sqrt{g}$$

and e^{klm} and e_{klm} are *Levi-Civita alternating symbols,*

(A.102) $$e^{123} = e^{312} = e^{231} = -e^{213} = -e^{321} = -e^{132} = 1 ,$$

and the other $e^{klm} = 0$. The symbols e_{klm} are defined similarly. Let us remind the reader of some of the important relations:

(A.103) $$\varepsilon_{pkl} \varepsilon^{qmn} = \begin{vmatrix} \delta^q_p & \delta^q_k & \delta^q_l \\ \delta^m_p & \delta^m_k & \delta^m_l \\ \delta^n_p & \delta^n_k & \delta^n_l \end{vmatrix} ,$$

(A.104) $$\varepsilon_{pkl} \varepsilon^{pmn} = \delta^m_k \delta^n_l - \delta^m_l \delta^n_k ,$$

(A.105) $$\varepsilon_{pkl} \varepsilon^{pkn} = 2\delta^n_l , \qquad \varepsilon_{pkl} \varepsilon^{pkl} = 6 .$$

The operators (A.98)—(A.100) are invariant with respect to a general transformation of coordinates.

It is sometimes advantageous to introduce the *nabla operator* ∇,

(A.106) $$\nabla = \boldsymbol{g}^k \frac{\partial}{\partial x^k} .$$

By using this symbol we can express Eqs (A.98)—(A.100) in the following form:

(A.107) $$\operatorname{grad} \Phi = \nabla \Phi = \boldsymbol{g}^k \Phi_{,k} ,$$

(A.108) $$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \boldsymbol{g}^k \cdot \frac{\partial}{\partial x^k}(v^l \boldsymbol{g}_l) = \boldsymbol{g}^k \cdot \boldsymbol{g}_l v^l{}_{;k} = v^k{}_{;k},$$

(A.109) $$\operatorname{rot} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \boldsymbol{g}^k \times \frac{\partial}{\partial x^k}(v^l g_l) = \boldsymbol{g}^k \times \boldsymbol{g}_l v^l{}_{;k} = \varepsilon^{klm} v_{m;l} \boldsymbol{g}_k .$$

If we use Eq. (A.94), we can express div $\boldsymbol{v}$ in a more convenient form:

(A.110) $$\operatorname{div} \boldsymbol{v} = v^k{}_{;k} = v^k{}_{,k} + \begin{Bmatrix} k \\ k \ l \end{Bmatrix} v^l = v^k{}_{,k} + v^k (\log \sqrt{g})_{,k} = = [\sqrt{(g)} v^k]_{,k} / \sqrt{g} .$$

Laplace's operator is

(A.111) $$\nabla^2 \Phi = \operatorname{div} \operatorname{grad} \Phi = (g^{kl} \Phi_{,l})_{;k} = g^{kl} (\Phi_{,l})_{;k} = [\sqrt{(g)} g^{kl} \Phi_{,l}]_{,k} / \sqrt{g} .$$

Let us generalize the above differential operations also for tensors of higher orders. The gradient, divergence and rotation of tensor $\mathbf{A}$ are defined by the relations

(A.112) $$\operatorname{grad} \mathbf{A} \equiv \nabla \mathbf{A},$$

(A.113) $$\operatorname{div} \mathbf{A} \equiv \nabla \cdot \mathbf{A},$$

(A.114) $$\operatorname{rot} \mathbf{A} \equiv \nabla \times \mathbf{A}.$$

If $\mathbf{A}$ is a tensor of order p,

(A.115) $$\mathbf{A} = A^{k_1 \dots k_p} \boldsymbol{g}_{k_1} \dots \boldsymbol{g}_{k_p} = A_{k_1 \dots k_p} \boldsymbol{g}^{k_1} \dots \boldsymbol{g}^{k_p},$$

and, consequently,

(A.116) $$\operatorname{grad} \mathbf{A} = A_{k_1 \dots k_p;m} \boldsymbol{g}^m \boldsymbol{g}^{k_1} \dots \boldsymbol{g}^{k_p} = (\operatorname{grad} \mathbf{A})_{mk_1 \dots k_p} \boldsymbol{g}^m \boldsymbol{g}^{k_1} \dots \boldsymbol{g}^{k_p},$$

(A.117) $$\operatorname{div} \mathbf{A} = A^{mk_2 \dots k_p}{}_{;m} \boldsymbol{g}_{k_2} \dots \boldsymbol{g}_{k_p} = (\operatorname{div} \mathbf{A})^{k_2 \dots k_p} \boldsymbol{g}_{k_2} \dots \boldsymbol{g}_{k_p},$$

(A.118) $$\operatorname{rot} \mathbf{A} = \varepsilon^{nmk_1} A_{k_1 \dots k_p;m} \boldsymbol{g}_n \boldsymbol{g}^{k_2} \dots \boldsymbol{g}^{k_p} = (\operatorname{rot} \mathbf{A})^n{}_{k_2 \dots k_p} \boldsymbol{g}_n \boldsymbol{g}^{k_2} \dots \boldsymbol{g}^{k_p}.$$

By lowering and raising the indices, we can express the above tensors in terms of associated components.

Example 12: The *gradient of vector* $\boldsymbol{v} = v_l \boldsymbol{g}^l$ is defined as

(A.119) $$\operatorname{grad} \boldsymbol{v} = v_{l;k} \boldsymbol{g}^k \boldsymbol{g}^l = (\operatorname{grad} \boldsymbol{v})_{kl} \boldsymbol{g}^k \boldsymbol{g}^l .$$

Example 13: For the 2nd-order tensor $\mathbf{A} = A^{kl} \boldsymbol{g}_k \boldsymbol{g}_l = A_{kl} \boldsymbol{g}^k \boldsymbol{g}^l$

(A.120) $$\operatorname{div} \mathbf{A} = A^{kl}{}_{;k} \boldsymbol{g}_l = (\operatorname{div} \mathbf{A})^l \boldsymbol{g}_l ,$$

(A.121) $$\operatorname{rot} \mathbf{A} = \varepsilon^{klm} A_{mn;l} \boldsymbol{g}_k \boldsymbol{g}^n = (\operatorname{rot} \mathbf{A})^k{}_n \boldsymbol{g}_k \boldsymbol{g}^n .$$

Lemma 6: Let Φ, Ψ be scalars, $\boldsymbol{u}$, $\boldsymbol{v}$ vectors and $\mathbf{A}$ a 2nd-order tensor. It then holds that

(A.122) $$\operatorname{grad}(\Phi\Psi) = \Phi \operatorname{grad}\Psi + \Psi \operatorname{grad}\Phi,$$

(A.123) $$\operatorname{div}(\Phi\boldsymbol{u}) = \Phi \operatorname{div}\boldsymbol{u} + \boldsymbol{u}.\operatorname{grad}\Phi,$$

(A.124) $$\operatorname{rot}(\Phi\boldsymbol{u}) = \Phi \operatorname{rot}\boldsymbol{u} + \operatorname{grad}\Phi \times \boldsymbol{u},$$

(A.125) $$\operatorname{grad}(\boldsymbol{u}.\boldsymbol{v}) = \operatorname{grad}\boldsymbol{u}.\boldsymbol{v} + \operatorname{grad}\boldsymbol{v}.\boldsymbol{u},$$

(A.126) $$\operatorname{div}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}.\operatorname{rot}\boldsymbol{u} - \boldsymbol{u}.\operatorname{rot}\boldsymbol{v},$$

(A.127) $$\operatorname{rot}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}.\operatorname{grad}\boldsymbol{u} - \boldsymbol{u}.\operatorname{grad}\boldsymbol{v} + \boldsymbol{u}\operatorname{div}\boldsymbol{v} - \boldsymbol{v}\operatorname{div}\boldsymbol{u},$$

(A.128) $$\boldsymbol{u}\times\operatorname{rot}\boldsymbol{v} = \operatorname{grad}\boldsymbol{v}.\boldsymbol{u} - \boldsymbol{u}.\operatorname{grad}\boldsymbol{v},$$

(A.129) $$\operatorname{rot}\operatorname{grad}\Phi = 0, \qquad \operatorname{div}\operatorname{rot}\boldsymbol{u} = 0,$$

(A.130) $$\operatorname{grad}\operatorname{div}\boldsymbol{u} = \operatorname{div}[(\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}],$$

(A.131) $$\operatorname{rot}\operatorname{rot}\boldsymbol{u} = \operatorname{grad}\operatorname{div}\boldsymbol{u} - \operatorname{div}\operatorname{grad}\boldsymbol{u},$$

(A.132) $$\operatorname{grad}(\Phi\boldsymbol{u}) = \Phi\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\Phi)\boldsymbol{u},$$

(A.133) $$\operatorname{div}(\Phi\mathbf{A}) = \Phi\operatorname{div}\mathbf{A} + \operatorname{grad}\Phi.\mathbf{A},$$

(A.134) $$\operatorname{rot}(\Phi\mathbf{A}) = \Phi\operatorname{rot}\mathbf{A} + \operatorname{grad}\Phi\times\mathbf{A},$$

(A.135) $$\operatorname{div}(\boldsymbol{u}\boldsymbol{v}) = \boldsymbol{v}\operatorname{div}\boldsymbol{u} + \boldsymbol{u}.\operatorname{grad}\boldsymbol{v},$$

(A.136) $$\operatorname{rot}(\boldsymbol{u}\boldsymbol{v}) = (\operatorname{rot}\boldsymbol{u})\boldsymbol{v} - \boldsymbol{u}\times\operatorname{grad}\boldsymbol{v}.$$

Proof:

$$\operatorname{grad}(\Phi\Psi) = (\Phi\Psi)_{,k}\boldsymbol{g}^k = \Phi\Psi_{,k}\boldsymbol{g}^k + \Psi\Phi_{,k}\boldsymbol{g}^k = \Phi\operatorname{grad}\Psi + \Psi\operatorname{grad}\Phi,$$

$$\operatorname{div}(\Phi\boldsymbol{u}) = (\Phi\boldsymbol{u})^k{}_{;k} = \Phi u^k{}_{;k} + u^k\Phi_{,k} = \Phi\operatorname{div}\boldsymbol{u} + \boldsymbol{u}.\operatorname{grad}\Phi,$$

$$\operatorname{rot}(\Phi\boldsymbol{u}) = \varepsilon^{klm}(\Phi\boldsymbol{u})_{m;l}\boldsymbol{g}_k = \varepsilon^{klm}u_m\Phi_{,l}\boldsymbol{g}_k + \Phi\varepsilon^{klm}u_{m;l}\boldsymbol{g}_k = \Phi\operatorname{rot}\boldsymbol{u} + \operatorname{grad}\Phi\times\boldsymbol{u},$$

$$\begin{aligned}\operatorname{grad}(\boldsymbol{u}.\boldsymbol{v}) &= (\boldsymbol{u}.\boldsymbol{v})_{;k}\boldsymbol{g}^k = (u^l v_l)_{;k}\boldsymbol{g}^k = u^l{}_{;k}v_l\boldsymbol{g}^k + u^l v_{l;k}\boldsymbol{g}^k = \\ &= (\operatorname{grad}\boldsymbol{u})^l{}_k v_l\boldsymbol{g}^k + (\operatorname{grad}\boldsymbol{v})_{kl}u^l\boldsymbol{g}^k = \operatorname{grad}\boldsymbol{u}.\boldsymbol{v} + \operatorname{grad}\boldsymbol{v}.\boldsymbol{u},\end{aligned}$$

$$\begin{aligned}\operatorname{div}(\boldsymbol{u}\times\boldsymbol{v}) &= (\boldsymbol{u}\times\boldsymbol{v})^k{}_{;k} = (\varepsilon^{klm}u_l v_m)_{;k} = \varepsilon^{klm}(u_l v_m)_{;k} = \varepsilon^{klm}(u_{l;k}v_m + u_l v_{m;k}) = \\ &= \varepsilon^{klm}u_{l;k}v_m - \varepsilon^{lkm}u_l v_{m;k} = \boldsymbol{v}.\operatorname{rot}\boldsymbol{u} - \boldsymbol{u}.\operatorname{rot}\boldsymbol{v},\end{aligned}$$

$$\begin{aligned}\operatorname{rot}(\boldsymbol{u}\times\boldsymbol{v}) &= \varepsilon^{klm}(\boldsymbol{u}\times\boldsymbol{v})_{m;l}\boldsymbol{g}_k = \varepsilon^{klm}(\varepsilon_{mpq}u^p v^q)_{;l}\boldsymbol{g}_k = \\ &= (\delta_p{}^k\delta_q{}^l - \delta_q{}^k\delta_p{}^l)(u^p{}_{;l}v^q + u^p v^q{}_{;l})\boldsymbol{g}_k = \\ &= (u^k{}_{;l}v^l - u^l v^k{}_{;l} + u^k v^l{}_{;l} - u^l{}_{;l}v^k)\boldsymbol{g}_k = \\ &= (v^l(\operatorname{grad}\boldsymbol{u})_l{}^k - u^l(\operatorname{grad}\boldsymbol{v})_l{}^k + u^k\operatorname{div}\boldsymbol{v} - v^k\operatorname{div}\boldsymbol{u})\boldsymbol{g}_k = \\ &= \boldsymbol{v}.\operatorname{grad}\boldsymbol{u} - \boldsymbol{u}.\operatorname{grad}\boldsymbol{v} + \boldsymbol{u}\operatorname{div}\boldsymbol{v} - \boldsymbol{v}\operatorname{div}\boldsymbol{u},\end{aligned}$$

