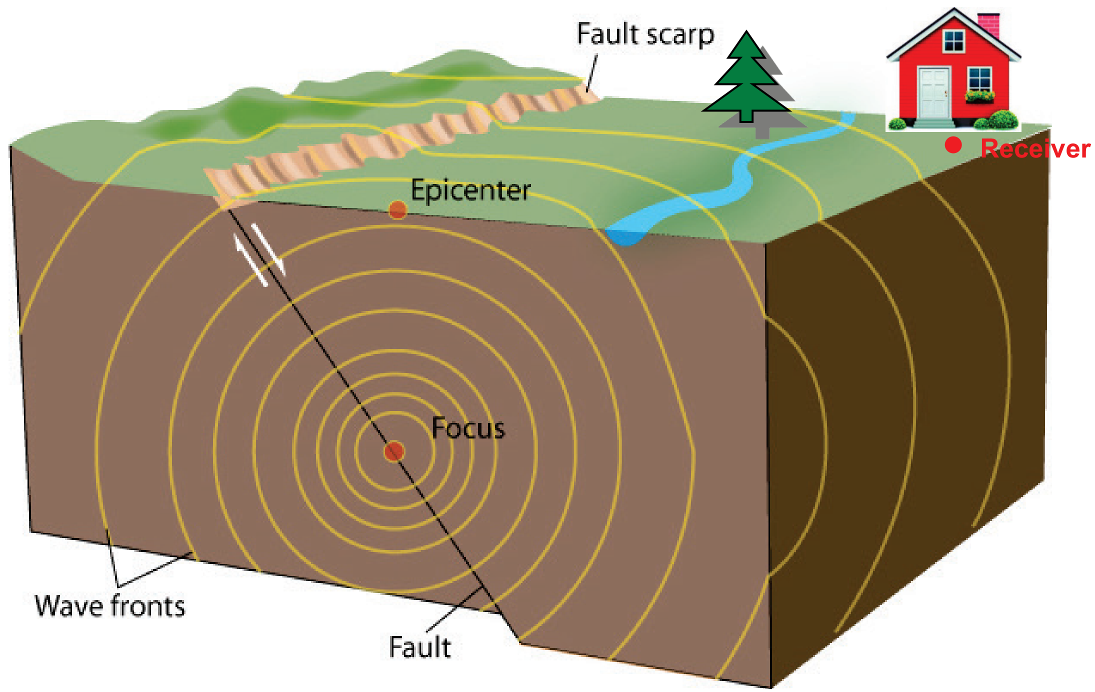
Propagation of seismic waves - theoretical background



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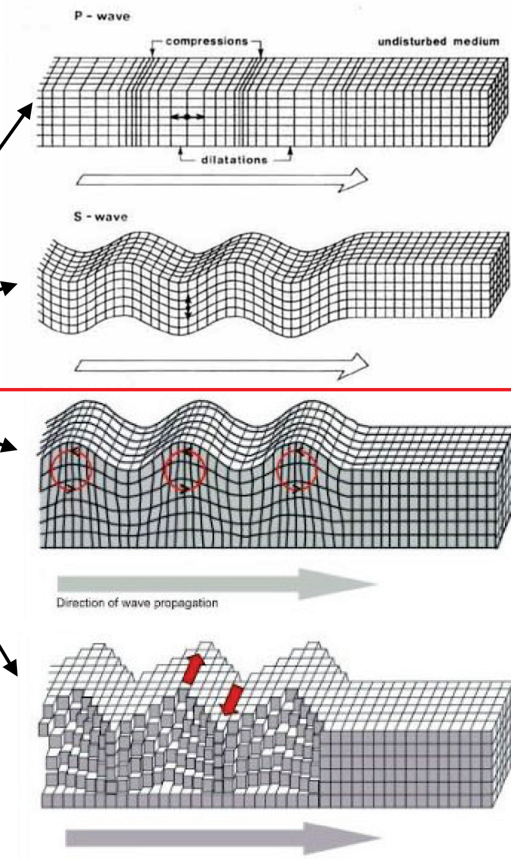
Seismic waves = waves in elastic continuum

elastic continuum: a model of the medium through which the waves propagate

Body waves in isotropic media

- elastic waves in isotropic media
  - body waves
    - longitudinal waves
    - transverse waves
  - surface waves
    - Rayleigh waves
    - Love waves



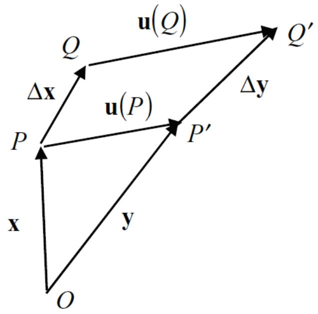
+ Body waves in acoustic and anisotropic media

In this course we rely on basic knowledge of continuum mechanics (students should be familiar with the underlined terms below).

A time-dependent perturbation of an elastic medium generates elastic waves emanating from the source region. These disturbances produce local changes in stress and strain.

To understand the propagation of elastic waves we need to describe kinematically the deformation of our medium and the resulting forces (stress). The relation between deformation and stress is governed by elastic constants.

The time-dependence of these disturbances will lead us to the elastic wave equation as a consequence of conservation of energy and momentum.



P,Q …. positions of two particles before deformation
P',Q'… positions of the particles after deformation
u ……. displacement vector

Basic assumptions:

1) Geological structure can be considered as an **elastic continuum** (despite of presence of discontinuities at all scales) – the parameters characterizing the continuum are the ‘effective’ parameters averaged over a portion of the real structure at the given scale.
(at different scales the parameters of the same material can be different)

2) Deformations are small => strain tensor can be linearized to the **infinitesimal strain tensor**

$$\left|\frac{\partial u_i}{\partial x_j}\right| << 1$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Why? We consider only **incremental** stress and strain, directly connected with the wave propagation.

3) Incremental **body forces** (per unit mass) associated to the wave propagation **are negligible**, i.e. the body forces acting in the medium before and after the seismic event are the same.

4) Stress-strain relation is linear – generalized **Hook’s law**

$$\tau_{ij} = c_{ijkl} e_{kl}$$

(in this course, it is moreover time-independent => no absorption)

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Not satisfied closer to the source – finite strain theory

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Not satisfied closer to the source

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(in this course, it is moreover time-independent => no absorption)

Not satisfied closer to the source

Under these simplifications, the wave propagation satisfies the following **equation of motion**

$$\cancel{F_i} + \frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Emloying the Hook's law we get

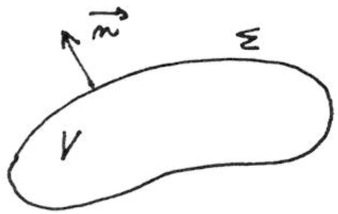
$$\frac{\partial}{\partial x_j}\left(\underbrace{c_{ijkl}}\frac{\partial u_k}{\partial x_l}\right) + \cancel{F_i} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Elastic constants (medium parameters, known)

Density (medium parameter, known)

which is to be solved for $u_i = u_i(\mathbf{x}, t)$

Uniqueness theorem



Assume a volume V bounded by a surface Σ.
When is the solution of

$$\rho \ddot{u}_i - \tau_{ij,j} = f_i$$

unique inside V?

At any point inside V and any time $t \geq t_0$ the solution is uniquely determined by:

1) Distribution of u_i and $\dot{u}_i$ inside V at t_0. *(initial conditions).*
+
2) Distribution of u_i and/or its spatial derivatives on Σ for any $t \geq t_0$. *(boundary conditions)*
+
3) Distribution of f_i (and heat – for adiabatic processes neglected) inside V at any time $t \geq t_0$.

Proof: Assume two different solutions $u_i^{(1)}$ and $u_i^{(2)}$ for the same forces and initial/boundary conditions.

Then, for $U_i = u_i^{(1)} - u_i^{(2)}$ it holds $U_i(t_0) = \dot{U}_i(t_0) = 0$, $U_i = U_{i,k} = 0$ on Σ and $f_i = 0$.

$$\frac{d\epsilon}{dt} = \iiint_V \dot{u}_i f_i dV + \iiint_V (\tau_{ij}\dot{u}_i)_{,j} dV$$

$$= \iiint_V \dot{u}_i f_i dV + \iint_\Sigma c_{ijkl} u_{k,l} n_j \dot{u}_i dS$$

$$\Downarrow$$

$\epsilon = \text{constant}$ for U_i

$$\epsilon = \frac{1}{2} \iiint_V (\rho \dot{u}_i \dot{u}_i + \tau_{ij} e_{ij}) dV$$

$$= \frac{1}{2} \iiint_V (\rho \dot{u}_i \dot{u}_i + c_{ijkl} u_{k,l} u_{i,j}) dV$$

$$\Downarrow \quad \text{(init.cond.)}$$

$$\epsilon = 0$$

$$W = K = 0$$

$$\Longrightarrow \dot{U}_i(x_m, t) = 0$$

$$U_i(x_m, t) = 0$$

$$\Updownarrow$$

$$u_i^{(1)}(x_m, t) = u_i^{(2)}(x_m, t)$$

The medium through which the waves propagate is characterized (besides of its density) by the elastic parameters c_{ijkl} .

a four-rank symmetric tensor with 81 elements but only **21** of them **are**, in general, **independent**. ….. *general anisotropy*

If less than 21 elements are independent …. *anisotropy with special symmetries (orthorombic, hexagonal, etc.)*