$$\boldsymbol{u}\times \operatorname{rot}\boldsymbol{v} = \varepsilon_{klm}u^l(\operatorname{rot}\boldsymbol{v})^m\boldsymbol{g}^k = \varepsilon_{mkl}\varepsilon^{mpq}u^l v_{q;p}\boldsymbol{g}^k = (\delta_k^p\delta_l^q - \delta_l^p\delta_k^q)u^l v_{q;p}\boldsymbol{g}^k =$$
$$= (u^l v_{l;k} - u^l v_{k;l})\boldsymbol{g}^k = (u^l(\operatorname{grad}\boldsymbol{v})_{kl} - u^l(\operatorname{grad}\boldsymbol{v})_{lk})\boldsymbol{g}^k = \operatorname{grad}\boldsymbol{v}\,.\,\boldsymbol{u} - \boldsymbol{u}\,.\operatorname{grad}\boldsymbol{v},$$

$$\operatorname{rot}\operatorname{grad}\Phi = \varepsilon^{klm}(\operatorname{grad}\Phi)_{m;l}\boldsymbol{g}_k = \varepsilon^{klm}\Phi_{,ml}\boldsymbol{g}_k = 0\,,$$

$$\operatorname{div}\operatorname{rot}\boldsymbol{u} = (\operatorname{rot}\boldsymbol{u})^k{}_{;k} = \varepsilon^{klm}u_{m;lk} = 0\,,$$

$$\operatorname{grad}\operatorname{div}\boldsymbol{u} = (\operatorname{div}\boldsymbol{u})_{;k}\boldsymbol{g}^k = u^l{}_{;lk}\boldsymbol{g}^k = u^l{}_{;kl}\boldsymbol{g}^k = [(\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}]^l{}_{k;l}\boldsymbol{g}^k = \operatorname{div}[(\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}]\,,$$

$$\operatorname{rot}\operatorname{rot}\boldsymbol{u} = \varepsilon_{klm}(\operatorname{rot}\boldsymbol{u})^m{}_;{}^l\boldsymbol{g}^k = \varepsilon_{mkl}\varepsilon^{mpq}u_{q;p}{}^l\boldsymbol{g}^k = (\delta_k{}^p\delta_l{}^q - \delta_l{}^p\delta_k{}^q)u_{q;p}{}^l\boldsymbol{g}^k =$$
$$= u_{l;k}{}^l\boldsymbol{g}^k - u_{k;l}{}^l\boldsymbol{g}^k = u^l{}_{;lk}\boldsymbol{g}^k - u_{k;l}{}^l\boldsymbol{g}^k = (\operatorname{div}\boldsymbol{u})_{;k}\boldsymbol{g}^k - (\operatorname{grad}\boldsymbol{u})^l{}_{k;l}\boldsymbol{g}^k =$$
$$= \operatorname{grad}\operatorname{div}\boldsymbol{u} - \operatorname{div}\operatorname{grad}\boldsymbol{u}\,,$$

$$\operatorname{grad}(\Phi\boldsymbol{u}) = (\Phi\boldsymbol{u})_{l;k}\boldsymbol{g}^k\boldsymbol{g}^l = (\Phi u_{l;k} + \Phi_{,k}u_l)\boldsymbol{g}^k\boldsymbol{g}^l =$$
$$= [\Phi(\operatorname{grad}\boldsymbol{u})_{kl} + (\operatorname{grad}\Phi)_k u_l]\boldsymbol{g}^k\boldsymbol{g}^l = \Phi\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\Phi)\boldsymbol{u}\,,$$

$$\operatorname{div}(\Phi\mathbf{A}) = (\Phi\mathbf{A})^{kl}{}_{;k}\boldsymbol{g}_l = \Phi A^{kl}{}_{;k}\boldsymbol{g}_l + A^{kl}\Phi_{,k}\boldsymbol{g}_l = \Phi\operatorname{div}\mathbf{A} + \operatorname{grad}\Phi\,.\,\mathbf{A}\,,$$

$$\operatorname{rot}(\Phi\mathbf{A}) = \varepsilon^{klm}(\Phi\mathbf{A})_{mn;l}\boldsymbol{g}_k\boldsymbol{g}^n = \Phi\varepsilon^{klm}A_{mn;l}\boldsymbol{g}_k\boldsymbol{g}^n + \varepsilon^{klm}\Phi_{,l}A_{mn}\boldsymbol{g}_k\boldsymbol{g}^n =$$
$$= \Phi\operatorname{rot}\mathbf{A} + \operatorname{grad}\Phi\times\mathbf{A}\,,$$

$$\operatorname{div}(\boldsymbol{u}\boldsymbol{v}) = (\boldsymbol{u}\boldsymbol{v})^k{}_{l;k}\boldsymbol{g}^l = (u^k v_l)_{;k}\boldsymbol{g}^l = (u^k{}_{;k}v_l + u^k v_{l;k})\boldsymbol{g}^l =$$
$$= [v_l\operatorname{div}\boldsymbol{u} + u^k(\operatorname{grad}\boldsymbol{v})_{kl}]\boldsymbol{g}^l = \boldsymbol{v}\operatorname{div}\boldsymbol{u} + \boldsymbol{u}\,.\operatorname{grad}\boldsymbol{v}\,,$$

$$\operatorname{rot}(\boldsymbol{u}\boldsymbol{v}) = \varepsilon^{klm}(u_{m;l}v_n + u_m v_{n;l})\boldsymbol{g}_k\boldsymbol{g}^n = (\operatorname{rot}\boldsymbol{u})\boldsymbol{v} - \boldsymbol{u}\times\operatorname{grad}\boldsymbol{v}$$

A.8. Orthogonal curvilinear coordinates

Let us first express the differential operators given above in general orthogonal curvilinear coordinates, and then in spherical coordinates. Since the physical components of tensors, and vectors, are the same in orthogonal curvilinear coordinates for all kinds of tensor components, we shall represent the tensors, and vectors, by physical components.

In *orthogonal curvilinear coordinates holds*:

(A.137)
$$g_{kl} = g^{kl} = 0 \quad \text{for } k \neq l, \qquad g^{\underline{kk}} = 1/g_{\underline{kk}},$$
$$\boldsymbol{g}^k = g^{\underline{kk}}\boldsymbol{g}_k, \qquad g = g_{11}g_{22}g_{33},$$

(A.138)
$$\mathrm{d}s^2 = g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2 + g_{33}(\mathrm{d}x^3)^2\,,$$
$$\begin{Bmatrix} k \\ l\ m \end{Bmatrix} = \frac{1}{g_{\underline{kk}}}[lm,k] = \frac{1}{2g_{\underline{kk}}}\left[\frac{\partial g_{kk}}{\partial x^m}\delta_{kl} + \frac{\partial g_{mm}}{\partial x^l}\delta_{km} - \frac{\partial g_{ll}}{\partial x^k}\delta_{lm}\right],$$

(A.139)
$$\operatorname{grad}\Phi = \frac{1}{(g_{11})^{1/2}}\frac{\partial\Phi}{\partial x^1}\boldsymbol{e}_1 + \frac{1}{(g_{22})^{1/2}}\frac{\partial\Phi}{\partial x^2}\boldsymbol{e}_2 + \frac{1}{(g_{33})^{1/2}}\frac{\partial\Phi}{\partial x^3}\boldsymbol{e}_3\,,$$

(A.140)
$$\operatorname{div} \boldsymbol{u} = \frac{1}{(g_{11}g_{22}g_{33})^{1/2}} \left\{ \frac{\partial}{\partial x^1}\left[(g_{22}g_{33})^{1/2} u^{(1)} \right] + \frac{\partial}{\partial x^2}\left[(g_{33}g_{11})^{1/2} u^{(2)} \right] + \frac{\partial}{\partial x^3}\left[(g_{11}g_{22})^{1/2} u^{(3)} \right] \right\},$$

(A.141)
$$\operatorname{rot} \boldsymbol{u} = \frac{1}{(g_{22}g_{33})^{1/2}} \left\{ \frac{\partial}{\partial x^2}\left[(g_{33})^{1/2} u^{(3)} \right] - \frac{\partial}{\partial x^3}\left[(g_{22})^{1/2} u^{(2)} \right] \right\} \boldsymbol{e}_1 + \frac{1}{(g_{33}g_{11})^{1/2}} \left\{ \frac{\partial}{\partial x^3}\left[(g_{11})^{1/2} u^{(1)} \right] - \frac{\partial}{\partial x^1}\left[(g_{33})^{1/2} u^{(3)} \right] \right\} \boldsymbol{e}_2 + \frac{1}{(g_{11}g_{22})^{1/2}} \left\{ \frac{\partial}{\partial x^1}\left[(g_{22})^{1/2} u^{(2)} \right] - \frac{\partial}{\partial x^2}\left[(g_{11})^{1/2} u^{(1)} \right] \right\} \boldsymbol{e}_3,$$

(A.142)
$$\nabla^2 \Phi = \frac{1}{(g_{11}g_{22}g_{33})^{1/2}} \left\{ \frac{\partial}{\partial x^1}\left[\frac{(g_{22}g_{33})^{1/2}}{(g_{11})^{1/2}} \frac{\partial \Phi}{\partial x^1} \right] + \frac{\partial}{\partial x^2}\left[\frac{(g_{33}g_{11})^{1/2}}{(g_{22})^{1/2}} \frac{\partial \Phi}{\partial x^2} \right] + \frac{\partial}{\partial x^3}\left[\frac{(g_{11}g_{22})^{1/2}}{(g_{33})^{1/2}} \frac{\partial \Phi}{\partial x^3} \right] \right\},$$

(A.143)
$$\operatorname{grad} \boldsymbol{u} = u^l{}_{;k} \boldsymbol{g}^k \boldsymbol{g}_l = (\operatorname{grad} \boldsymbol{u})_k{}^l \boldsymbol{g}^k \boldsymbol{g}_l = (\operatorname{grad} \boldsymbol{u})_{(k)}{}^{(l)} \boldsymbol{e}_k \boldsymbol{e}_l,$$

(A.144)
$$(\operatorname{grad} \boldsymbol{u})_{(k)}{}^{(l)} = \frac{1}{(g_{kk})^{1/2}} \frac{\partial u^{(l)}}{\partial x^k} - \frac{u^{(k)}}{(g_{kk}g_{ll})^{1/2}} \frac{\partial (g_{kk})^{1/2}}{\partial x^l} + \frac{\delta_{kl}}{(g_{kk})^{1/2}} \sum_{m=1}^{3} \frac{u^{(m)}}{(g_{mm})^{1/2}} \frac{\partial (g_{ll})^{1/2}}{\partial x^m},$$

(A.145)
$$\operatorname{div} \mathbf{A} = A^k{}_{l;k} \boldsymbol{g}^l = (\operatorname{div} \mathbf{A})_l \boldsymbol{g}^l = (\operatorname{div} \mathbf{A})_{(l)} \boldsymbol{l}_l,$$

(A.146)
$$(\operatorname{div} \mathbf{A})_{(l)} = \sum_{k=1}^{3} \left\{ \frac{1}{(g)^{1/2}} \frac{\partial}{\partial x^k}\left[A^{(k)}{}_{(l)} \frac{(g)^{1/2}}{(g_{kk})^{1/2}} \right] + \frac{1}{(g_{kk}g_{ll})^{1/2}} \frac{\partial (g_{ll})^{1/2}}{\partial x^k} A^{(k)}{}_{(l)} - \frac{1}{(g_{kk}g_{ll})^{1/2}} \frac{\partial (g_{kk})^{1/2}}{\partial x^l} A^{(k)}{}_{(k)} \right\}.$$

Note: Sometimes it is advantageous in orthogonal curvilinear coordinates to introduce *Lame's coefficients* H_k by

(A.147)
$$H_k = (g_{kk})^{1/2}.$$

Example 14: Express above relation in spherical coordinates r, ϑ, φ.

The definition relation between Cartesian and *spherical coordinates* is

(A.148)
$$y^1 = r \sin \vartheta \cos \varphi, \quad y^2 = r \sin \vartheta \sin \varphi, \quad y^3 = r \cos \vartheta.$$

Lamé's coefficients read

(A.149) $$H_r = 1,\; H_\vartheta = r,\; H_\varphi = r\sin\vartheta.$$

Christoffel's symbols of the 2nd kind,

(A.150) $$\begin{Bmatrix} r \\ \vartheta\ \vartheta \end{Bmatrix} = -r,\; \begin{Bmatrix} r \\ \varphi\ \varphi \end{Bmatrix} = -r\sin^2\vartheta,\; \begin{Bmatrix} \vartheta \\ \mathrm{r}\ \vartheta \end{Bmatrix} = 1/\mathrm{r},$$
$$\begin{Bmatrix} \vartheta \\ \varphi\ \varphi \end{Bmatrix} = -\sin\vartheta\cos\vartheta,\; \begin{Bmatrix} \varphi \\ r\ \varphi \end{Bmatrix} = 1/r,$$
$$\begin{Bmatrix} \varphi \\ \vartheta\ \varphi \end{Bmatrix} = \cot\vartheta,\ \text{others} \begin{Bmatrix} k \\ l\ m \end{Bmatrix} = 0.$$

(A.151) $$\operatorname{grad}\Phi = \frac{\partial\Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta}\boldsymbol{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\Phi}{\partial\varphi}\boldsymbol{e}_\varphi,$$

(A.152) $$\operatorname{div}\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(u_\vartheta\sin\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial u_\varphi}{\partial\varphi},$$

(A.153) $$\operatorname{rot}\boldsymbol{u} = \frac{1}{r\sin\vartheta}\left[\frac{\partial(u_\varphi\sin\vartheta)}{\partial\vartheta} - \frac{\partial u_\vartheta}{\partial\varphi}\right]\boldsymbol{e}_r +$$
$$+\left[\frac{1}{r\sin\vartheta}\frac{\partial \mathrm{u_r}}{\partial\varphi} - \frac{1}{r}\frac{\partial(ru_\varphi)}{\partial r}\right]\boldsymbol{e}_\vartheta + \frac{1}{r}\left[\frac{\partial(ru_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial\vartheta}\right]\boldsymbol{e}_\varphi,$$

(A.154) $$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}.$$

A.9. Tensor of small deformations and equations of motion of the continuum in curvilinear orthogonal coordinates

We shall introduce the *tensor of small deformations* $\mathbf{e}$ as (see Supplement B)

(A.155) $$\mathbf{e} = \tfrac{1}{2}[\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^{\mathrm{T}}],$$

where $\boldsymbol{u}$ is the *displacement vector*. Using Eq. (A.144), we can express this tensor in terms of the physical components,

(A.156) $$\mathbf{e} = e^{(k)}{}_{(l)}\boldsymbol{e}_k\boldsymbol{e}_l,$$

(A.157) $$e^{(k)}{}_{(l)} = e^k{}_l(g_{\underline{kk}}/g_{\underline{ll}})^{1/2} =$$
$$= \frac{1}{2}\left\{\frac{(g_{\underline{kk}})^{1/2}}{(g_{\underline{ll}})^{1/2}}\frac{\partial}{\partial x^l}\left[\frac{u^{(k)}}{(g_{\underline{kk}})^{1/2}}\right] + \frac{g_{\underline{ll}})^{1/2}}{(g_{\underline{kk}})^{1/2}}\frac{\partial}{\partial x^k}\left[\frac{u^{(l)}}{(g_{\underline{ll}})^{1/2}}\right]\right\} +$$
$$+ \frac{\delta_{kl}}{(g_{\underline{ll}})^{1/2}}\sum_{m=1}^{3}\frac{u^{(m)}}{(g_{\underline{mm}})^{1/2}}\frac{\partial(g_{\underline{kk}})^{1/2}}{\partial x^m}.$$

Example 16: Express the components of the tensor of small deformations in spherical coordinates r, ϑ, φ.

Let us denote the physical components of the displacement vector $\boldsymbol{u}$ by the symbols (u, v, w). Then

$$\text{(A.158)} \qquad \begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, \quad e_{\vartheta\vartheta} = \frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{u}{r}, \\ e_{\varphi\varphi} &= \frac{1}{r\sin\vartheta}\frac{\partial w}{\partial \varphi} + \frac{u}{r} + \frac{v}{r}\cot\vartheta, \\ 2e_{r\vartheta} &= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \vartheta} - \frac{v}{r}, \\ 2e_{r\varphi} &= \frac{\partial w}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial u}{\partial \varphi} - \frac{w}{r}, \\ 2e_{\vartheta\varphi} &= \frac{1}{r}\left(\frac{\partial w}{\partial \vartheta} + \frac{1}{\sin\vartheta}\frac{\partial v}{\partial \varphi} - w\cot\vartheta\right), \end{aligned}$$

where e_{rr}, $e_{r\vartheta}$, ..., $e_{\vartheta\varphi}$ are the physical components of the tensor of small deformations $\mathbf{e}$.

We shall express the *equations of motion* of the continuum in vectorial form (see Supplement B),

$$\text{(A.159)} \qquad \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} = \varrho\boldsymbol{f} + \operatorname{div}\mathbf{t},$$

where ϱ is the *density*, $\boldsymbol{f}$ the body force per unit mass, $\boldsymbol{u}$ the *displacement vector* and $\mathbf{t}$ *Cauchy's stress tensor*. If we use Eq. (A.146), the 1th equation of motion, expressed in terms of physical components will read

$$\text{(A.160)} \qquad \begin{aligned} \sum_{k=1}^{3} \left\{ \frac{1}{(g)^{1/2}} \frac{\partial}{\partial x^k}\left[t^{(k)}{}_{(l)} \frac{(g)^{1/2}}{(g_{\underline{kk}})^{1/2}} \right] + \frac{1}{(g_{\underline{kk}}g_{\underline{ll}})^{1/2}} \frac{\partial (g_{\underline{ll}})^{1/2}}{\partial x^k} t^{(l)}{}_{(k)} - \right. \\ \left. - \frac{1}{(g_{\underline{kk}}g_{\underline{ll}})^{1/2}} \frac{\partial (g_{\underline{kk}})^{1/2}}{\partial x^l} t^{(k)}{}_{(k)} \right\} + \varrho f_{(l)} = \varrho \frac{\mathrm{d}^2 u_{(l)}}{\mathrm{d}t^2}. \end{aligned}$$

Example 17: Express the equations of motion of the continuum in spherical coordinates r, ϑ, φ.

Let us denote the physical components of Cauchy's stress tensor, of the body force and displacement vector in spherical coordinates by the symbols $(t_{rr}, t_{r\vartheta}, \ldots t_{\vartheta\varphi})$, $(f_r, f_\vartheta, f_\varphi)$ and (u, v, w). In these coordinates the equations of motion of the continuum can be expressed as follows:

$$\text{(A.161)} \qquad \begin{aligned} \varrho\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} &= \varrho f_r + \frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\vartheta}}{\partial \vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial t_{r\varphi}}{\partial \varphi} + \\ &\quad + \frac{1}{r}(2t_{rr} - t_{\vartheta\vartheta} - t_{\varphi\varphi} + t_{r\vartheta}\cot\vartheta), \end{aligned}$$

$$\varrho\frac{d^2v}{dt^2} = \varrho f_\vartheta + \frac{\partial t_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\vartheta\vartheta}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial t_{\vartheta\varphi}}{\partial\varphi} +$$
$$+ \frac{1}{r}[3t_{r\vartheta} + (t_{\vartheta\vartheta} - t_{\varphi\varphi})\cot\vartheta],$$
$$\varrho\frac{d^2w}{dt^2} = \varrho f_\varphi + \frac{\partial t_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial t_{\vartheta\varphi}}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial t_{\varphi\varphi}}{\partial\varphi} +$$
$$+ \frac{1}{r}(3t_{r\varphi} + 2t_{\vartheta\varphi}\cot\vartheta).$$

A.10. Two-point tensor field

Definition 10: The quantities $A^k{}_K(\boldsymbol{x}, \boldsymbol{X})$ that transform like tensors with respect to the indices k and K under transformation of the coordinate systems x^k and X^K, are referred to as *two-point tensors.*

Therefore, if

(A.162) $$x^{k'} = x^{k'}(\boldsymbol{x}),\quad X^{K'} = X^{K'}(\boldsymbol{X})$$

are differentiable transformations of coordinates, and if

(A.163) $$A^{k'}{}_{K'}(\boldsymbol{x}', \boldsymbol{X}') = A^m{}_M(\boldsymbol{x}, \boldsymbol{X})\frac{\partial x^{k'}}{\partial x^m}\frac{\partial X^M}{\partial X^{K'}},$$

then $A^k{}_K$ is a two point tensor. If $\boldsymbol{g}_k$ and $\boldsymbol{G}_K$ are base vectors and $\boldsymbol{g}^k$ and $\boldsymbol{G}^K$ vectors reciprocal to the former in coordinate systems x^k and X^K, then $A^k{}_K$ are components of the tensor

(A.164) $$\mathbf{A}(\boldsymbol{x}, \boldsymbol{X}) = A^k{}_K(\boldsymbol{x}, \boldsymbol{X})\,\boldsymbol{g}_k(\boldsymbol{x})\,\boldsymbol{G}^K(\boldsymbol{X}).$$

An example of a two-point tensor are *shifters* defined by

(A.165) $$g^k{}_K(\boldsymbol{x}, \boldsymbol{X}) = \boldsymbol{g}^k(\boldsymbol{x}) \,.\, \boldsymbol{G}_K(\boldsymbol{X}),$$
$$g^K{}_k(\boldsymbol{x}, \boldsymbol{X}) = \boldsymbol{G}^K(\boldsymbol{X}) \,.\, \boldsymbol{g}_k(\boldsymbol{x}).$$

Another example are *deformation gradients*

(A.166) $$x^k{}_{,K} = \frac{\partial x^k}{\partial X^K},\quad X^K{}_{,k} = \frac{\partial X^K}{\partial x^k}.$$

The two-point tensor character of these quantities is implied by the relation

(A.167) $$\frac{\partial x^{k'}}{\partial X^{K'}} = \frac{\partial x^{k'}}{\partial x^m}\frac{\partial x^m}{\partial X^M}\frac{\partial X^M}{\partial X^{K'}},$$

where we have made use of rule of chaine of differentiation. Equation (A.167) has the form of (A.163). Multiple-point tensors of higher orders are similarly defined.

Definition 11: The *total covariant derivative* of the two-point tensor $A^k{}_K(\boldsymbol{x}, \boldsymbol{X})$, when $\boldsymbol{x}$ is related to $\boldsymbol{X}$ by the a mapping $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$, is defined by

(A.168) $$A^k{}_{K:L} = A^k{}_{K;L} + A^k{}_{K;l} x^l{}_{,L},$$

where $A^k{}_{K;L}$ is the covariant partial derivative of $A^k{}_K$ with respect to metric G_{KL} at the fixed point $\boldsymbol{x}$, and $A^k{}_{K;l}$ is the covariant partial derivative with respect to metric g_{kl} at the fixed point $\boldsymbol{X}$, i.e.,

(A.169) $$A^k{}_{K;L} = \frac{\partial A^k{}_K}{\partial X^L} - \left\{ \begin{matrix} M \\ L\ K \end{matrix} \right\} A^k{}_M, \qquad A^k{}_{K;l} = \frac{\partial A^k{}_K}{\partial x^l} + \left\{ \begin{matrix} k \\ l\ m \end{matrix} \right\} A^m{}_K.$$

Therefore,

(A.170) $$A^k{}_{K:L} = \frac{\partial A^k{}_K}{\partial X^L} - \left\{ \begin{matrix} M \\ L\ K \end{matrix} \right\} A^k{}_M + \left[\frac{\partial A^k{}_K}{\partial x^l} + \left\{ \begin{matrix} k \\ l\ m \end{matrix} \right\} A^m{}_K \right] \frac{\partial x^l}{\partial X^L}.$$

Note that this result is produced by differentiating Eq. (164) with respect to X^K and by using Eqs (A.80) and (A.85) to express the derivatives of vectors $\boldsymbol{g}_k$ and $\boldsymbol{G}^K$. Therefore,

(A.171) $$\partial \mathbf{A}/\partial X^K = A^k{}_{L:K} \boldsymbol{g}_k \boldsymbol{G}^L.$$

By using Eq. (A.170) for $x^k{}_{,K}(\boldsymbol{X})$, where the vector $\boldsymbol{x}$ is missing in the argument $x^k{}_{,K}$, we arrive at

(A.172) $$(x^k{}_{,K})_{:L} = \frac{\partial^2 x^k}{\partial X^L \partial X^K} - \left\{ \begin{matrix} M \\ L\ K \end{matrix} \right\} \frac{\partial x^k}{\partial X^M} + \left\{ \begin{matrix} k \\ l\ m \end{matrix} \right\} \frac{\partial x^m}{\partial X^K} \frac{\partial x^l}{\partial X^L}.$$

Note that (A.168) is a generalization of the total derivative of the scalar function of two variables, $\Phi(x, X)$ with $x = x(X)$, i.e.,

(A.173) $$\frac{\mathrm{d}\Phi}{\mathrm{d}X} = \frac{\partial \Phi}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \Phi}{\partial X}.$$

The same formal rules apply to the total covariant derivative as to the covariant partial derivative, e.g.,

(A.174) $$g^k{}_{K:M} = G_{KL:m} = g_{kl:M} = 0, \qquad (A^k{}_K B^L{}_l)_{:M} = A^k{}_{K:M} B^L{}_l + A^k{}_K B^L{}_{l:M}, \qquad (A^k{}_K + B^k{}_K)_{:M} = A^k{}_{K:M} + B^k{}_{K:M}.$$

For other accounts, see [58].

Let S be an oriented surface in three-dimensional space represented in *Gaussian form*,

(A.175) $$\boldsymbol{x} = \boldsymbol{x}(p^a) \qquad a = 1, 2, \quad \text{or}$$
$$x^k = x^k(p^1, p^2) \qquad k = 1, 2, 3,$$

where p^1, p^2 are curvilinear coordinates on surface S and x^k are space curvilinear coordinates of point $\boldsymbol{x}$ on surface S. Assume $\boldsymbol{n}(\boldsymbol{x})$ to be the *unit normal* external to surface S at point $\boldsymbol{x}$ on S.