To describe special anisotropic symmetries it is useful to introduce the so called **Voight notation** for the elastic paramenters c_{ijkl} :

$$c_{ijkl} \longleftrightarrow C_{\alpha\beta}$$

matrix 6x6

How the indices are assigned:

1 ↔ 1,1; 2 ↔ 2,2; 3 ↔ 3,3; 4 ↔ 2,3; 5 ↔ 3,1; 6 ↔ 1,2

$$C_{\alpha\beta} = C_{\beta\alpha}$$

Various types of anisotropic symmetries

Orthorombic: ***9** independent parameters*

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix}$$

Hexagonal: ***5** independent parameters*

Transverse isotropy – one axis of symmetry; medium invariant to any rotation around it

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{pmatrix}$$

Isotropy: ***2** independent parameters*

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{11}-2C_{44} & C_{11}-2C_{44} & 0 & 0 & 0 \\ & C_{11} & C_{11}-2C_{44} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix}$$

In the Lame's notation – λ,μ … Lame's parameters

$$C_{\alpha\beta} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{pmatrix}$$

Acoustic case (fluids):
$\mu=0$, only ***1** parameter* λ

Acoustic medium – equation of motion

e.g., Earth's core, oceans, often used in seismic prospection …

Hook's law

$$\tau_{ij} = \lambda\theta\delta_{ij} = \frac{1}{\kappa}\theta\delta_{ij}, \qquad \theta = u_{i,i}$$

$\frac{1}{\kappa}\theta$ = - *p* ….. pressure

κ: compressibility

(comma is used to separate component indices from differentiation indices)

Equation of motion then yields

$$p_{,i} + \rho\frac{\partial v_i}{\partial t} = F_i$$

while taking the time derivative of the Hook's law gives $v_{k,k} + \kappa\frac{\partial p}{\partial t} = 0$.

Eliminating v_i from the set of the two equations we get the **scalar acoustic eq.**

$$(\rho^{-1}p_{,i})_{,i} + f^p = \kappa\ddot{p}\ , \quad \text{where} \quad f^p = -(\rho^{-1}F_i)_{,i}$$

In the vector notation:

$$\nabla\cdot(\rho^{-1}\nabla p) + f^p = \kappa\ddot{p}$$

…should be solved for p (numerically)

An analytical solution is known only for **homogeneous medium** (ρ=const, κ=const). Then the equation of motion reads

$$\boxed{\nabla^2 p + \rho f^p = c^{-2}\ddot{p}}, \qquad c = \sqrt{1/\rho\kappa}$$

c: acoustic velocity

Plane wave … the simplest solution of the eq. of motion in homogeneous media

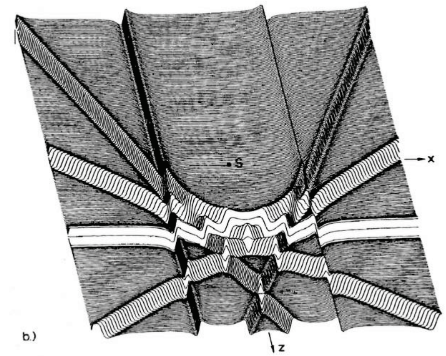
- in reality does not exist (the existence conditioned by homogeneous unbounded medium)

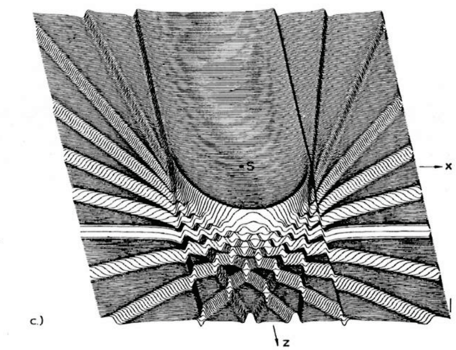
Why bother?

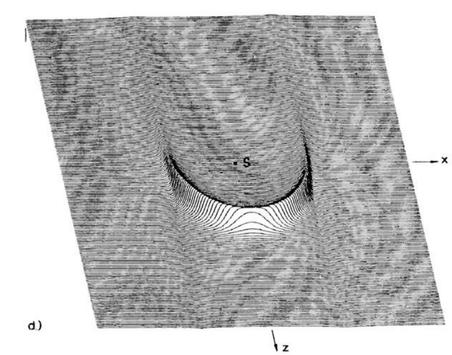
1) It may be a good approximation of the real wave field for a distant source in a simple (close to homogeneous) structure.

2) High-frequency asymptotic methods (e.g., ray theory) are based on a generalization of the plane wave solution.

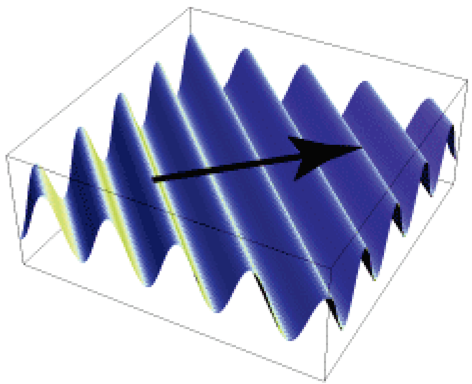
3) A general wave with a curved wavefront can be represented as an integral superposition of plane waves (Weyl integral) →







Plane wave … the simplest solution of the eq. of motion in homogeneous media



Harmonic plane wave:

Circular frequency $\omega = 2\pi f$

$$p(\vec{x}, t) = P \exp[-i\omega(t - T(\vec{x}))]$$

Constant scalar amplitude (can be complex-valued)

Propagation time linear function of coordinates

$$T(\vec{x}) = p_i x_i$$

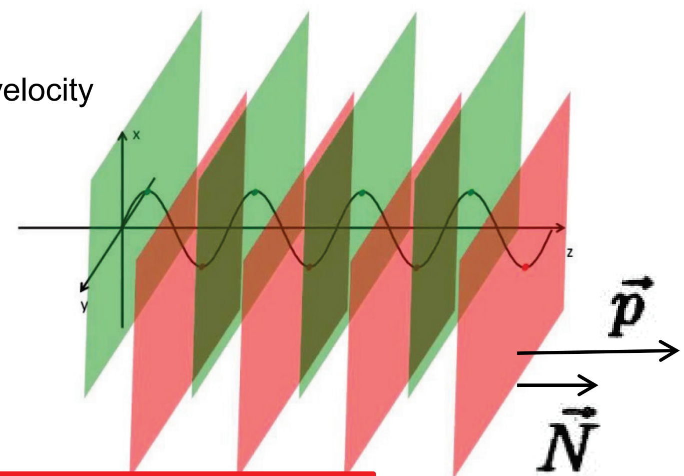
equation $t = T(\vec{x})$ represents a moving plane, here called the *wavefront*.

We shall denote the unit normal to the wavefront $\vec{N}$

$$N_i = p_i / \sqrt{p_k p_k} \qquad \vec{p} = \vec{N}/C$$

C: Phase velocity

$\vec{p}$: Slowness vector



Inserting the solution into the acoustic eq. yields

$$p_1^2 + p_2^2 + p_3^2 = 1/c^2 \qquad \text{where } c = (\rho\kappa)^{-1/2}$$

Existence condition for the plane wave

$$C = c = (\rho\kappa)^{-1/2}$$

It does not depend on the frequency and on the direction of propagation $\vec{N}$.

Plane wave … the simplest solution of the eq. of motion in homogeneous media

The same conclusions can be made (in the same way) for a **transient** plane wave:

$$p(\vec{x},t) = PF(t - T(\vec{x}))$$

Constant scalar amplitude (can be complex-valued)

Propagation time linear function of coordinates $T(\vec{x}) = p_i x_i$

the so called analytical signal

The analytic signal is any complex-valued function whose real and imaginary parts form a Hilbert pair (the imaginary part is a Hilbert's transform $\mathcal{H}$ of the real one), i.e.,

$$F(t) = f(t) + \mathrm{i}\mathcal{H}[f(t)] = f(t) + \mathrm{i}\frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{f(\xi)}{t-\xi}\mathrm{d}\xi\ , \qquad (2.5)$$

where $f(t)$ is a real-valued function and P.V. means that the integral is taken in the sense of the Cauchy principal value. Note that the exponential function $\exp(\mathrm{i}t) = \cos(t) + \mathrm{i}\sin(t)$ is the special case of the analytic signal.

An advantage of the analytic signal – one-sided spectrum. However, only its real part has the physical meaning of pressure.

Anisotropic medium – equation of motion

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \cancel{F_i} = \rho\frac{\partial^2 u_i}{\partial t^2}$$

… equation for general anisotropic medium, should be solved for u_i (numerically; analytical solution is not known)

→ This is not a subject of this course

An analytical solution is known only for **homogeneous medium** (ρ=const, c_{ijkl}=const). Then the equation of motion reads

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + \cancel{F_i} = \rho\frac{\partial^2 u_i}{\partial t^2}$$

The simplest solution of the eq. of motion in homogeneous media is a **plane wave**

$$\vec{u}(\vec{x},t) = \vec{U}F(t - T(x_i)), \quad T(x_i) = \frac{N_i x_i}{C} = p_i x_i$$

- $\vec{U}$: *Constant vector amplitude (can be complex-valued)*
- F: Analytical signal (for harmonic waves exp)
- $T(x_i)$: *Propagation time linear function of coordinates*
- N_i: Unit normal to the wavefront
- C: Phase velocity
- p_i: Slowness vector

Plane waves in anisotropic medium

We insert the plane-wave solution into the elastodynamic equation and employ the Christoffel formalism. That is we insert $\vec{u}(\vec{x},t) = \vec{U}F(t - T(x_i))$ into the eq.

$$a_{ijkl}u_{k,lj} = \ddot{u}_i \;, \quad i = 1,2,3 \;, \quad a_{ijkl} = c_{ijkl}/\rho$$

We get $\quad a_{ijkl}p_j p_l U_k - U_i = 0 \;, \quad i = 1,2,3 \;.$

Denoting $\Gamma_{ik} = a_{ijkl}p_j p_l$ (Christoffel matrix) we come to the Christoffel equation $\Gamma_{ik}U_k - U_i = 0$,

i.e., $\quad (\Gamma_{ik} - \delta_{ik})U_k = 0$

(Alternatively, introducing $\bar{\Gamma}_{ik} = a_{ijkl}N_j N_l$ we get the Christoffel equation in the form

$$(\bar{\Gamma}_{ik} - C^2\delta_{ik})U_k = 0$$

This is a system of three linear equations for U_1, U_2, U_3. If U_i and p_i satisfy the Christoffel eq. then the plane wave is a solution of the elastodynamic equation.