We shall refer to vector $\boldsymbol{v}$ on S as a *vector tangential* to S, if $\boldsymbol{n} . \boldsymbol{v} = 0$ at every point $\boldsymbol{x}$ on S. We shall refer to the 2nd-order tensor $\mathbf{A}$ on S as a *tensor tangential* to S, if $\boldsymbol{n} . \mathbf{A} = \mathbf{A} . \boldsymbol{n} = 0$ at every point $\boldsymbol{x}$ on S. If $\mathbf{I}$ is a *three-dimensional identical tensor*, $I^k{}_l = \delta^k{}_l$, i.e. a 2nd-order tensor such that $\boldsymbol{v} . \mathbf{I} = \mathbf{I} . \boldsymbol{v} = \boldsymbol{v}$ and $\mathbf{A} . \mathbf{I} = \mathbf{I} . \mathbf{A} = \mathbf{A}$ for any vector $\boldsymbol{v}$ and 2nd-order tensor $\mathbf{A}$, then the equation

(A.176) $$\mathbf{I}_s = \mathbf{I} - \boldsymbol{n}\boldsymbol{n}$$

defines the tangential 2nd-order tensor on S which we refer to as the *surface identical tensor* since, if $\boldsymbol{v}$ is the vector tangential to S and $\mathbf{A}$ the 2nd-order tensor tangential to S, then $\boldsymbol{v} . \mathbf{I}_s = \mathbf{I}_s . \boldsymbol{v} = \boldsymbol{v}$, $\mathbf{A} . \mathbf{I}_s = \mathbf{I}_s . \mathbf{A} = \mathbf{A}$.

The projection of vector $\boldsymbol{v}$ on surface S is the tangential vector $\boldsymbol{v}_s$,

(A.177) $$\boldsymbol{v}_s = \boldsymbol{v} . \mathbf{I}_s = \mathbf{I}_s . \boldsymbol{v} .$$

If $\boldsymbol{v}$ is the vector tangential to S, then $\boldsymbol{v}_s = \boldsymbol{v}$. Assume grad to be the gradient operator in three-dimensional space. The *surface gradient*, grad_s, at point $\boldsymbol{x}$ on surface S is defined as the projection of operator grad onto surface S,

(A.178) $$\text{grad}_s = \mathbf{I}_s . \text{grad} = \text{grad} - \boldsymbol{n}(\boldsymbol{n} . \text{grad}) .$$

For example, if φ is a scalar field on S,

(A.179) $$\text{grad}_s \varphi = \text{grad}\, \varphi - \boldsymbol{n}(\boldsymbol{n} . \text{grad}\, \varphi) .$$

Since grad_s only contains derivatives in the direction tangential to surface S, the operator grad_s may be applied to any field, defined on surface S, regardless of whether this field is defined elsewhere in space or not.

Assume $\mathbf{Q}$ to be a scalar, vector or tensor field defined on surface S. If we move this field from point $\boldsymbol{x}$ on S to a point infinitesimally close, $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$, also lying on S, the field $\mathbf{Q}$ will change by the value $\mathrm{d}\mathbf{Q}$:

(A.180) $$\mathrm{d}\mathbf{Q} = \mathrm{d}\boldsymbol{x} . \text{grad}_s \mathbf{Q} .$$

The projection of the 2nd-order tensor $\mathbf{A}$ on surface S is tensor

(A.181) $$\mathbf{A}_s = \mathbf{I}_s . \mathbf{A} = \mathbf{A} - \boldsymbol{n}\boldsymbol{n} . \mathbf{A} = \mathbf{A} - \boldsymbol{n}(\boldsymbol{n} . \mathbf{A}) .$$

Tensor $\mathbf{A}_s$ is, in general, not tangential to surface S. If $\mathbf{A} = \text{grad}\, \boldsymbol{v}$, the surface gradient of vector $\boldsymbol{v}$ is defined bys the relation

$$\text{(A.182)} \qquad \text{grad}_s\, \boldsymbol{v} \equiv \text{grad}\, \boldsymbol{v} - \boldsymbol{n}(\boldsymbol{n} \,.\, \text{grad}\, \boldsymbol{v}) \,.$$

The *surface divergence* of vector $\boldsymbol{v}$ is defined as

$$\text{(A.183)} \qquad \text{div}_s\, \boldsymbol{v} \equiv \text{tr}(\text{grad}_s\, \boldsymbol{v}) = \text{div}\, \boldsymbol{v} - \boldsymbol{n} \,.\, \text{grad}\, \boldsymbol{v} \,.\, \boldsymbol{n} \,.$$

Using (A.132), (A.123) and (A.125), it is easy to prove the identities

$$\text{(A.184)} \qquad \begin{aligned} \text{grad}_s(\varphi \boldsymbol{v}) &= \varphi\, \text{grad}_s\, \boldsymbol{v} + (\text{grad}_s\, \varphi)\, \boldsymbol{v} \,, \\ \text{div}_s(\varphi \boldsymbol{v}) &= \varphi\, \text{div}_s\, \boldsymbol{v} + \boldsymbol{v} \,.\, \text{grad}_s\, \varphi \,, \\ \text{grad}_s(\boldsymbol{u} \,.\, \boldsymbol{v}) &= \text{grad}_s\, \boldsymbol{u} \,.\, \boldsymbol{v} + \text{grad}_s\, \boldsymbol{v} \,.\, \boldsymbol{u} \,. \end{aligned}$$

SUPPLEMENT B. FUNDAMENTAL RELATIONS OF THE THEORY OF ELASTICITY

This supplement is devoted to a brief recapitulation of the fundamental relations of the theory of elastic bodies. Strain geometry is described with the aid of the theory of differential geometry, and laws of conservation are described in natural (deformed) and reference (undeformed) systems of coordinates.

A detailed discussion of theory of elasticity and continuum physics is given in [28, 37, 64, 73, 74, 76, 91, 95, 96, 99, 109, 129, 130]. Our brief description follows books of Eringen [56—60].

B.1. Strain tensor

B.1.1. Coordinates, deformation, motion

Consider an continuum body at two different states of time. In the first, assume the body to be unstrained, in pre-strain state, or the initial undeformed state. In the second, assume the body to be strained, in the post-strain state, or deformed state. Assume the undeformed body B to have volume V and surface S. Assume the deformed body b to have volume v and surface s. The position of material point P in body B will be described by the curvilinear coordinates X^1, X^2, X^3, or by the position vector $\boldsymbol{P}$ (also $\boldsymbol{X}$) which extends from the origin O of the coordinates to point P. In the deformed state, assume the material point p to be represented by a new set of curvilinear coordinates x^1, x^2, x^3, or by a position vector $\boldsymbol{p}$ (also $\boldsymbol{x}$) that extends from the origin o of the new coordinates to point p. Often it is advantageous to select these two systems of coordinates. The coordinates X^K are called the *Lagrangian* or *material* coordinates and x^k the *Eulerian* or *spatial* coordinates.

The motion of the body carries various material points through various spatial positions. This is expressed by

$$x^k = x^k(X^K, t)\,, \quad X^K = X^K(x^k, t) \tag{B.1}$$

for $k = 1, 2, 3$ and $K = 1, 2, 3$. (B.1) can be abbreviated to read

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)\,, \quad \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)\,. \tag{B.2}$$

Equation $(B.1)_1$ states that at time t a material point X^K of B occupies the spatial position x^k in b. Equation $(B.1)_2$ describes the opposite.

We shall assume that functions x^k and X^K are continuously differentiable at least up to the first order in the neighbourhood of point $P(\boldsymbol{X})$, or $p(\boldsymbol{x})$, and that the Jacobian of transformation is not identically zero, i.e.

$$j \equiv \det\left(\frac{\partial x^k}{\partial X^K}\right) \neq 0 \tag{B.3}$$

describes unique inverse transformations.

The assumptions mentioned express the *axiom of continuity*, the consequence of which is, on the one hand, the *axiom of indestructability* of matter, i.e. no region of a finite positive volume can be deformed into a region of zero volume, and, on the other, the *axiom of impenetrability* of matter, i.e. under motion every volume is again transformed into a volume, every surface into a surface, and every curve into a curve. However, in some cases it must be assumed that, within a particular interval of time, there may exist singular surfaces, curves and points in which the axiom of continuity is not satisfied.

We shall denote the quantities relating to the undeformed body B by capital letters, to the deformed body b by lower-case letters. The components of vectors and tensors relative to coordinates X^K will have capital Roman letter indices, those relative to coordinates x^k lower-case Roman letters. For example, $G_{KL}(\boldsymbol{X})$ and $g_{kl}(\boldsymbol{x})$ are the covariant metric tensors in B and b, respectively.

B.1.2. Base vectors, metric tensors, shifters

The position vector $\boldsymbol{P}$ of point P in B and the position vector $\boldsymbol{p}$ of point p in b are expressed in Cartesian coordinates Y^K and y^k as

$$\boldsymbol{P} = Y^K \boldsymbol{I}_K\,, \quad \boldsymbol{p} = y^k \boldsymbol{i}_k\,, \tag{B.4}$$

where $\boldsymbol{I}_K$ and $\boldsymbol{i}_k$ are unit base vectors in Cartesian coordinates Y^K and y^k. We are again going to use Einstein's summation rule, i.e. we summate from one to three over every diagonally repeated index (Fig. B1).

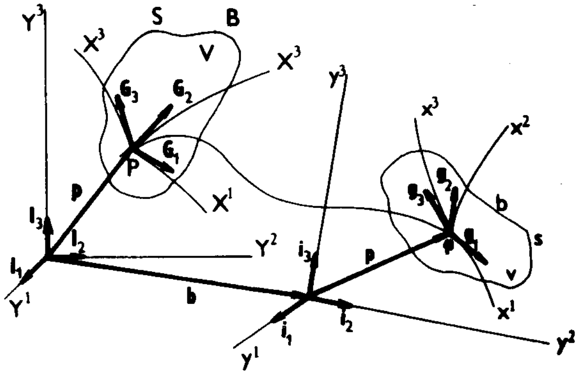

Fig. B1. Coordinate systems for an undeformed body B and a deformed body b.

We shall introduce the *base vectors* $\boldsymbol{G}_K(\boldsymbol{X})$ and $\boldsymbol{g}_k(\boldsymbol{x})$ at X^K and x^k, respectively,

(B.5) $$\boldsymbol{G}_K(\boldsymbol{X}) = \frac{\partial \boldsymbol{P}}{\partial X^K} = \frac{\partial Y^M}{\partial X^K}\boldsymbol{I}_M, \quad \boldsymbol{g}_k(\boldsymbol{x}) = \frac{\partial \boldsymbol{p}}{\partial x^k} = \frac{\partial y^m}{\partial x^k}\boldsymbol{i}_m .$$

The infinitesimal differential vectors d$\boldsymbol{P}$ at point P and d$\boldsymbol{p}$ at point p are

(B.6) $$\mathrm{d}\boldsymbol{P} = \frac{\partial \boldsymbol{P}}{\partial X^K}\mathrm{d}X^K = \boldsymbol{G}_K \mathrm{d}X^K, \quad \mathrm{d}\boldsymbol{p} = \frac{\partial \boldsymbol{p}}{\partial x^k}\mathrm{d}x^k = \boldsymbol{g}_k \,\mathrm{d}x^k .$$

The base vectors $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ are tangential to the coordinate lines X^K and x^k.

The squares of the lenghts in B and b are

(B.7) $$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{P}.\mathrm{d}\boldsymbol{P} = G_{KL}\,\mathrm{d}X^K \mathrm{d}X^L, \quad \mathrm{d}s^2 = \mathrm{d}\boldsymbol{p}.\mathrm{d}\boldsymbol{p} = g_{kl}\,\mathrm{d}x^k \,\mathrm{d}x^l ,$$

respectively, where

(B.8) $$G_{KL}(\boldsymbol{X}) = \boldsymbol{G}_K . \boldsymbol{G}_L = \frac{\partial Y^M}{\partial X^K}\frac{\partial Y^N}{\partial X^L}\delta_{MN}, \quad g_{kl}(\boldsymbol{x}) = \boldsymbol{g}_k . \boldsymbol{g}_l = \frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^l}\delta_{mn}$$

are metric tensors in B and b, respectively. Kronecker's delta symbols δ_{MN}, δ_{mn}, δ^M_N and δ^m_n are equal to unity if their indices are the same and to zero if they are different.

The *reciprocal base vectors* $\boldsymbol{G}^K(\boldsymbol{X})$ and $\boldsymbol{g}^k(\boldsymbol{x})$ are defined by the equations

(B.9) $$\boldsymbol{G}^K . \boldsymbol{G}_L = \delta^K_L, \quad \boldsymbol{g}^k . \boldsymbol{g}_l = \delta^k_l .$$

The solution to these equations reads

(B.10) $$\boldsymbol{G}^K = G^{KL}\boldsymbol{G}_L, \quad \boldsymbol{g}^k = g^{kl}\boldsymbol{g}_l ,$$

where

(B.11) $$G^{KL} = \frac{\text{alg. cofactor } G_{KL}}{\det(G_{KL})}, \quad g^{kl} = \frac{\text{alg. cofactor } g_{kl}}{\det(g_{kl})}.$$

The scalar product of (B.10) with vectors $\boldsymbol{G}^L$ and $\boldsymbol{g}^l$ yields

(B.12) $$G^{KL} = \boldsymbol{G}^K . \boldsymbol{G}^L, \quad g^{kl} = \boldsymbol{g}^k . \boldsymbol{g}^l .$$

The representation of vectors and tensors with respect to coordinates X^K or x^k is separated, e.g. the components of the position vectors $\boldsymbol{P}$ and $\boldsymbol{p}$ in coordinates X^K and x^k are

(B.13) $$P^K = \boldsymbol{P} . \boldsymbol{G}^K, \quad p^k = \boldsymbol{p} . \boldsymbol{g}^k .$$

We would like to express the vectors and tensors from one coordinate system in terms of their projection into the other coordinate system and vice versa. For this purpose, let us shift vector $\boldsymbol{p}$ parallelly to point $P(\boldsymbol{X})$. If p^K are the components of vector $\boldsymbol{p}$ in X^K,

(B.14) $$\boldsymbol{p} = p^K \boldsymbol{G}_K(\boldsymbol{X}) = p^k \boldsymbol{g}_k(\boldsymbol{x}) .$$

The scalar product of (B.14) with the vectors $\boldsymbol{G}^L$ and $\boldsymbol{g}^l$ yields

(B.15) $$p^K = g^K{}_k p^k, \quad p^k = g^k{}_K p^K ,$$

where

(B.16) $$g^K{}_k(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{G}^K(\boldsymbol{X}) . \boldsymbol{g}_k(\boldsymbol{x}), \quad g^k{}_K(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}^k(\boldsymbol{x}) . \boldsymbol{G}_K(\boldsymbol{X})$$

are so-called *shifters*. These are two-point tensor (see A.10); i.e., they transform as tensors with respect to indices K and k under transformation of coordinates X^K and x^k. With the aid of shifters it is possible to express vectors and tensors from one coordinate system with the aid of their projection into another coordinate system.

In very much the same way we now define

(B.17) $$g_{Kk}(\boldsymbol{X}, \boldsymbol{x}) = g_{kK}(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}_k(\boldsymbol{x}) . \boldsymbol{G}_K(\boldsymbol{X}),$$
$$g^{Kk}(\boldsymbol{X}, \boldsymbol{x}) = g^{kK}(\boldsymbol{X}, \boldsymbol{x}) = \boldsymbol{g}^k(\boldsymbol{x}) . \boldsymbol{G}^K(\boldsymbol{X}).$$

By raising and lowering the capital-letter indices with the aid of tensors G^{KL} and G_{KL}, and by raising and lowering the lower-case indices with the aid of metric tensors g^{kl} and g_{kl}, we arrive at

(B.18) $$g_{Kk} = g_{kl} g^l{}_K = G_{KL} g^L{}_k = g_{kl} G_{KL} g^{lL},$$
$$g^{Kk} = g^{kl} g^K{}_l = G^{KL} g^k{}_L = g^{kl} G^{KL} g_{lL},$$
$$g^K{}_k = g_{kl} g^{lK} = G^{KL} g_{kL} = g_{kl} G^{KL} g^l{}_L,$$
$$g^K{}_k g^l{}_K = \delta^l{}_k, \quad g^K{}_k g^k{}_L = \delta^K{}_L .$$

By substituting (B.5) into (B.17) we obtain

(B.19) $$g_{Kk} = \delta_{Ll}\frac{\partial Y^L}{\partial X^K}\frac{\partial y^l}{\partial x^k}, \quad \delta_{Ll} = \boldsymbol{i}_L \cdot \boldsymbol{i}_l .$$

This equation implies not only the two-point character of tensor g_{Kk}, but also the relation

(B.20) $$g_{Kk} = \delta_{Kk}, \quad g^K{}_k = \delta^K{}_k ,$$

provided both coordinates X^K and x^k are Cartesian.

B.1.3. Deformation gradients, deformation tensors

Equation (B.1) for a fixed time yields

(B.21) $$\mathrm{d}x^k = x^k{}_{,K}\mathrm{d}X^K, \quad \mathrm{d}X^K = X^K{}_{,k}\mathrm{d}x^k ,$$

where the indices following the commas represent partial derivatives with respect to X^K, if the index is a capital letter, and with respect to x^k, if the index is a lower-case letter, i.e.,

(B.22) $$x^k{}_{,K} = \frac{\partial x^k}{\partial X^K}, \quad X^K{}_{,k} = \frac{\partial X^K}{\partial x^k}.$$

The quantities defined by Eq. (B.22) are referred to as *deformation gradients.* According to the chain rule of partial differentiation,

(B.23) $$x^k{}_{,K}X^K{}_{,l} = \delta^k{}_l, \quad X^K{}_{,k}x^k{}_{,L} = \delta^K{}_L .$$

Each of these systems represents nine linear algebraic equations for nine unknowns $x^k{}_{,K}$ or $X^K{}_{,k}$. Since the Jacobian of transformation is non-zero by assumption, there exists a unique solution to these equations. According to Cramer's determinant rule,

(B.24) $$X^K{}_{,k} = \frac{\text{alg. cofactor } x^k{}_{,K}}{j} = \frac{1}{2j}e^{KLM}e_{klm}x^l{}_{,L}x^m{}_{,M},$$

where e^{KLM} and e_{klm} are Levi-Civita's permutation symbols and

(B.25) $$j \equiv \det(x^k{}_{,K}) = \frac{1}{3!}e^{KLM}e_{klm}x^k{}_{,K}x^l{}_{,L}x^m{}_{,M}.$$

By differentiating (B.24) and (B.25) we obtain two important *Jacobi's identities*:

(B.26) $$(jX^K{}_{,k})_{,K} = 0 \text{ and } (j^{-1}x^k{}_{,K})_{,k} = 0,$$
$$\frac{\partial j}{\partial x^k{}_{,K}} = \text{alg. cofactor } x^k{}_{,K} = jX^K{}_{,k}.$$

By substituting (B.21) into (B.7) we obtain

(B.27) $$\mathrm{d}S^2 = c_{kl}(\boldsymbol{x}, t)\,\mathrm{d}x^k\,\mathrm{d}x^l,\ \ \mathrm{d}s^2 = C_{KL}(\boldsymbol{X}, t)\,\mathrm{d}X^K\,\mathrm{d}X^L,$$

where

(B.28) $$\begin{aligned} c_{kl}(\boldsymbol{x}, t) &= G_{KL}(\boldsymbol{X})\,X^K{}_{,k}X^L{}_{,l}, \\ C_{KL}(\boldsymbol{X}, t) &= g_{kl}(\boldsymbol{x})\,x^k{}_{,K}x^l{}_{,L} \end{aligned}$$

are *Cauchy's and Green's deformation tensors.* Both tensors are symmetric, $c_{kl} = c_{lk}$, $C_{KL} = C_{LK}$, and both are positive definite. Equations (B.28) indicate that the metric tensor $G_{KL}(\boldsymbol{X})$ transforms to tensor $c_{kl}(\boldsymbol{x}, t)$ through the motion. Tensor C_{KL} can be said to do the same in inverse motion.

New base vectors, so-called *Cauchy's and Green's base vectors* $\boldsymbol{c}_k(\boldsymbol{x}, t)$ and $\boldsymbol{C}_K(\boldsymbol{X}, t)$, can be defined with respect to these two new tensors:

(B.29) $$\begin{aligned} \boldsymbol{c}_k(\boldsymbol{x}, t) &= \frac{\partial \boldsymbol{P}}{\partial x^k} = \frac{\partial \boldsymbol{P}}{\partial X^K}\frac{\partial X^K}{\partial x^k} = \boldsymbol{G}_K(\boldsymbol{X})\,X^K{}_{,k}, \\ \boldsymbol{C}_K(\boldsymbol{X}, t) &= \frac{\partial \boldsymbol{p}}{\partial X^K} = \frac{\partial \boldsymbol{p}}{\partial x^k}\frac{\partial x^k}{\partial X^K} = \boldsymbol{g}_k(\boldsymbol{x})\,x^k{}_{,K}. \end{aligned}$$

This immediately yields

(B.30) $$c_{kl} = c_{lk} = \boldsymbol{c}_k\,.\,\boldsymbol{c}_l,\ \ C_{KL} = C_{LK} = \boldsymbol{C}_K\,.\,\boldsymbol{C}_L\,.$$

Equations (B.29) indicate that the base vectors $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ deform to vectors $\boldsymbol{c}_k$ and $\boldsymbol{C}_K$ through the motion.

We now have two different representations for the differential vectors $\mathrm{d}\boldsymbol{P}$ and $\mathrm{d}\boldsymbol{p}$. One in coordinate system X^K and the other in x^k, i.e.,

(B.31) $$\begin{aligned} \mathrm{d}\boldsymbol{P} &= \boldsymbol{G}_K(\boldsymbol{X})\,\mathrm{d}X^K = \boldsymbol{c}_k(\boldsymbol{x}, t)\,\mathrm{d}x^k, \\ \mathrm{d}\boldsymbol{p} &= \boldsymbol{C}_K(\boldsymbol{X}, t)\,\mathrm{d}X^K = \boldsymbol{g}_k(\boldsymbol{x})\,\mathrm{d}x^k. \end{aligned}$$

Similarly, the square of length elements are

(B.32) $$\begin{aligned} \mathrm{d}S^2 &= G_{KL}(\boldsymbol{X})\,\mathrm{d}X^K\,\mathrm{d}X^L = c_{kl}(\boldsymbol{x}, t)\,\mathrm{d}x^k\,\mathrm{d}x^l, \\ \mathrm{d}s^2 &= C_{KL}(\boldsymbol{X}, t)\,\mathrm{d}X^K\,\mathrm{d}X^L = g_{kl}(\boldsymbol{x})\,\mathrm{d}x^k\,\mathrm{d}x^l. \end{aligned}$$

B.1.4. Strain tensors, displacement vectors

Lagrange's and Euler's strain tensors are defined as

(B.33) $$\begin{aligned} E_{KL} &= \tfrac{1}{2}[C_{KL}(\boldsymbol{X}, t) - G_{KL}(\boldsymbol{X})], \\ e_{kl} &= \tfrac{1}{2}[g_{kl}(\boldsymbol{x}) - c_{kl}(\boldsymbol{x}, t)]. \end{aligned}$$

(B.32) and (B.33) then yield the following important relation

(B.34) $$ds^2 - dS^2 = 2E_{KL}(\boldsymbol{X}, t)\, dX^K dX^L = 2e_{kl}(\boldsymbol{x}, t)\, dx^k dx^l .$$

When the body undergoes only a *rigid displacement* there will be no change in the differential length in which case the difference $ds^2 - dS^2$ given by (B.34) vanishes. If this is true for all directions dX^K and dx^k, then E_{KL} and e_{kl} vanish. Therefore, these tensors represent a measure of deformation of the body.

Equation (B.34) immediately yields

(B.35) $$E_{KL} = e_{kl} x^k{}_{,K} x^l{}_{,L}, \quad e_{kl} = E_{KL} X^K{}_{,k} X^L{}_{,l} .$$

These relations indicate that E_{KL} and e_{kl} are 2nd-order tensors.

Strain tensors can also be expressed in terms of the *displacement vector* $\boldsymbol{u}$, defined as the vector extending from point P of the undeformed body B to its spatial point p of the deformed body b (see Fig. B2):

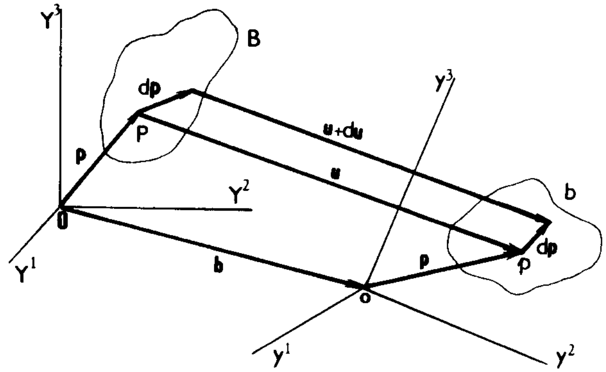

Fig. B2. Displacement vector.

(B.36) $$\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{P} + \boldsymbol{b} .$$

The displacement vector can be represented by Lagrange's or Euler's components U^K and u^k,

(B.37) $$\boldsymbol{u} = U^K \boldsymbol{G}_K = u^k \boldsymbol{g}_k .$$

The scalar product of both sides of Eq. (B.36) with vectors $\boldsymbol{G}^K$ and $\boldsymbol{g}^k$ yields

(B.38) $$U^K = p^K - P^K + B^K, \quad u^k = p^k - P^k + b^k ,$$

where p^K, P^K, B^K and p^k, P^k, b^k are the components of vectors $\boldsymbol{p}$, $\boldsymbol{P}$ and $\boldsymbol{b}$ in X^K and x^k, respectively.