We will find p_i and U_i from eigenvalues G and eigenvectors g_i of the Christoffel matrix. That is, we will seek G (eigenvalue) from the equation

$$\det\,(\Gamma_{ik} - G\delta_{ik}) = 0 = \det\begin{pmatrix} \Gamma_{11} - G & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{12} & \Gamma_{22} - G & \Gamma_{23} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} - G \end{pmatrix}.$$

When we find G, we put G=1 to make the equation consistent with the Christoffel equation. (In the case of $\bar{\Gamma}_{ik}$, we put $\bar{G} = C^2$ (phase velocity)). **… existence condition for the plane wave**

Plane waves in anisotropic medium (continuation)

We use the standard algebraic procedure to find the eigenvalues. The equation $\det(\Gamma_{ik} - G\delta_{ik}) = 0$ represents the **cubic algebraic eq**. $G^3 - PG^2 + QG - R = 0$, where P, Q and R are the invariants of the matrix $\boldsymbol{\Gamma}$:

$$P = \mathrm{tr}\,\boldsymbol{\Gamma}\,, \quad R = \det\,\boldsymbol{\Gamma}\,,$$

$$Q = \det\begin{pmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{12} & \Gamma_{22}\end{pmatrix} + \det\begin{pmatrix}\Gamma_{22} & \Gamma_{23}\\ \Gamma_{23} & \Gamma_{33}\end{pmatrix} + \det\begin{pmatrix}\Gamma_{11} & \Gamma_{13}\\ \Gamma_{13} & \Gamma_{33}\end{pmatrix}.$$

→ 3 roots (in general, sought numerically)

It is obvious that matrix Γ_{ij} has three eigenvalues. We denote them G_m, $m = 1, 2, 3$.
As matrix $\boldsymbol{\Gamma}$ is symmetric and positive definite, all the three eigenvalues are real-valued and positive.

The equations $G_m(p_i)=1$ (or $\bar{G}_m(N_i)= C^2$), m=1 or 2 or 3, represents a constraint for $\boldsymbol{p}$. Only the waves satifying these conditions represent the solutions of the elastodynamic eq. Thus, in anisotropic media, there may exist **three independent waves** propagating in a given direction at different speeds $C^{(1)}, C^{(2)}, C^{(3)}$.

$$C^{(m)^2} = \bar{G}_m(N_i) \quad \Rightarrow \boxed{C^{(m)} = \sqrt{\bar{G}_m(N_i)}}$$

↓ **phase velocity depends on direction (anisotropy)**

qP, qS1 and qS2 waves
(qS1 and qS2 phase velocities sometimes close to each other or even coincide – singular regions)

Eigenvectors: $\vec{g}^{(m)}$, $m = 1, 2, 3$ (mutually perpendicular) are solutions of the equation

$$(\Gamma_{ik} - G_m\delta_{ik})g_k^{(m)} = 0\,, \quad i = 1, 2, 3\,,$$ with the normalization condition $g_k^{(m)}g_k^{(m)} = 1$.

A comparison with the Christoffel condition yields $\vec{U} = A\vec{g}^{(m)}$ Thus, the final plane wave solution is:

→ linear polarization

$$\boxed{\vec{u}(x_i, t) = A\vec{g}^{(m)}F(t - N_i x_i/C^{(m)})\,, \; m = 1 \text{ or } 2 \text{ or } 3}.$$

Isotropic medium – equation of motion

Kronecker's delta: $= 0, \quad k \neq l$; $= 1, \quad k = l$

Introducing the Lame's coefficients λ and μ we get the following eq. of motion in isotropic media

$$c_{ijkl} = \lambda \delta_{kl}\delta_{ij} + \mu(\delta_{ik}\delta_{jl} + \delta_{jl}\delta_{il})$$

$$\lambda_{,i} u_{k,k} + \mu_{,l} u_{i,l} + \mu_{,k} u_{k,i} + \lambda u_{l,li} + \mu u_{i,ll} + \mu u_{j,ij} + F_i = \rho u_{i,tt}$$

… equation for general isotropic medium, should be solved for u_i (numerically; analytical solution is not known)

This is not a subject of this course

An analytical solution is known only for **homogeneous medium** (ρ=const, λ,μ =const). Then the equation of motion reads

$$(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} + F_i = \rho u_{i,tt}$$

The simplest solution of the eq. of motion in homogeneous media is a **plane wave**

$$\vec{u}(\vec{x}, t) = \vec{U} F(t - T(x_i)), \quad T(x_i) = \frac{N_i x_i}{C} = p_i x_i$$

Constant vector amplitude (can be complex-valued)

Analytical signal (for harmonic waves exp)

Propagation time linear function of coordinates

Unit normal to the wavefront

Slowness vector

Phase velocity

Plane waves in isotropic medium

In analogy to the anisotropic case, we insert the plane-wave solution into the elastodynamic equation and employ the Christoffel formalism. This again leads to the eigenvalue problem, but now the Christoffel matrix has the form:

$$\Gamma_{ik} = \frac{\lambda + \mu}{\rho} p_i p_k + \frac{\mu}{\rho} \delta_{ik} p_l p_l$$

The eigenvalues G can be determined from relation

$$\det \begin{pmatrix} \frac{\lambda+\mu}{\rho} p_1^2 + \frac{\mu}{\rho} p_i p_i - G & \frac{\lambda+\mu}{\rho} p_1 p_2 & \frac{\lambda+\mu}{\rho} p_1 p_3 \\ \frac{\lambda+\mu}{\rho} p_1 p_2 & \frac{\lambda+\mu}{\rho} p_2^2 + \frac{\mu}{\rho} p_i p_i - G & \frac{\lambda+\mu}{\rho} p_2 p_3 \\ \frac{\lambda+\mu}{\rho} p_1 p_3 & \frac{\lambda+\mu}{\rho} p_2 p_3 & \frac{\lambda+\mu}{\rho} p_3^2 + \frac{\mu}{\rho} p_i p_i - G \end{pmatrix} = 0 .$$

This eq. can be solved analyticaly. It can be shown that

$$\det\,(\ldots) = \left(\frac{\mu}{\rho} p_i p_i - G\right)^2 \left(\frac{\lambda + 2\mu}{\rho} p_i p_i - G\right) = (\alpha^2 p_i p_i - G)(\beta^2 p_i p_i - G)^2 \;\; = 0 ,$$

with

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} \quad \text{and} \quad \beta = \left(\frac{\mu}{\rho}\right)^{1/2} .$$

It means that two eigenvalues coincide, i.e., instead of three different values we obtain only two. Thus, **only two plane waves can propagate in a given direction in isotropic media.** One coresponding to the eigenvalues

$$G_1(p_i) = G_2(p_i) = \beta^2 p_i p_i ,$$

and the other one corresponding to

$$G_3(p_i) = \alpha^2 p_i p_i$$

Plane waves in isotropic medium (continuation)

The constraints $G_m(p_i) = 1$ yield **two existence conditions**

$$p_1^2 + p_2^2 + p_3^2 = 1/\alpha^2$$ … **P wave (primary)**

$$p_1^2 + p_2^2 + p_3^2 = 1/\beta^2$$ … **S wave (secondary),** $\alpha > \beta$

$$= |\mathrm{p}|^2 = \left(\frac{1}{c}\right)^2$$

The existence conditions yield the phase velocity of P wave as $c = \alpha$ and that of S wave $c = \beta$.

Eigenvecors:

P wave …….

$$\left[\frac{\lambda+\mu}{\rho} p_i p_k + \frac{\mu}{\rho}\delta_{ik} p_l p_l - \delta_{ik}\right] g_k^{(3)} = 0$$

. Multiplying this eq. by $g_i^{(3)}$ and inserting $p_i = N_i/\alpha$ we get $\vec{g}^{(3)} = \pm\vec{N}$.

… **linear polarization, perpendicular to the wavefront (compressional, longitudinal wave)**

S wave …….

$\mathrm{g}^{(1)}, \mathrm{g}^{(2)} \perp \mathrm{g}^{(3)}$

(the vectors cannot be determined uniquely because of the coinciding eigenvalues)

… **polarized in the plane of the wavefront (transverse, shear wave)**

Final solution for P

$$\vec{u}(\vec{x}, t) = A\vec{N}F(t - N_i x_i/\alpha) .$$

Final solution for S

$$\vec{u}(\vec{x}, t) = (B\vec{e}_1 + C\vec{e}_2)F(t - N_i x_i/\beta) .$$

elliptical polarization in the plane of the wavefront (in a special case of real-valued *B* and *C*, the polarization is linear).