Let us express the strain tensors in terms of the displacement vector. By substituting $(B.28)_2$ and $(B.29)_2$ into $(B_{33})_1$ we can express Lagrange's strain tensor as

(B.39) $$E_{KL} = \tfrac{1}{2}(\boldsymbol{g}_k \cdot \boldsymbol{g}_l x^k{}_{,K} x^l{}_{,L} - G_{KL}).$$

Substituting from $(B.5)_2$ into the last equation yields

(B.40) $$E_{KL} = \frac{1}{2}\left(\frac{\partial \boldsymbol{p}}{\partial X^K} \cdot \frac{\partial \boldsymbol{p}}{\partial X^L} - G_{KL}\right).$$

If we also make use of (B.36) we obtain

(B.41) $$E_{KL} = \tfrac{1}{2}[(U_{M;K}\boldsymbol{G}^M + \boldsymbol{G}_K) . (U_{M;L}\boldsymbol{G}^M + \boldsymbol{G}_L) - G_{KL}],$$

in which the semi-colon indicates covariant partial differentiation, $\partial \boldsymbol{u}/\partial X^K = U_{M;K}\boldsymbol{G}^M$. After some algebra, (B.41) yields

(B.42) $$E_{KL} = \tfrac{1}{2}(U_{K;L} + U_{L;K} + U_{M;K}U^M{}_{;L}).$$

Euler's strain tensor can be expressed in very much the same way:

(B.43) $$e_{kl} = \tfrac{1}{2}(u_{k;l} + u_{l;k} - u_{m;k}u^m{}_{;l}).$$

B.1.5. Changes of lengths and angles

Let us demonstrate the geometric significance of the components of the strain tensor. According to (B.31), the parallelepiped with sides $\boldsymbol{G}_1\,\mathrm{d}X^1$, $\boldsymbol{G}_2\,\mathrm{d}X^2$, $\boldsymbol{G}_3\,\mathrm{d}X^3$, located at point $P(\boldsymbol{X})$ deforms into a parallelepiped with sides $\boldsymbol{C}_1\,\mathrm{d}X^1$, $\boldsymbol{C}_2\,\mathrm{d}X^2$, $\boldsymbol{C}_3\,\mathrm{d}X^3$, located at point $p(\boldsymbol{x})$. It holds that

(B.44) $$\mathrm{d}\boldsymbol{X} = \boldsymbol{G}_K \mathrm{d}X^K, \ \mathrm{d}\boldsymbol{x} = \boldsymbol{C}_K \mathrm{d}X^K, \ \boldsymbol{C}_K = \boldsymbol{g}_k x^k_{,K}.$$

The unit vectors $\boldsymbol{N}$ and $\boldsymbol{n}$ along $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$ are defined as

(B.45) $$N^K = \frac{\mathrm{d}X^K}{|\mathrm{d}\boldsymbol{X}|} = \frac{\mathrm{d}X^K}{\mathrm{d}S}, \ n^k = \frac{\mathrm{d}x^k}{|\mathrm{d}\boldsymbol{x}|} = \frac{\mathrm{d}x^k}{\mathrm{d}s},$$

where $\mathrm{d}S$ and $\mathrm{d}s$ are the lengths of vectors $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$. The *relative change of the length* of vector $\boldsymbol{N}$ is defined by

(B.46) $$E_{(\boldsymbol{N})} = e_{(\boldsymbol{n})} = \frac{\mathrm{d}s - \mathrm{d}S}{\mathrm{d}S}.$$

Let us express Langrange's strain tensor in terms of the quantity $E_{(\boldsymbol{N})}$. Equations (B.34) and (B.45) yield

(B.47) $$2E_{KL}N^K N^L = \frac{\mathrm{d}s^2 - \mathrm{d}S^2}{\mathrm{d}S^2} = E_{(\boldsymbol{N})}(E_{(\boldsymbol{N})} + 2).$$

If $\boldsymbol{N}$ is a vector tangential to coordinate line X^1, $N^1 = \mathrm{d}X^1/\mathrm{d}S = 1/(G_{11})^{1/2}$, $N^2 = N^3 = 0$, then

(B.48) $$2E_{11}/G_{11} = E_{(1)}(E_{(1)} + 2).$$

The last equation can also be expressed as

(B.49) $$E_{(1)} = -1 + (1 + 2E_{11}/G_{11})^{1/2}.$$

If the strains are small, $E_{(1)} \ll 1$, the following approximate relation applies:

(B.50) $$E_{11}/G_{11} \doteq E_{(1)}.$$

Analogous relations also hold for components E_{22} and E_{33}.

Now, assume $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ to be unit vectors along $\mathrm{d}\boldsymbol{X}_1$, $\mathrm{d}\boldsymbol{X}_2$ at point $\boldsymbol{X}$, and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ unit vectors along $\mathrm{d}\boldsymbol{x}_1$, $\mathrm{d}\boldsymbol{x}_2$ at point $\boldsymbol{x}$. The angle $\Theta_{(\boldsymbol{N}_1,\boldsymbol{N}_2)}$ between $\mathrm{d}\boldsymbol{X}_1$ and $\mathrm{d}\boldsymbol{X}_2$ deforms into angle $\vartheta_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}$ between $\mathrm{d}\boldsymbol{x}_1$ and $\mathrm{d}\boldsymbol{x}_2$ (see Fig. B3). We also have

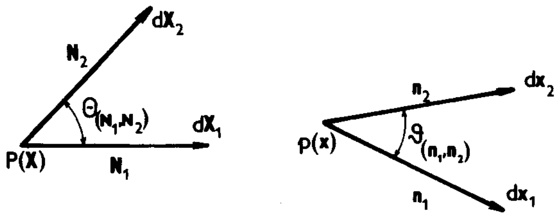

Fig. B3. Angle change.

(B.51) $$\boldsymbol{N}_a = \frac{\mathrm{d}\boldsymbol{X}_a}{|\mathrm{d}\boldsymbol{X}_a|}, \quad \boldsymbol{n}_a = \frac{\mathrm{d}\boldsymbol{x}_a}{|\mathrm{d}\boldsymbol{x}_a|}, \quad a = 1, 2.$$

Let us now calculate the angles $\Theta_{(\boldsymbol{N}_1,\boldsymbol{N}_2)}$ and $\vartheta_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}$ from

(B.52) $$\cos\Theta_{(\boldsymbol{N}_1,\boldsymbol{N}_2)} = \boldsymbol{N}_1 . \boldsymbol{N}_2 = \frac{\mathrm{d}\boldsymbol{X}_1}{|\mathrm{d}\boldsymbol{X}_1|} \cdot \frac{\mathrm{d}\boldsymbol{X}_2}{|\mathrm{d}\boldsymbol{X}_2|} = \frac{G_{KL}\,\mathrm{d}X^K{}_1\,\mathrm{d}X^L{}_2}{|\mathrm{d}\boldsymbol{X}_1|\,|\mathrm{d}\boldsymbol{X}_2|} = G_{KL}N^K{}_1N^{L2}.$$

Similarly,

(B.53) $$\cos\vartheta_{(\boldsymbol{n}_1,\boldsymbol{n}_2)} = \boldsymbol{n}_1 . \boldsymbol{n}_2 = \frac{\mathrm{d}\boldsymbol{x}_1}{|\mathrm{d}\boldsymbol{x}_1|} \cdot \frac{\mathrm{d}\boldsymbol{x}_2}{|\mathrm{d}\boldsymbol{x}_2|} = \frac{C_{KL}\,\mathrm{d}X^K{}_1\,\mathrm{d}X^L{}_2}{(C_{MN}\,\mathrm{d}X^M{}_1\,\mathrm{d}X^N{}_2)^{1/2}(C_{PQ}\,\mathrm{d}X^P{}_1\,\mathrm{d}X^Q{}_2)^{1/2}} = \frac{C_{KL}N^K{}_1N^L{}_2}{(E_{(\boldsymbol{N}_1)} + 1)(E_{(\boldsymbol{N}_2)} + 1)}.$$

The difference $\Theta_{(\boldsymbol{N}_1,\boldsymbol{N}_2)} - \vartheta_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}$ determines the *change of the angles* of directions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ due to the motion

(B.54) $$\Gamma_{(\boldsymbol{N}_1,\boldsymbol{N}_2)} = \gamma_{(\boldsymbol{n}_1,\boldsymbol{n}_2)} = \Theta_{(\boldsymbol{N}_1,\boldsymbol{N}_2)} - \vartheta_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}.$$

Here we have again dual representation, Γ and γ, for the same physical quantity, i.e. the change of angle of two directions is denoted differently in Lagrange's and Euler's representations. (B.53) and (B.54) yield

$$\sin \Gamma_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = H \sin \Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} - (1 - H^2)^{1/2} \cos \vartheta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)}, \tag{B.55}$$

which, for the orthogonal directions before deformation, $\Theta_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = \frac{1}{2}\pi$, reduces to

$$\sin \Gamma_{(\boldsymbol{N}_1, \boldsymbol{N}_2)} = H = \frac{C_{KL} N^K{}_1 N^L{}_2}{(E_{(\boldsymbol{N}_1)} + 1)(E_{(\boldsymbol{N}_2)} + 1)}. \tag{B.56}$$

If we eliminate directions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ from Eqs (B.52) and (B.53), we obtain

$$\begin{aligned} \cos \Theta_{(KL)} &= G_{KL}/(G_{\underline{KK}} G_{\underline{LL}})^{1/2}, \\ \cos \vartheta_{(KL)} &= C_{KL}/(C_{\underline{KK}} C_{\underline{LL}})^{1/2} = \\ &= (G_{KL} + 2E_{KL})/[(G_{\underline{KK}} + 2E_{\underline{KK}})(G_{\underline{LL}} + 2E_{\underline{LL}})]^{1/2}. \end{aligned} \tag{B.57}$$

If X^K are Cartesian coordinates, (B.57) will simplify to

$$\begin{aligned} \cos \Theta_{(KL)} &= \delta_{KL}, \\ \cos \vartheta_{(KL)} &= \sin \Gamma_{(KL)} = \\ &= (\delta_{KL} + 2E_{KL})/[(1 + 2E_{\underline{KK}})(1 + 2E_{\underline{LL}})]^{1/2}. \end{aligned} \tag{B.58}$$

By using (B.49) in $(B.58)_2$ we may also write

$$2E_{KL} = (1 + E_{(K)})(1 + E_{(L)}) \sin \Gamma_{(KL)} \quad \text{for} \quad K \neq L. \tag{B.59}$$

In the case of small strains, $E_{(K)} \ll 1$, $E_{(L)} \ll 1$, the following approximate relation applies

$$2E_{KL} \doteq \sin \Gamma_{(KL)} \doteq \Gamma_{(KL)}. \tag{B.60}$$

B.1.6. Changes of areas and volumes

The element of area bounded by vectors $\boldsymbol{G}_1 \, \mathrm{d}X^1$ and $\boldsymbol{G}_2 \, \mathrm{d}X^2$ after deformation change to the area bounded by vectors $\boldsymbol{C}_1 \, \mathrm{d}X^1$ and $\boldsymbol{C}_2 \, \mathrm{d}X^2$. The deformed area is thus given by

$$\mathrm{d}\boldsymbol{a}_3 = \boldsymbol{C}_1 \, \mathrm{d}X^1 \times \boldsymbol{C}_2 \, \mathrm{d}X^2 = x^k{}_{,1} x^l{}_{,2} \boldsymbol{g}_k \times \boldsymbol{g}_l \, \mathrm{d}X^1 \, \mathrm{d}X^2. \tag{B.61}$$

However,

$$\boldsymbol{g}_k \times \boldsymbol{g}_l = \varepsilon_{klm} \boldsymbol{g}^m = g^{1/2} e_{klm} \boldsymbol{g}^m, \tag{B.62}$$

where e_{klm} is Levi-Civita's alternating symbol. By substituting (B.62) into (B.61) we obtain

$$\mathrm{d}\boldsymbol{a}_3 = g^{1/2} x^k{}_{,1} x^l{}_{,2} e_{klm} \boldsymbol{g}^m \mathrm{d}X^1 \, \mathrm{d}X^2. \tag{B.63}$$

The element of area prior to deformation is

(B.64) $$\mathrm{d}\boldsymbol{A}_3 = \boldsymbol{G}_1 \times \boldsymbol{G}_2 \,\mathrm{d}X^1\,\mathrm{d}X^2 = G^{1/2}\boldsymbol{G}^3\,\mathrm{d}X^1\,\mathrm{d}X^2 .$$

Consequently,

(B.65) $$\mathrm{d}A_3 = G^{1/2}\,\mathrm{d}X^1\,\mathrm{d}X^2 .$$

By substituting (B.65) into (B.63) we obtain

(B.66) $$\mathrm{d}\boldsymbol{a}_3 = (g/G)^{1/2} x^k{}_{,1} x^l{}_{,2} e_{klm} \boldsymbol{g}^m \,\mathrm{d}A_3 .$$

Equation (B.24) yields

(B.67) $$jX^3{}_{,m} = e_{klm} x^k{}_{,1} x^l{}_{,2} ,$$

so that

(B.68) $$\mathrm{d}\boldsymbol{a}_3 = J X^3{}_{,m} \boldsymbol{g}^m \,\mathrm{d}A_3 ,$$

where

(B.69) $$J = (g/G)^{1/2} j .$$

Similar relations also hold for $\mathrm{d}\boldsymbol{a}_1$ and $\mathrm{d}\boldsymbol{a}_2$. Therefore,

(B.70) $$\mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{a}_1 + \mathrm{d}\boldsymbol{a}_2 + \mathrm{d}\boldsymbol{a}_3 = JX^K{}_{,k} \boldsymbol{g}^k \,\mathrm{d}A_K ,$$

the kth component of which yields the important relation

(B.71) $$\mathrm{d}a_k = JX^K{}_{,k} \,\mathrm{d}A_K .$$

Let us also determine the change of volume under deformation. The deformed volume element is