A, *B*, and *C* are arbitrary complex-valued constants. $\vec{e}_1$, $\vec{e}_2$ are arbitrary mutually perpendicular vectors situated in the plane of the wavefront.

Plane waves in isotropic medium (continuation)

The constraints $G_m(p_i) = 1$ yield **two existence conditions**

$$p_1^2 + p_2^2 + p_3^2 = 1/\alpha^2$$ … **P wave (primary)**

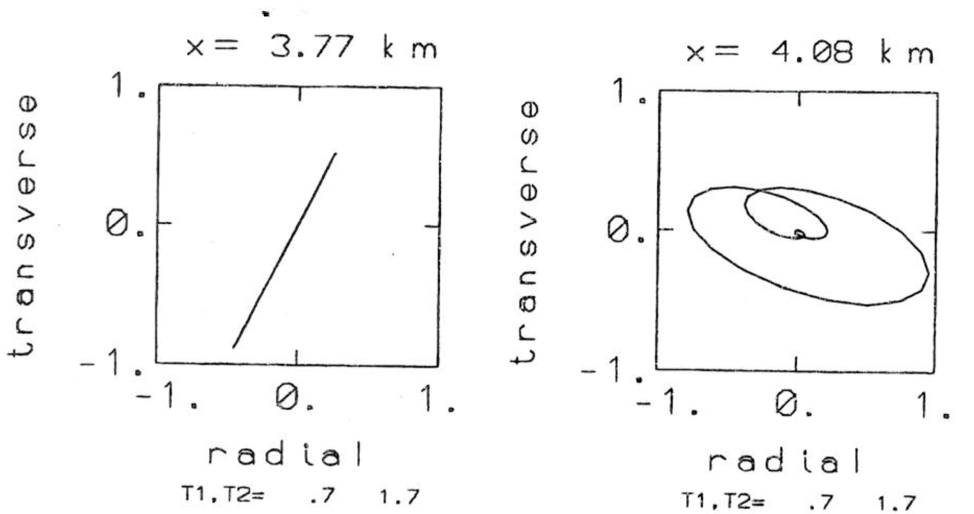
$$p_1^2 + p_2^2 + p_3^2 = 1/\beta^2$$ … **S wave (secondary),** $\alpha > \beta$

$$= |\mathrm{p}|^2 = \left(\frac{1}{c}\right)^2$$

The existence conditions yield the phase velocity of P wave as $c = \alpha$ and that of S wave $c = \beta$.

Eigenvecors:

P wav



plying this eq. by $g_i^{(3)}$ and inserting

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S wav

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Final

→ **elliptical polarization in the plane of the wavefront (in a special case of real-valued *B* and *C*, the polarization is linear).**

Final solution for S $$\vec{u}(\vec{x},t) = (B\vec{e}_1 + C\vec{e}_2)F(t - N_i x_i/\beta) .$$

A, *B*, and *C* are arbitrary complex-valued constants. $\vec{e}_1$, $\vec{e}_2$ are arbitrary mutually perpendicular vectors situated in the plane of the wavefront.

Energy of elastic waves (coontinuum mechanics basics)

Total energy of a wave = kinetic energy + strain energy (potential energy)

Density of strain energy (per unit volume)

$$W = \frac{1}{2}\tau_{ij}e_{ij} \quad = \frac{1}{2}c_{ijkl}e_{ij}e_{kl}$$

Density of kinetic energy

$$K = \frac{1}{2}\rho\dot{u}_i\dot{u}_i$$

Density of total energy

$$E = W + K = \frac{1}{2}c_{ijkl}e_{ij}e_{kl} + \frac{1}{2}\rho\dot{u}_i\dot{u}_i$$

Density of energy flux

$$S_i = -\tau_{ij}\dot{u}_j = -c_{ijkl}e_{kl}\dot{u}_j = -c_{ijkl}u_{k,l}\dot{u}_j$$

Velocity of energy flux = group velocity

$$v_i^g = \frac{S_i}{E}$$

Elastodynamic equation yields the conservation of elastic energy (for vanishing body forces)

$$\partial E/\partial t + \nabla \cdot \vec{S} = 0$$

Elastic energy flux

Total energy in a volume V and its time derivative

$$\epsilon = \iiint_V E\,dV = \frac{1}{2}\iiint_V \left(\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \tau_{ij}e_{ij}\right) dV\,, \quad \frac{d\epsilon}{dt} = \iiint_V \frac{\partial E}{\partial t} dV = \iiint_V \left[\rho \frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial t^2} + \tau_{ij}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right)\right] dV$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2}\tau_{ij}\, e_{ij}\right) = \frac{1}{2} c_{ijkl} \frac{\partial}{\partial t}\left(e_{kl}\, e_{ij}\right) = \frac{1}{2} c_{ijkl}\left(\dot{e}_{kl} e_{ij} + e_{kl}\dot{e}_{ij}\right) = c_{ijkl}\, e_{kl}\, \dot{e}_{ij} = \quad (c_{ijkl} = c_{klij})$$

$$= \tau_{ij}\, \dot{e}_{ij} = \tau_{ij}\frac{\partial}{\partial t}\left(\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right) = \tau_{ij}\, \dot{u}_{i,j}$$

This term can be expressed as

$$S_j = -\tau_{ij}\, \dot{u}_i$$

$$\iiint_V \tau_{ij}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right) dV = \iiint_V \frac{\partial}{\partial x_j}\left(\tau_{ij}\frac{\partial u_i}{\partial t}\right) dV - \iiint_V \frac{\partial u_i}{\partial t}\tau_{ij,j}\, dV$$

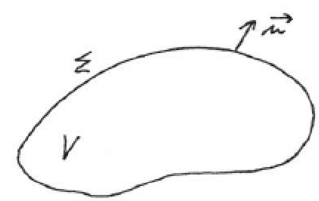
$$= -\iiint_V \frac{\partial S_j}{\partial x_j} dV - \iiint_V \frac{\partial u_i}{\partial t}\left(\rho\frac{\partial^2 u_i}{\partial t^2} - f_i\right) dV$$

Thus

$$\frac{d\epsilon}{dt} + \iiint_V \operatorname{div} \vec{S}\, dV = \iiint_V \frac{\partial u_i}{\partial t} f_i\, dV \qquad (= 0 \quad \text{for} \quad f_i = 0)$$

From the Gauss theorem we get

$$\frac{d\epsilon}{dt} + \iiint_V \operatorname{div} \vec{S}\, dV = 0 \Leftrightarrow \frac{d\epsilon}{dt} + \iint_\Sigma S_j n_j\, d\Sigma = 0$$



That means that the change of elastic energy is compensated by a flux of the vector ***S*** (energy flux, analogous to the Poynting vector) across the surface Σ.

Direction of ***S*** … direction in which energy flows.
Magnitude of ***S*** … amount of energy passed through a unit area perpendicular to ***S*** within a time unit.

Energy of plane waves in acoustic media

Real parts of the complex solutions for pressure and particle velocity can be written as (we consider p_i real)

$$p = \frac{1}{2}(PF + P^*F^*)\,, \quad v_i = \frac{1}{2}\frac{p_i}{\rho}(PF + P^*F^*) \quad .$$

Keeping in mind that in the acoustic case, $p = -k\theta\,, \quad \tau_{ij} = -p\delta_{ij} \quad ,$

↳ $u_{i,i}$

we can write

$$\begin{aligned}
W &= \frac{1}{2}\tau_{ij}u_{i,j} = -\frac{1}{2}pu_{i,i} = \frac{1}{2k}p^2 = \frac{1}{2}\kappa p^2 = \frac{1}{8}\kappa(PF + P^*F^*)^2\,, \\
K &= \frac{1}{2}\rho\dot{u}_i\dot{u}_i = \frac{1}{2}\rho v_i v_i = \frac{1}{8}(\rho c^2)^{-1}(PF + P^*F^*)^2\,, \\
S_i &= -\tau_{ij}\dot{u}_j = p\delta_{ij}v_j = pv_i = \frac{1}{4}p_i\rho^{-1}(PF + P^*F^*)^2\,,
\end{aligned}$$

Since $c^2 = (\kappa\rho)^{-1}$,

$$\boxed{W = K}\,,$$

i.e., **the strain energy of a plane acoustic wave equals at any time to its kinetic energy.**

The total energy is

$$E = \frac{1}{4}(\rho c^2)^{-1}(PF + P^*F^*)^2 = \frac{1}{4}\kappa(PF + P^*F^*)^2 \quad ,$$

which yields the group velocity as

$$\boxed{v_i^{(g)} = S_i/E = c^2 p_i = cN_i = c_i}\,,$$

i.e., in acoustic media, the phase and group velocities coincide in direction and magnitude (energy flows perpendicularly to the wavefront).