(B.72) $$\begin{aligned}\mathrm{d}v &= \mathrm{d}\boldsymbol{a}_3 . \boldsymbol{C}_3 \,\mathrm{d}X^3 = JX^3{}_{,k} \boldsymbol{g}^k . \boldsymbol{g}_m x^m{}_{,3} \,\mathrm{d}A_3 \,\mathrm{d}X^3 = \\ &= JX^3{}_{,k} x^m{}_{,3} \delta^k{}_m \,\mathrm{d}A_3 \,\mathrm{d}X^3 = J\,\mathrm{d}A_3 \,\mathrm{d}X^3 .\end{aligned}$$

The undeformed volume element is

(B.73) $$\mathrm{d}V = \mathrm{d}\boldsymbol{A}_3 . \boldsymbol{G}_3 \,\mathrm{d}X^3 = \boldsymbol{G}^3 . \boldsymbol{G}_3 \,\mathrm{d}A_3 \,\mathrm{d}X^3 = \mathrm{d}A_3 \,\mathrm{d}X^3 .$$

Finally, (B.72) and (B.73) yield the following important relation:

(B.74) $$\mathrm{d}v = J\,\mathrm{d}V .$$

B.2. Stress Tensor

B.2.1. Stress vector and tensor

We shall denote the surface force per unit surface in the deformed body with external normal $\boldsymbol{n}$ by $\boldsymbol{t}_{(n)}$ and refer to it as the *stress vector*. In particular, the stress vector which acts on the kth unit coordinate surface from the side of the external normal, will be denoted by $\boldsymbol{t}_k$; we shall refer to its lth component, t_{kl}, as the *stress tensor*:

(B.75) $$\boldsymbol{t}_k = t_{kl}\boldsymbol{g}^l .$$

To be able to find the relation between the components of the stress tensor t_{kl} and the components of the stress vector $\boldsymbol{t}_{(n)}$, acting on any surface in any point of the continuum, let us consider the condition of equilibrium of an infinitesimal tetrahedron, volume Δv whose three sides $\Delta a^{(k)}$ lie in the coordinate surfaces passing through point p, and the fourth side Δa is perpendicular to $\boldsymbol{n}$ (see Fig. B4). The equation of equilibrium of the acting forces can be estimated with the aid of the mean-value theorem,

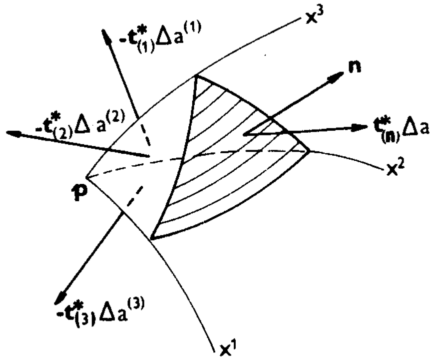

Fig. B4. Tetrahedron.

(B.76) $$\frac{\mathrm{d}}{\mathrm{d}t}(\varrho^*\boldsymbol{v}^*\Delta \boldsymbol{v}) = \boldsymbol{t}^*_{(n)}\Delta a - \boldsymbol{t}^*_{(k)}\Delta a^{(k)} + \varrho^*\boldsymbol{f}^*\Delta \boldsymbol{v},$$

where ϱ^*, $\boldsymbol{v}^*$ and $\boldsymbol{f}^*$ are the density, velocity and body force per unit mass at some interior point of the tetrahedron, $\boldsymbol{t}^*_{(n)}$ and $\boldsymbol{t}^*_{(k)}$ are the values of the stress vector $\boldsymbol{t}_{(n)}$ on surface Δa and on the coordinate surfaces $\Delta a^{(k)}$. The limiting transition for $\Delta v \to 0$ yields

(B.77) $$\boldsymbol{t}_{(n)}\,\mathrm{d}a = \boldsymbol{t}_{(k)}\,\mathrm{d}a^{(k)} .$$

However,

(B.78) $$\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\,\mathrm{d}a = \sum_k \mathrm{d}a^{(k)}\boldsymbol{g}_k/(g_{\underline{kk}})^{1/2} = \mathrm{d}a^k\boldsymbol{g}_k .$$

The last equation yields

(B.79) $$\mathrm{d}a^{(k)}/(g_{\underline{kk}})^{1/2} = \mathrm{d}a^k = n^k\,\mathrm{d}a .$$

$$(B.80) \qquad \boldsymbol{t}_{(n)} = \sum_k \boldsymbol{t}_{(k)} (g_{\underline{kk}})^{1/2} n^k = \boldsymbol{t}_{(k)} n^{(k)} = \boldsymbol{t}_k n^k = \boldsymbol{t}^k n_k ,$$

where $n^{(k)}$ is the physical component of the vector of the external normal $\boldsymbol{n}$ and

$$(B.81) \qquad \boldsymbol{t}_k = \boldsymbol{t}_{(k)} (g_{\underline{kk}})^{1/2}, \quad \boldsymbol{t}^k = g^{kl} \boldsymbol{t}_l, \quad n^{(k)} = n^k (g_{\underline{kk}})^{1/2} .$$

Substituting (B.75) into (B.80) leads to

$$(B.82) \qquad \boldsymbol{t}_{(n)} = t_{kl} n^k \boldsymbol{g}^l, \quad \text{or} \quad \boldsymbol{t}_{(n)l} = \boldsymbol{t}_{kl} n^k .$$

We can see that the stress vector, acting on any surface, is fully described by the components of the stress tensor at this point. Equation (B.80) also yields

$$(B.83) \qquad \boldsymbol{t}_{(-n)} = -\boldsymbol{t}_{(n)} .$$

B.2.2. Equations of motion in integral form

Independently of the geometry of strain and rheological relations, the following laws of conservation are postulated in continuum mechanics.

Axiom 1 (Conservative of Mass): The total mass of a body does not change with motion.

The existence of a continuous function of mass density ϱ is postulated in continuum mechanics. The total mass is given by the expression

$$(B.84) \qquad M = \int_V \varrho \, \mathrm{d}V, \qquad 0 \le \varrho < \infty ,$$

where the integration is taken over the material volume of the body.

The law of mass conservation in turn postulates that the initial total mass of a body is equal to the total mass of the body at any other time, i.e.

$$(B.85) \qquad \int_V \varrho_0 \, \mathrm{d}V = \int_v \varrho \, \mathrm{d}v .$$

By using the transformation relation $\mathrm{d}v = J \, \mathrm{d}V$, we may write

$$(B.86) \qquad \int_V (\varrho_0 - \varrho J) \, \mathrm{d}V = 0 .$$

Alternatively, we may take the material derivative of (B.85). Thus

$$(B.87) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \int_v \varrho \, \mathrm{d}v = 0 .$$

The law of mass conservation may thus be mathematically expressed as either Eq. (B.86) or Eq. (B.87).

Axiom 2 (Balance of Momentum): The time rate of change of the total momentum of a body is equal to the resultant of external forces $\boldsymbol{F}$ *acting on the body.*

Mathematically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{v}\,\mathrm{d}v = \boldsymbol{F}, \tag{B.88}$$

where the l.h.s. represents the time rate of change of the total momentum of the body. The external forces acting on a body are the body forces such as gravity, on the one hand, and surface forces, generated by contact of the body with other bodies, on the other. Consequently,

$$\boldsymbol{F} = \int_s \boldsymbol{t}_{(n)}\,\mathrm{d}a + \int_v \varrho \boldsymbol{f}\,\mathrm{d}v, \tag{B.89}$$

where $\boldsymbol{t}_{(n)}$ is the stress vector per unit area of the surface s with external normal $\boldsymbol{n}$. The body force $\boldsymbol{f}$ refers to unit mass. The balance of momentum thus takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{v}\,\mathrm{d}v = \int_s \boldsymbol{t}_{(n)}\,\mathrm{d}a + \int_v \varrho \boldsymbol{f}\,\mathrm{d}v. \tag{B.90}$$

Axiom 3 (Balance of Moment of Momentum): The time rate of change of the moment of momentum of a body is equal to the resultant moment of all external forces.

Mathematically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \varrho \boldsymbol{p} \times \boldsymbol{v}\,\mathrm{d}v = \int_s \boldsymbol{p} \times \boldsymbol{t}_{(n)}\,\mathrm{d}a + \int_v \varrho \boldsymbol{p} \times \boldsymbol{f}\,\mathrm{d}v, \tag{B.91}$$

where the l.h.s. is the time rate of change of the total moment of momentum of the body about the origin. The surface integral on the r.h.s. of (B.91) is the resultant moment of the surface forces about the origin, and the volume integral is the resultant moment of the body forces about the origin.

Let us emphasize that these relations do not follow from similar equations for a system of mass points and a rigid body, but that they are independent physical laws.

B.2.3. Equations of motion in differential form

The two following integral theorems [57, 59] are important for deriving the equations of motion in differential form.

Consider a continuum, volume v, intersected by surface of discontinuity $\sigma(t)$

moving at velocity $\boldsymbol{v}$ (see Fig. B.5). The material derivative of the volume integral of tensor field $\boldsymbol{\Phi}$ then reads

$$\text{(B.92)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{v-\sigma} \boldsymbol{\Phi}\,\mathrm{d}v = \int_{v-\sigma}\left[\frac{\partial \boldsymbol{\Phi}}{\partial t} + \operatorname{div}(\boldsymbol{\Phi}\boldsymbol{v})\right]\mathrm{d}v + \int_{\sigma}[\boldsymbol{\Phi}(\boldsymbol{v} - \nu)]_{-}^{+}\,.\,\mathrm{d}\boldsymbol{a}\,.$$

The Green-Gauss theorem generalized for a 2nd-order tensor field, $\boldsymbol{\tau} = \tau^{kl}\boldsymbol{g}_k\boldsymbol{g}_l$ is

$$\text{(B.93)} \qquad \int_{v-\sigma} \operatorname{div}\boldsymbol{\tau}\,\mathrm{d}v + \int_{\sigma}[\boldsymbol{\tau}]_{-}^{+}\,.\,\boldsymbol{n}\,\mathrm{d}a = \int_{s-\sigma}\boldsymbol{\tau}\,.\,\boldsymbol{n}\,\mathrm{d}a\,.$$

By volume integral over $v - \sigma$ we understand the volume integral over volume v excluding the material points lying on the surface of discontinuity σ. The same applies to the surface integral over $s - \sigma$. Therefore (see Fig. B5).

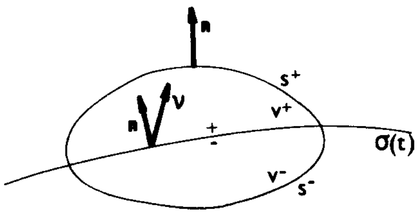

Fig. B5. Region with discontinuity surface.

$$v - \sigma = v^{+} + v^{-}\,, \quad s - \sigma = s^{+} + s^{-}\,.$$

The symbol $[\;]_{-}^{+}$ indicates a jump of the function in brackets at boundary σ,

$$[f]_{-}^{+} = f^{+} - f^{-}\,.$$

Let us apply these two theorems to balance laws postulated in the preceding section. If we put $\boldsymbol{\Phi} = \varrho$ in (B.92), we shall obtain the law of mass conservation in the following form:

$$\text{(B.94)} \qquad \int_{v-\sigma}\left[\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v})\right]\mathrm{d}\boldsymbol{v} + \int_{\sigma}[\varrho(\boldsymbol{v} - \nu)]_{-}^{+}\,.\,\mathrm{d}\boldsymbol{a} = 0\,.$$

For the last equation to hold in any part of the body and on any surface of discontinuity, the integrands in both integrals must be equal to zero:

$$\text{(B.95)} \qquad \begin{aligned} &\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0 \quad \text{in } v - \sigma\,,\\ &[\varrho(\boldsymbol{v} - \nu)]_{-}^{+}\,.\,\boldsymbol{n} = 0 \quad \text{on } \sigma\,. \end{aligned}$$

These equations express "locally" the law of mass conservation in continuum together with the boundary condition. Equation (B.95)$_1$ is called the *equation of continuity*. It is none other than the material derivative of

(B.96) $$\varrho_0 = \varrho J\,.$$

In virtue of Eq. (B.80), the equation of global balance of momentum now reads

(B.97) $$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v-\sigma} \varrho \boldsymbol{v}\,\mathrm{d}v = \int_{s-\sigma} \boldsymbol{t}^k n_k\,\mathrm{d}a + \int_{v-\sigma} \varrho \boldsymbol{f}\,\mathrm{d}v\,.$$

However,

(B.98) $$\boldsymbol{t}^k n_k = t^{kl} n_k \boldsymbol{g}_l = (\boldsymbol{n}\,.\,\mathbf{t})^l \boldsymbol{g}_l = \boldsymbol{n}\,.\,\mathbf{t} = \mathbf{t}\,.\,\boldsymbol{n}\,,$$

since, as we shall show in the next, the stress tensor is symmetric. Using Eqs (B.92) and (B.93) $\boldsymbol{\Phi} = \varrho\boldsymbol{v}$ and $\boldsymbol{\tau} = \mathbf{t}$, we obtain

(B.99) $$\int_{v-\sigma}\left[\frac{\partial(\varrho\boldsymbol{v})}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}\boldsymbol{v}) - \operatorname{div}\mathbf{t} - \varrho\boldsymbol{f}\right]\mathrm{d}v + \\ + \int_\sigma [\varrho\boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \mathbf{t}]^+_- \,.\, \boldsymbol{n}\,\mathrm{d}a = 0$$

This is postulated to be valid for all parts of the body. Thus the integrounds vanish separately.

(B.100) $$\begin{aligned} &\operatorname{div}\mathbf{t} + \varrho(\boldsymbol{f} - \boldsymbol{a}) = 0 && \text{in } v - \sigma, \\ &[\varrho\boldsymbol{v}(\boldsymbol{v} - \boldsymbol{\nu}) - \mathbf{t}]^+_- \,.\, \boldsymbol{n} = 0 && \text{on } \sigma, \end{aligned}$$

where

(B.101) $$\boldsymbol{a} = \partial\boldsymbol{v}/\partial t + \boldsymbol{v}\,.\operatorname{grad}\boldsymbol{v}\,.$$

These equations express "locally" the balance of momentum together with the boundary condition. Equation (B.100)$_1$ is frequently referred to as *Cauchy's first law of motion*, and the stress tensor $\mathbf{t}$, which occurs in it and which is referred to the deformed body, as *Cauchy's stress tensor*.
Equation (B.100)$_1$ in component form reads

(B.102) $$t^{lk}{}_{;l} + \varrho(f^k - a^k) = 0\,.$$

By lowering the indices we obtain the associated equation

(B.103) $$\begin{aligned} t^l{}_{k;l} + \varrho(f_k - a_k) = 0\,, \\ t_{lk;}{}^l + \varrho(f_k - a_k) = 0\,. \end{aligned}$$

By substituting Eq. (B.80) into the equation of balance of the moment of momentum (B.91) and using Eqs. (B.92), (B.93) and (B.100), we arrive at

(B.104) $$\boldsymbol{g}_k \times \boldsymbol{t}^k = 0 \quad \text{in } v - \sigma .$$

The associated jump conditions have already been expressed by Eqs (B.95)$_2$ and (B.100)$_2$. The substitution of (B.75) and (B.62) into (B.104) yields

(B.105) $$t^{kl} = t^{lk}, \; t^k{}_l = t_l{}^k,$$

which is the expression for *Cauchy's second law of motion.*

To conclude, let us express Cauchy's first law of motion in terms of the physical components of vectors and tensors. Equation (B.103)$_1$ will read

(B.106) $$t^l{}_{k,l} + \begin{Bmatrix} l \\ m \; l \end{Bmatrix} t^m{}_k - \begin{Bmatrix} m \\ k \; l \end{Bmatrix} t^l{}_m + \varrho g_{kl}(f^l - a^l) = 0 .$$

Vectors f^l and a^l are expressed in terms of the physical components $f^{(l)}$ and $a^{(l)}$ in Eq. (A.70)$_1$, the stress tensor $t^k{}_l = t_l{}^k$ in terms of the physical components $t^{(k)}{}_{(l)}$ in Eq. (A.76)$_1$. If we now use Eq. (A.94), Eq. (B.106) can be modified to read

(B.107) $$\sum_{k=1}^{3} \left\{ \frac{\partial}{\partial x^k} \left[t_{(l)}{}^{(k)} \frac{(g_{ll})^{1/2}}{(g_{kk})^{1/2}} \right] + t_{(l)}{}^{(k)} \frac{(g_{ll})^{1/2}}{(g_{kk})^{1/2}} \frac{\partial}{\partial x^k} [\log (g)^{1/2}] - \right.$$
$$\left. - \sum_{m=1}^{3} \left[\begin{Bmatrix} k \\ l \; m \end{Bmatrix} t_{(k)}{}^{(m)} \frac{(g_{kk})^{1/2}}{(g_{mm})^{1/2}} \right] + \varrho \frac{g_{kl}}{(g_{kk})^{1/2}} [f^{(k)} - a^{(k)}] \right\} = 0 .$$

This equation is valid in any curvilinear coordinate system provided the stress tensor is symmetric. If the curvilinear coordinates are orthogonal, Eq. (B.107) converts to Eq. (A.160).

B.2.4. Equations of motion in the reference coordinate system

Cauchy's equations of motion have been expressed in terms of Euler's coordinates. However, in many cases it is convenient to formulate the problem in the reference (Lagrange's) coordinate system.

Let us now, therefore, express the equations of motion in the reference system X^K. Equation (B.96) followed from the law of mass conservation:

(B.108) $$\varrho_0 = \varrho J, \; J = (g/G)^{1/2} j, \; j = \det (x^k_{,K}) .$$

Let us introduce the stress vector $\boldsymbol{T}^K$ at spatial point $\boldsymbol{x}$ and time t relative to the underformed surface $\mathrm{d}A_K$, located at point $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$:

(B.109) $$\boldsymbol{t}_{(n)} \mathrm{d}a = \boldsymbol{t}^k \mathrm{d}a_k = \boldsymbol{T}^K \mathrm{d}A_K .$$

By using Eq. (B.71) we obtain

(B.110) $$\boldsymbol{t}^k = J^{-1} x^k_{,K} \boldsymbol{T}^K, \; \boldsymbol{T}^K = J X^K_{,k} \boldsymbol{t}^k .$$

Let us introduce the *Piola-Kirchhoff pseudostress tensor* T^{Kl} and T^{KL} by

(B.111) $$\boldsymbol{T}^K = T^{Kl}\boldsymbol{g}_l = T^{KL}x^l_{,L}\boldsymbol{g}_l\,.$$

Equations (B.110) and (B.75) then yield

(B.112) $$T^{Kl} = JX^K{}_{,k}t^{kl}\,,$$
$$T^{KL} = T^{Kl}X^L{}_{,l} = JX^K{}_{,k}X^L{}_{,l}t^{kl}\,.$$

Equations (B.109) and (B.111)$_1$ indicate that T^{Kl} expresses the stress at $\boldsymbol{x}$ measured per unit undeformed area at $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$. From (B.112) it also follows that

(B.113) $$t^{kl} = J^{-1}x^k{}_{,K}T^{Kl} = J^{-1}x^k{}_{,K}x^l{}_{,L}T^{KL}\,.$$

The equations of motion (B.102) can be expressed in terms of the components T^{Kl} as

(B.114) $$T^{Kk}{}_{,K} + T^{Km}\begin{Bmatrix} k \\ m\ l \end{Bmatrix}x^l{}_{,K} + T^{Kk}\begin{Bmatrix} L \\ L\ K \end{Bmatrix} + \varrho_0(f^k - a^k) = 0\,.$$

If we introduce total covariant derivatives of the two-point tensor field $T^{Kk}(\boldsymbol{X}, \boldsymbol{x})$ — refer to Supplement A — Eq. (B.114) can be expressed in a more consise form

(B.115) $$T^{Kk}{}_{:K} + \varrho_0(f^k - a^k) = 0\,.$$

Cauchy's second law of motion now has a more complicated form,

(B.116) $$T^{Kk}x^l{}_{,K} = T^{Kl}x^k{}_{,K}\,.$$

The equations of motion, expressed in terms of the components T^{KL}, now read

(B.117) $$(T^{KL}x^k{}_{,L})_{,K} + \left(\begin{Bmatrix} k \\ m\ l \end{Bmatrix}x^m{}_{,L}x^k{}_{,K} + \right.$$
$$\left. + \begin{Bmatrix} M \\ M\ K \end{Bmatrix}x^k{}_{,L}\right)T^{KL} + \varrho_0(f^k - a^k) = 0\,,$$
$$T^{KL} = T^{LK}\,.$$

It is easy to prove that, if the deformations are small, there is no difference between the equations of motion expressed in Euler's and Lagrange's coordinates.

To be able to express the jump conditions in the reference system, we shall first derive the relation for the external normals $\boldsymbol{n}$ and $\boldsymbol{N}$ of the deformed and undeformed surfaces s and S. With a view to (B.71) we have

(B.118) $$\mathrm{d}a_k = JX^K{}_{,k}\mathrm{d}A_K\,.$$

However,

(B.119) $$n_k = \mathrm{d}a_k/\mathrm{d}a = \mathrm{d}a_k/(\mathrm{d}a^l\,\mathrm{d}a_l)^{1/2}\,,$$
$$N_K = \mathrm{d}A_K/\mathrm{d}A = \mathrm{d}A_K/(\mathrm{d}A^L\,\mathrm{d}A_L)^{1/2}\,,$$

and, therefore,

(B.120)
$$n_k = JX^K{}_{,k}N_K \mathrm{d}A/\mathrm{d}a\,.$$

By using (B.118) we obtain

(B.121)
$$\mathrm{d}A/\mathrm{d}a = J^{-1}(C^{KL}N_K N_L)^{-1/2},$$

where

(B.122)
$$C^{KL} = g^{kl}X^K{}_{,k}X^L{}_{,l}$$

is *Piola's deformation tensor*. Finally, we obtain

(B.123)
$$n_k = (C^{KL}N_K N_L)^{-1/2}X^M{}_{,k}N_M\,.$$

By substituting Eqs (B.109) and (B.120) into $(B.95)_2$ and $(B.100)_2$, we arrive at the jump conditions in the reference system:

(B.124)
$$\left[\varrho_0(v^k - \nu^k)\,X^K{}_{,k}N_K\frac{\mathrm{d}A}{\mathrm{d}a}\right]_-^+ = 0 \quad \text{on } \Sigma,$$

(B.125)
$$\left[[\varrho_0 \boldsymbol{v}(v^k - \nu^k)\,X^K{}_{,k} - \boldsymbol{T}^K]\,N_K\frac{\mathrm{d}A}{\mathrm{d}a}\right]_-^+ = 0 \quad \text{on } \Sigma.$$

At a solid surface of discontinuity (solid elastic substance — solid elastic substance boundary) it also holds that

(B.126)
$$[\mathrm{d}a]_-^+ = [\mathrm{d}A]_-^+ = 0$$

and conditions (B.124) and (B.125) can be expressed as

(B.127)
$$[\varrho_0(v^k - \nu^k)\,X^K{}_{,k}]_-^+\,N_K = 0 \quad \text{on } \Sigma,$$

(B.128)
$$[\varrho_0 \boldsymbol{v}(v^k - \nu^k)\,X^K{}_{,k} - \boldsymbol{T}^K]_-^+\,N_K = 0 \quad \text{on } \Sigma.$$

However, at a liquid surface of discontinuity (solid elastic substance — liquid boundary) only the following holds (see Fig. B6):

(B.129)
$$[\mathrm{d}a]_-^+ = 0$$

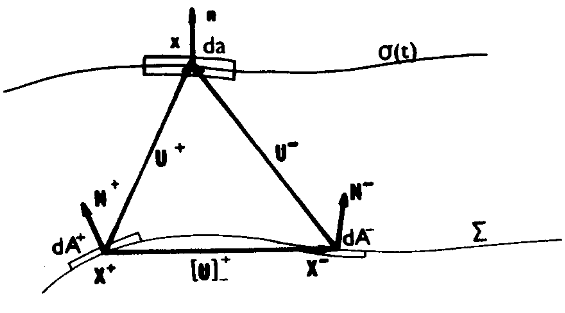

Fig. B6. Liquid boundary before and after deformation.

and conditions (B.124) and (B.125) can be expressed as

(B.130) $$[\varrho_0(v^k - \nu^k) X^K{}_{,k} \mathrm{d}A]^+_- = 0 \quad \text{on } \Sigma,$$

(B.131) $$[[\varrho_0 \nu(v^k - \nu^k) X^K{}_{,k} - t^K] N_K \mathrm{d}A]^+_- = 0 \quad \text{on } \Sigma.$$

SUPPLEMENT C. LIMITING VALUE OF FUNCTION $z_n(x)$

Equation (8.10) defines function $z_n(x)$,

(C.1) $$z_n(x) = x j_{n+1}(x)/j_n(x),$$

where $j_n(x)$ is a spherical Bessel function of the 1st kind,

(C.2) $$j_n(x) = \sqrt{\left(\frac{\pi}{2x}\right)} J_{n+\frac{1}{2}}(x)$$

and $J_n(x)$ is Bassel's function of the 1st kind. Let us seek to determine the limiting value of function $z_n(x)$ for $n \to \infty$ for a fixed value of x. According to [1],

(C.3) $$\lim_{n\to\infty} J_n(x) = \frac{1}{\sqrt{(2\pi x)}} \left(\frac{\mathrm{e}x}{2n}\right)^n \quad \text{for fixed } x,$$

where $\mathrm{e} \doteq 2.718\,281\,828$. This yields the limiting value of function $z_n(x)$ for a fixed x,

(C.4) $$\lim_{n\to\infty} z_n(x) = \frac{\mathrm{e}x^2}{2n+3}\left(\frac{n+\frac{1}{2}}{n+\frac{3}{2}}\right)^{n+1}.$$

However, according to [125], for any finite number a

(C.5) $$\lim_{n\to\infty} (1 + a/\mathrm{n})^{\mathrm{n}} = e^a.$$

Equation (C.4) can then be modified to read

(C.6) $$\lim_{n\to\infty} z_n(x) = \frac{\mathrm{e}x^2}{2n+3} \lim_{n\to\infty} \frac{1+\frac{1}{2n}}{1+\frac{3}{2n}} \lim_{n\to\infty} \frac{\left(1+\frac{1}{2n}\right)^n}{\left(1+\frac{3}{2n}\right)^n}.$$

The first limiting value on the r.h.s. of (C.6) is equal to 1, the second limit is 1/e. Finally,

(C.7) $$\lim_{n\to\infty} z_n(x) = \frac{x^2}{2n+3}.$$

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