Energy of plane waves in isotropic media

We proceed in analogy to the case of the acoustic medium, but separately for P and S waves. Real parts of the solution are

$$u_i = \frac{1}{2}(AF + A^*F^*)N_i$$ for the P wave and $$u_i = \frac{1}{2}[(BF + B^*F^*)g_i^{(1)} + (CF + C^*F^*)g_i^{(2)}]$$ for the S wave. Energy quantities are

$$W_P = \frac{1}{8}\rho(AF' + A^*F^{*'})^2, \qquad W_S = \frac{1}{8}\rho[(BF' + B^*F^{*'})^2 + (CF' + C^*F^{*'})^2],$$

$$K_P = \frac{1}{8}\rho(AF' + A^*F^{*'})^2, \qquad K_S = \frac{1}{8}\rho[(BF' + B^*F^{*'})^2 + (CF' + C^*F^{*'})^2],$$

$$S_{Pi} = \frac{1}{4}\rho\alpha N_i(AF' + A^*F^{*'})^2, \qquad S_{Si} = \frac{1}{4}\rho\beta N_i[(BF' + B^*F^{*'})^2 + (CF' + C^*F^{*'})^2],$$

$$E_P = \frac{1}{4}\rho(AF' + A^*F^{*'})^2, \qquad K_S = \frac{1}{4}\rho[(BF' + B^*F^{*'})^2 + (CF' + C^*F^{*'})^2],$$

from which

$$W = K$$

In isotropic media, **the strain energy of a plane wave equals at any time to its kinetic energy.**

and

$$v_{Pi}^{(g)} = S_{Pi}/E_P = \alpha N_i, \quad v_{Si}^{(g)} = S_{Si}/E_S = \beta N_i.$$

In isotropic media, the **phase and group velocities coincide in direction and magnitude** (energy flows perpendicularly to the wavefront).

Energy of plane waves in anisotropic media

We proceed in analogy to the case of the acoustic or isotropic medium. $u_i = \frac{1}{2}(U_i F + U_i^* F^*) = \frac{1}{2}(AF + A^* F^*)g_i$

Taking into account that

$$u_{i,j} = -\tfrac{1}{2}(U_i p_j F' + U_i^* p_j F^{*\prime}),$$

$$e_{ij} = -\tfrac{1}{4}(U_i p_j + U_j p_i)F' - \tfrac{1}{4}(U_i^* p_j + U_j^* p_i)F^*$$

$$\tau_{ij} = -\tfrac{1}{2} c_{ijkl} p_l (U_k F' + U_k^* F^{*\prime})$$

Notation for c_{ijkl}/ρ

we get

$$W = \frac{1}{2}\rho a_{ijkl} u_{i,j} u_{k,l} = \frac{1}{8}\rho a_{ijkl} C^{-2}(U_i N_j F' + U_i^* N_j F^{*\prime})(U_k N_l F' + U_k^* N_l F^{*\prime})$$

$$= \frac{1}{8}\rho C^{-2}\Gamma_{ik}(AF' + A^* F^{*\prime})^2 g_i g_k,$$

$$K = \frac{1}{2}\rho \dot{u}_i \dot{u}_i = \frac{1}{8}\rho (AF' + A^* F^{*\prime})^2,$$

$$S_i = -\rho a_{ijkl} u_{k,l} \dot{u}_j = \frac{1}{4}\rho a_{ijkl} C^{-1} N_l (AF' + A^* F^{*\prime})^2 g_j g_k.$$

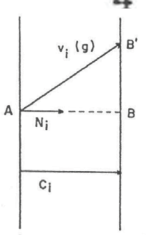
If we consider the Christoffel equation, multiplied by the vector g_i, we get

$$\overline{\Gamma}_{ik} g_i g_k - C^2 = 0 \Rightarrow C^{-2}\overline{\Gamma}_{ik} g_i g_k = 1 \Rightarrow \boxed{W = K}$$

In anisotropic media, **the strain energy of a plane wave equals at any time to its kinetic energy.**

The total energy and the group velocity are

$$E = \frac{1}{4}\rho (AF' + A^* F^{*\prime})^2 \quad \text{and} \quad v_i^{(g)} = S_i / E = C^{-1} a_{ijkl} N_l g_j g_k = a_{ijkl} p_l g_j g_k \Rightarrow \boxed{\vec{v}^{\,g} \neq \vec{c}}$$



It can be shown that

$$v_i^{(g)} p_i = 1 \Rightarrow v_i^{(g)} N_i = c \Rightarrow |\vec{v}^{\,g}| \geq |\vec{c}|$$

phase and group velocities differ from each other (energy flux is not perpendicular to the wavefront)

Plane waves in acoustic, isotropic and anisotropic media – basic differences

ANISOTROPIC	ISOTROPIC	ACOUTIC
3 independent waves in a given direction ***N*** qP, qS1, qS2 (S-wave splitting)	2 independent waves in a given direction ***N*** P, S	1 wave in a given direction ***N*** pressure, **"**P" (particle velocity)
S-wave singularities Phase velocities of qS1 and qS2 are equal in certain directions	No singularities	No singularities
All waves linearly polarized in a general direction not connected with the wavefront or group velocity direction	P wave linearly polarized perpendicularly to the wavefront, S wave elliptically polarized in the plane of the wavefront (linear polarization only in special cases)	P wave linearly polarized
v^g, $\mathcal{C}$ vary with the direction of ***N***	v^g, $\mathcal{C}$ independent on the direction of ***N***	v^g, $\mathcal{C}$ independent on the direction of ***N***
$\vec{v}^g \neq \vec{\mathcal{C}}, \quad \vec{\mathcal{C}} = \mathcal{C}\vec{N}$ $\lvert\vec{v}^g\rvert \geq \lvert\vec{\mathcal{C}}\rvert$	$\vec{v}^g = \vec{\mathcal{C}}$	$\vec{v}^g = \vec{\mathcal{C}}$
Energy flux **is not** perpendicular to the wavefront	Energy flux **is** perpendicular to the wavefront	Energy flux **is** perpendicular to the wavefront

Plane waves in acoustic, isotropic and anisotropic media – basic differences

ANISOTROPIC

3 independent waves
in a given direction ***N***
qP, qS1, qS2
(S-wave splitting)

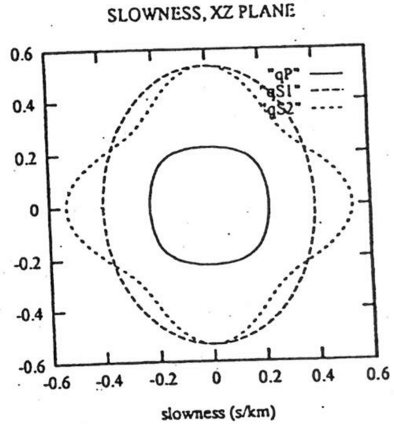
S-wave singularities
Phase velocities of qS1 and qS2
are equal in certain directions

All waves linearly polarized
in a general direction not
connected with the wavefront
or group velocity direction

v^g, $\mathcal{C}$ vary with the direction
of ***N***

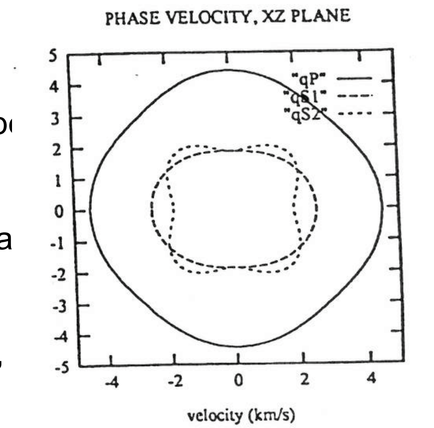
$$\vec{v}^g \neq \vec{\mathcal{C}}, \quad \vec{\mathcal{C}} = \mathcal{C}\vec{N}$$
$$|\vec{v}^g| \geq |\vec{\mathcal{C}}|$$

Energy flux **is not**
perpendicular to the wavefront



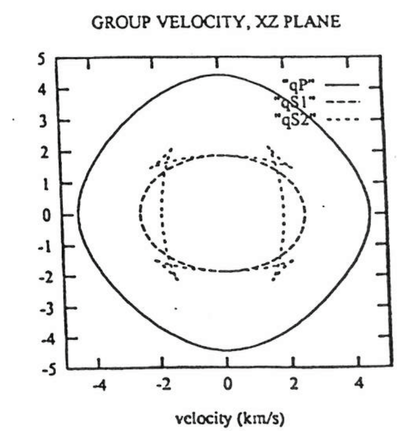
p

(linea



ses)

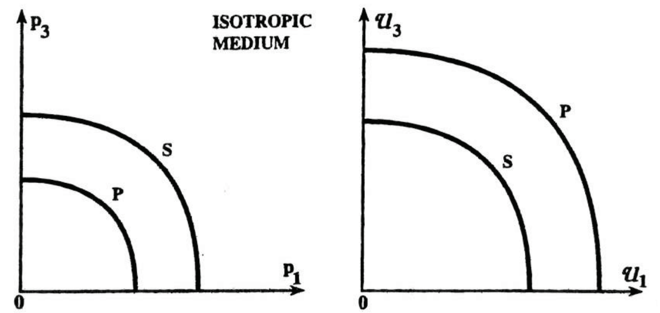
v^g,



t

ACOUTIC

1 wave
in a given direction ***N***
pressure, "P" (particle velocity)



v^g, $\mathcal{C}$ independent on the direction
of ***N***

$$\vec{v}^g = \vec{\mathcal{C}}$$

Energy flux **is**
perpendicular to the wavefront

Plane waves in acoustic, isotropic and anisotropic media – basic differences

ANISOTROPIC	ISOTROPIC	ACOUTIC
3 independent waves in a given direction ***N*** qP, qS1, qS2 (S-wave splitting)	2 independent waves in a given direction ***N*** P, S	1 wave in a given direction ***N*** pressure, **"**P" (particle velocity)
S-wave singularities Phase velocities of qS1 and qS2 are equal in certain directions	No singularities	No singularities
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v^g, $\mathcal{C}$ vary with the direction of ***N***	v^g, $\mathcal{C}$ independent on the direction of ***N***	v^g, $\mathcal{C}$ independent on the direction of ***N***
$\vec{v}^g \neq \vec{\mathcal{C}}, \quad \vec{\mathcal{C}} = \mathcal{C}\vec{N}$ $\lvert\vec{v}^g\rvert \geq \lvert\vec{\mathcal{C}}\rvert$	$\vec{v}^g = \vec{\mathcal{C}}$	$\vec{v}^g = \vec{\mathcal{C}}$
Energy flux **is not** perpendicular to the wavefront	Energy flux **is** perpendicular to the wavefront	Energy flux **is** perpendicular to the wavefront

Inhomogeneous waves

$$u_i = U_i \exp[-\mathrm{i}\omega(t - p_i x_i)]$$

real-valued ….. homogeneous waves; complex-valued …. inhomogeneous (evanescent) waves

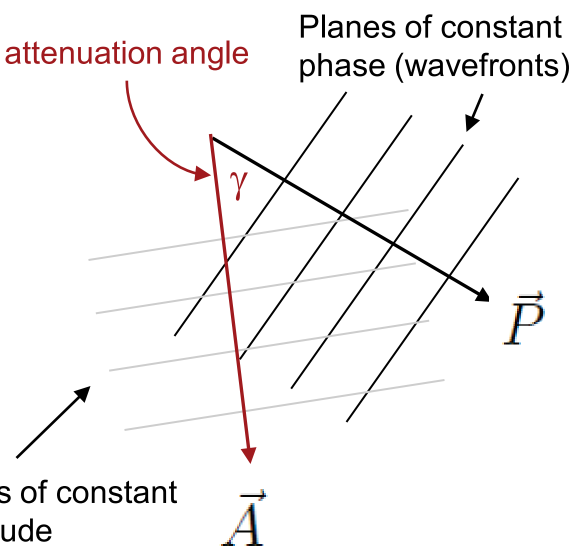
$$p_i = P_i + iA_i$$

P_i → Propagation vector

A_i → Atenuation vector

attenuation

$$u_i(\mathbf{x}, t) = U_i \exp(-\omega A_i x_i) \exp[-\mathrm{i}\omega(t - P_i x_i)]$$



Amplitude is no more constant along the wavefront

The existence condition in isotropic (acoustic) media:

$$p_i p_i = \frac{1}{\mathcal{C}^2}$$

$$P_i P_i - A_i A_i + 2\mathrm{i} A_i P_i = \frac{1}{\mathcal{C}^2}$$

→ real-valued in a non-dissipative medium

$$A_i P_i = 0 \Rightarrow \vec{P} \perp \vec{A}, \quad \gamma = \pm 90^\circ$$

$$\frac{1}{\mathcal{C}_{inh}^2} = P_i P_i = \frac{1}{\mathcal{C}^2} + A_i A_i > \frac{1}{\mathcal{C}^2}$$

$$\mathcal{C}_{inh} < \mathcal{C}$$

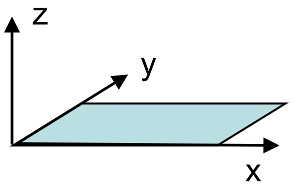
The inhomogeneous wave is always slower than the homogeneous one

Elastic waves at interfaces – boundary conditions (different for different interface types)

SOLID-SOLID (welded contact)

Continuous displacement $\mathbf{u}^{above} = \mathbf{u}^{below}$

Continuous traction $\mathbf{T}^{above} = \mathbf{T}^{below}$



$$[u_x] = [u_y] = [u_z] = 0$$

$$[T_x] = [T_y] = [T_z] = 0$$

$$\Downarrow$$

$$[\tau_{zx}] = [\tau_{zy}] = [\tau_{zz}] = 0$$

SOLID-FLUID (no cavitation)

$$[u_z] = 0$$

$$[T_z] = 0 \Rightarrow [\tau_{zz}] = 0$$

$$T_x = T_y = 0 \Rightarrow \tau_{zx} = \tau_{zy} = 0$$

FLUID-FLUID

$$[u_z] = 0$$

$$[T_z] = 0 \Rightarrow [\tau_{zz}] = 0$$

$$T_x = T_y = 0 \Rightarrow \tau_{zx} = \tau_{zy} = 0$$

$$[v_z] = 0$$

$$[p] = 0$$

VACUUM-SOLID (free surface)

$$T_x = T_y = T_z = 0 \Rightarrow \tau_{zx} = \tau_{zy} = \tau_{zz} = 0$$

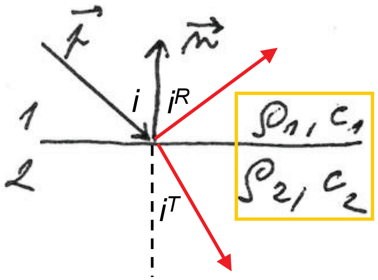
no condition for displacement

VACUUM-FLUID (free surface)

$$T_z = 0 \Rightarrow \tau_{zz} = 0$$

For the acoustic case $p = 0$

Plane waves at a plane interface – acoustic halfspaces



Incident wave (homogeneous)

$$p(\vec{x},t) = P\,F(t - p_z x_z)\;;\quad v_i(\vec{x},t) = \frac{1}{\rho} P\, p_i\, F(t - p_z x_z)$$

$$p_z = \frac{N_z}{c_1}$$

Generated waves G — R … reflected; T… transmitted

In the acoustic case, only one reflected and one transmitted wave can exist

$$p^G(\vec{x},t) = P^G F^G(t - p_z^G x_z - \varphi^G)\;;\quad v_i^G = \frac{1}{\rho_G} P^G p_i^G F^G(t - p_z^G x_z - \varphi^G)$$

$$p_z^R = \frac{N_z^R}{c_1}\,,\quad p_z^T = \frac{N_z^T}{c_2}\;;\quad \rho_G = \begin{cases} \rho_1 \;\ldots\; G = R \\ \rho_2 \;\ldots\; G = T \end{cases}$$

Material parameters (known)

Known quantities (incident wave)

Unknown quantities (reflected and transmitted wave)

Plane waves at a plane interface – acoustic halfspaces (continuation)

Boundary conditions

$$P F(t - p_k x_k) + P^R F^R(t - p_k^R x_k - \varphi^R) = P^T F^T(t - p_k^T x_k - \varphi^T)$$

$$\frac{1}{\rho_1} P \dot{F}(t - p_k x_k) p_l x_l + \frac{1}{\rho_1} P^R \dot{F}^R(t - p_k^R x_k^R - \varphi^R) p_l^R n_l = \frac{1}{\rho_2} P^T \dot{F}^T(t - p_k^T x_k - \varphi^T) p_l^T n_l$$

hold for any ***x*** and *t*. P, P^R, P^T do not depend on ***x*** and *t* (plane waves). Thus we can conclude:

1) $F(t - p_k x_k) = F^R(t - p_k^R x_k^R - \phi^R) = F^T(t - p_k^T x_k^T - \phi^T)$

i.e., the same analytical signal for all the waves involved in the R/T problem.

$$p_k x_k = p_k^R x_k^R - \phi^R = p_k^T x_k^T - \phi^T$$

Taking into account two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ on the interface, we get

$$p_k(x_{1k} - x_{2k}) = p_k^R(x_{1k} - x_{2k}) = p_k^T(x_{1k} - x_{2k})$$

2) **Tangential components** of $\mathbf{p}$, $\mathbf{p}^R$ and $\mathbf{p}^T$ **are the same.** → $\dfrac{\sin i}{c_1} = \dfrac{\sin i^G}{c_G}$ (Snell's law)

$\Downarrow$

$$p_k x_k = p_k^R x_k = p_k^T x_k$$

$\Downarrow$

3) $\phi^R = \phi^T = 0$ … **no phase shift at the interface**

$\Downarrow$

$\boldsymbol{p}^R$ and $\boldsymbol{p}^T$ remains in the plane of incidence determined by $\boldsymbol{p}$ and $\boldsymbol{n}$.

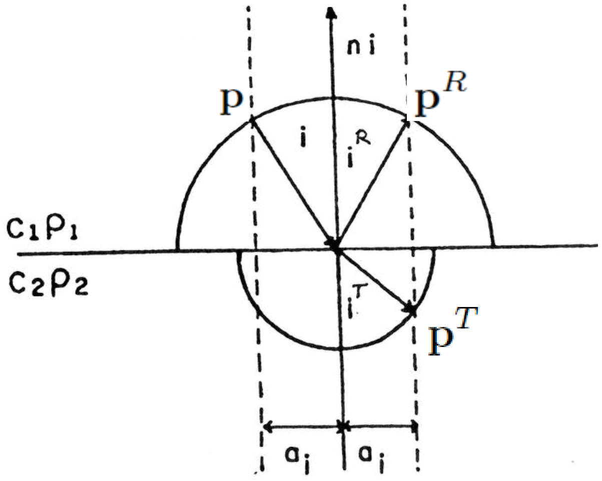
Plane waves at a plane interface – acoustic halfspaces (continuation)

Transformation of slowness across the interface:

Analyzing the boundary conditions we have found that tangential components of $\mathbf{p}$, $\mathbf{p}^R$ and $\mathbf{p}^T$ are the same.

Incident wave (known) *reflected* *transmitted*

What remains (in kinematics of the problem) is to determine the normal components of $\mathbf{p}^R$ and $\mathbf{p}^T$. To do this we use the **existence condition** (constraint for the magnitude of the slowness vectors)



$$\mathbf{p} = \mathbf{a} + \sigma \mathbf{n} \qquad \mathbf{p}^G = \mathbf{a} + \sigma^G \mathbf{n}$$

$$\sigma = \mathbf{p}\mathbf{n}$$
Incident wave (known)

$$\sigma^G = \mathbf{p}^G \mathbf{n}$$
we seek for

$$p_i p_i = a_i a_i + (p_m n_m)^2 = \frac{1}{c_1^2}$$

$$p_i^T p_i^T = a_i a_i + (p_m^T n_m)^2 = \frac{1}{c_2^2}$$

$\Downarrow$

$$p_m^T n_m = \pm\sqrt{\frac{1}{c_2^2} - a_i a_i} = \pm\sqrt{\frac{1}{c_2^2} - \frac{1}{c_1^2} + (p_m n_m)^2}$$

$\Downarrow$

Similarly, for the reflected wave we get ($c_G = c_1$)

$$p_k^T = p_k - \left[p_m n_m + \sqrt{\frac{1}{c_2^2} - \frac{1}{c_1^2} + (p_m n_m)^2} \right] n_k$$

Transmitted wave

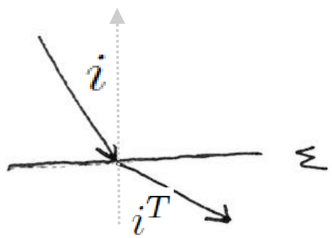
$$p_k^R = p_k - 2(p_m n_m) n_k$$

Reflected wave

Plane waves at a plane interface – acoustic halfspaces (continuation)

$$p_k^T = p_k - \left[p_m n_m + \sqrt{\frac{1}{c_2^2} - \frac{1}{c_1^2} + (p_m n_m)^2} \right] n_k$$

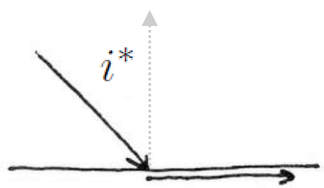
may be negative if $c_2 > c_1$ ⟹ complex-valued slowness ⟹ **inhomogeneous wave**



homogeneous

$$\frac{c_2}{c_1} \sin i = \sin i^T \leq 1$$

p^T real-valued
no attenuation



homogeneous

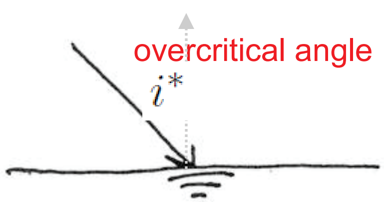
$$\sin i^* = \frac{c_1}{c_2}$$

$$i^* = \arcsin\left(\frac{c_1}{c_2}\right), \quad i^T = 90^\circ$$

critical angle

p^T real-valued
no attenuation

overcritical angle



inhomogeneous

$$\sin i > \frac{c_1}{c_2}$$

p^T complex-valued

Amplitude decays away from the interface

$$\frac{1}{C_{inh}^2} = P_i P_i = a_i a_i = \frac{\sin^2 i}{c_1^2} \Rightarrow C_{inh} = \frac{c_1}{\sin i}$$

decays with increasing angle of incidence

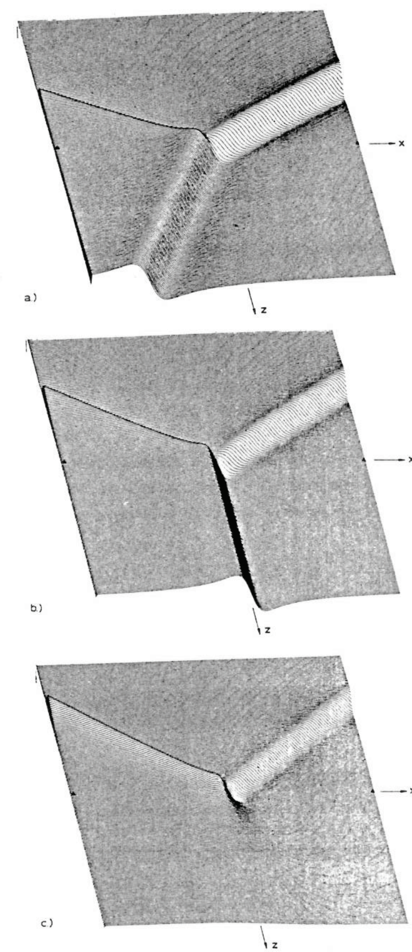


Fig. 2.2: Transient plane waves at a planar acoustic interface:
(a) Subcritical incidence
(b) Critical incidence
(c) Supercritical incidence.

Plane waves at a plane interface – acoustic halfspaces (continuation)

Transformation of amplitude across the interface:

Boundary conditions (divided by *F*) yield

$$P + P^R = P^T$$

$$\rho_1^{-1} P p_l n_l + \rho_1^{-1} P^R \underbrace{p_l^R n_l}_{= -p_l n_l} = \rho_2^{-1} p_l^T n_l P^T$$

Introducing the R/T coefficients for pressure

$$R^R = \frac{P^R}{P} \quad \text{and} \quad R^T = \frac{P^T}{P}$$

we get

$$R^R - R^T = -1$$

$$\rho_1^{-1} p_l n_l R^R + \rho_2^{-1} p_l^T n_l R^T = \rho_1^{-1} p_l n_l$$

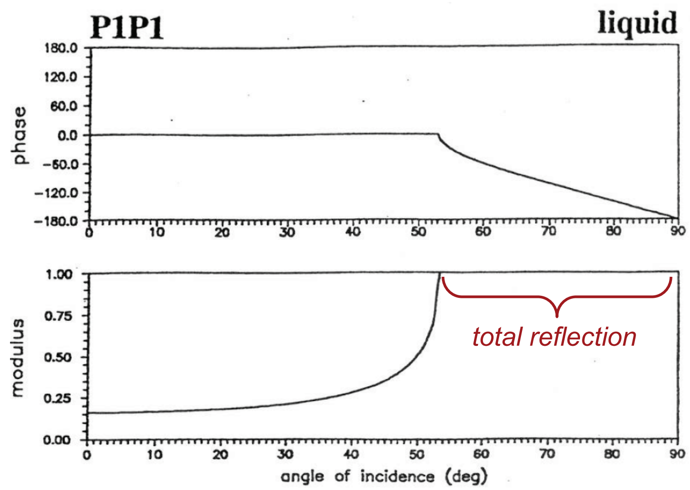
This set of equations has the analytical solution

$$R^R = \frac{\rho_2 p_l n_l - \rho_1 p_l^T n_l}{\rho_2 p_l n_l + \rho_1 p_l^T n_l} = \frac{\rho_2 c_2 \cos i - \rho_1 c_1 \cos i^T}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i^T}$$

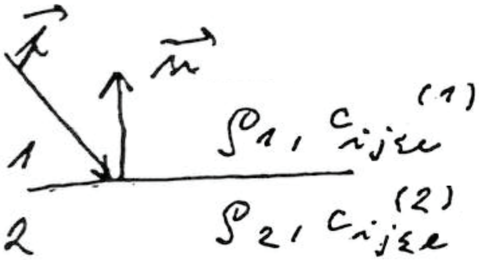
$$R^T = \frac{2 \rho_2 p_l n_l}{\rho_2 p_l n_l + \rho_1 p_l^T n_l} = \frac{2 \rho_2 c_2 \cos i}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i^T}$$

Example:

c_1=2000 m/s, c_2=2500 m/s (c_2>c_1)
ρ_1=1500 kg/m^3, ρ_2=1661 kg/m^3



Plane waves at a plane interface – anisotropic halfspaces



Incident wave (known) – 3 possibilities (qP,qS1 or qS2)

$$u_i(x_m,t) = A g_i F(t - p_k x_k)$$

Generated waves (unknown) – 6 possibilities (qP,qS1 or qS2 reflected and trasmitted)

$$u_i^{(j)}(x_m,t) = A^{(j)} g_i^{(j)} F^{(j)}(t - p_k^{(j)} x_k - \varphi^{(j)}), \quad j = \underbrace{1, 2, 3}_{\text{reflection}}, \underbrace{4, 5, 6}_{\text{transmission}}$$

Boundary conditions require: $[u_i] = 0 \; ; \; [T_i] = 0$

$$T_i = \tau_{ij} n_j = c_{ijkl} n_j u_{k,l}$$

e.g., for the incident wave $T_i = -A c_{ijkl}^{(1)} n_j g_k p_l \dot{F}(t - p_m x_m)$

$$-A^{(1)} g_i^{(1)} F^{(1)} - A^{(2)} g_i^{(2)} F^{(2)} - A^{(3)} g_i^{(3)} F^{(3)} + A^{(4)} g_i^{(4)} F^{(4)} + A^{(5)} g_i^{(5)} F^{(5)} + A^{(6)} g_i^{(6)} F^{(6)} = A g_i F$$

$$-A^{(1)} X_i^{(1)} \dot{F}^{(1)} - A^{(2)} X_i^{(2)} \dot{F}^{(2)} - A^{(3)} X_i^{(3)} \dot{F}^{(3)} + A^{(4)} X_i^{(4)} \dot{F}^{(4)} + A^{(5)} X_i^{(5)} \dot{F}^{(5)} + A^{(6)} X_i^{(6)} \dot{F}^{(6)} = A X_i \dot{F}$$

$$X_i = c_{ijkl} n_j g_k p_l$$, etc.

BC must not depend on position and time. Thus (similarly to the acoustic case):

1) Analytical signals of all the involved waves are the same
2) Tangential components of slowness vectors of all the waves are the same .
3) Slowness vectors of all the waves are situated in the plane of incidence.
4) There is no phase shift at the interface.

Plane waves at a plane interface – anisotropic halfspaces (continuation)

The slowness vector of any of the generated wave is: $p_i^{(j)} = a_i + \sigma^{(j)} n_i$

- a_i: tangential comp. (known)
- $\sigma^{(j)} n_i$: normal comp. (unknown) to be determined from the existence condition

$$\det(\Gamma_{ik} - \delta_{ik}) = \det\left(\frac{c_{ijkl}}{\rho} p_j^{(j)} p_l^{(j)} - \delta_{ik}\right) = 0$$

⬇

$$\det\left[\frac{c_{ijkl}}{\rho}(a_j + \sigma^{(j)} n_j)(a_l + \sigma^{(j)} n_l) - \delta_i k\right] = 0$$

$\frac{c_{ijkl}}{\rho}$: different for halfspace 1 and halfspace 2

… a six-order equation for σ – solved numerically;
it has six roots but we need only three – we have to exclude three of them

Selection criteria

1) For complex-valued roots (inhomogeneous waves) – **the wave must decay away from the interface**

$$Im(\sigma) n_m x_m > 0$$

$n_m x_m$: differs in sign for reflection and and transmission

2) For real-valued roots (homogeneous waves) we decide according to the energy flux direction, i.e., **energy must flow into the right halfspace** (1 for reflection and 2 for transmission)

$$n_i v_i^g = \begin{cases} n_i \dfrac{c_{ijkl}^{(1)}}{\rho_1} p_l g_j g_k \le 0 & \text{… transmission} \\ n_i \dfrac{c_{ijkl}^{(2)}}{\rho_2} p_l g_j g_k \ge 0 & \text{… reflection} \end{cases}$$

$p_l g_j g_k$: all these quantities are found numerically

Plane waves at a plane interface – anisotropic halfspaces (continuation)

Dynamics of the R/T problem – how to find amplitudes of the generated waves? The scalar amplitude coefficients are found from BC.

Divided by F (the same for all the waves), BC lead to the set of equations

$$-A^{(1)}g_i^{(1)} - A^{(2)}g_i^{(2)} - A^{(3)}g_i^{(3)} + A^{(4)}g_i^{(4)} + A^{(5)}g_i^{(5)} + A^{(6)}g_i^{(6)} = Ag_i ,$$

$$-A^{(1)}X_i^{(1)} - A^{(2)}X_i^{(2)} - A^{(3)}X_i^{(3)} + A^{(4)}X_i^{(4)} + A^{(5)}X_i^{(5)} + A^{(6)}X_i^{(6)} = AX_i$$

($X_i = c_{ijkl} n_j g_k p_l$ etc.)

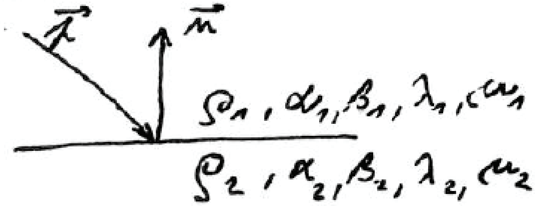
This is a set of six algebraic equations for six unknowns $A^{(j)}$ (solved numerically). It can be solved directly for the amplitude coefficients or we can introduce R/T coefficients and solve the equations for them.

$$R_{mn} = \frac{A^{(m)}}{A}, \quad m = 1,2,3,4,5,6 \quad \text{… generated waves}$$

$$n = 1,2,3 \quad \text{… incident waves}$$

Altogether we have 18 R/T coefficients, 9 for reflection and 9 for transmission. They are usually arranged into two 3x3 matrices of the R/T coefficients.

Plane waves at a plane interface – isotropic halfspaces



The incident wave is either P $u_i(x_m,t) = AN_iF(t - p_kx_k)$

or S $u_i(x_m,t) = (Bg_i^{(1)} + Cg_i^{(2)})F(t - p_kx_k)$.

From each of them we can have, in general, **four generated waves** which are to be determined.

BC (two solid halfspaces in welded contact):

$$[u_i] = 0 \;,\; [T_i] = 0 \;;\; T_i = \tau_{ij}n_j = \lambda n_i u_{k,k} + \mu n_j(u_{i,j} + u_{j,i})$$

P waves… $T_i = -A(\lambda_1 n_i N_k p_k + 2\mu_1 n_j p_j N_i)\dot{F}(t - p_kx_k)$

S waves… $T_i = -[B(g_i^{(1)}p_j + g_j^{(1)}p_i) + C(g_i^{(2)}p_j + g_j^{(2)}p_i)]\mu_1 n_j \dot{F}(t - p_kx_k)$

BC yield

1) Analytical signals of all the involved waves are the same
2) Tangential components of slowness vectors of all the waves are the same . →
3) Slowness vectors of all the waves are situated in the plane of incidence.
4) There is no phase shift at the interface.

$$\frac{\sin i_P^R}{\alpha_1} = \frac{\sin i_S^R}{\beta_1} = \frac{\sin i_P^T}{\alpha_2} = \frac{\sin i_S^T}{\beta_2} = \frac{\sin i}{V} \;;\; V = \begin{cases}\alpha_1 \\ \beta_1\end{cases}$$

Snell's law
(can be used to find the direction in which the waves leaves the interface)

The magnitude of the slowness vector can be determined employing the existence condition

$$p_i^G p_i^G = \frac{1}{\tilde{V}^2}, \quad \tilde{V} = \alpha_1, \beta_1, \alpha_2, \beta_2 .$$

The slowness is then

$$p_i^G = p_i - \{(p_m n_m) \pm [\underbrace{\tilde{V}^{-2} - V^{-2} + (p_m n_m)^2}]^{1/2}\} n_i .$$

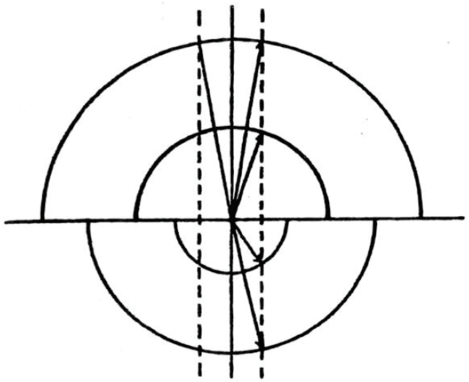
can be negative; even three inhomogeneous waves may exist simultaneously

Specifically, for monotypic reflection

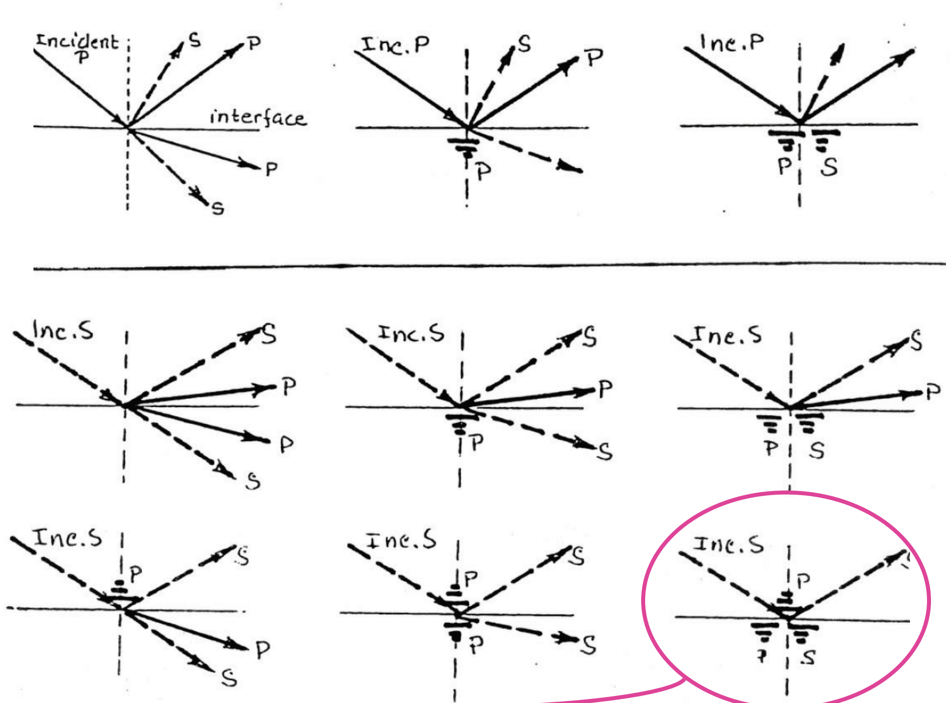
$$\tilde{V} = V \Rightarrow p_k^R = p_k - 2(p_m n_m)n_k$$ → always homogeneous

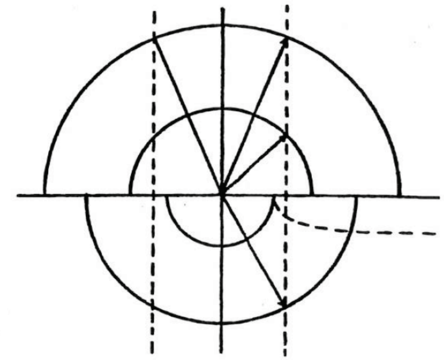
Plane waves at a plane interface – isotropic halfspaces (continuation)

In isotropic media, many jinds of inhomogeneous waves can exist:

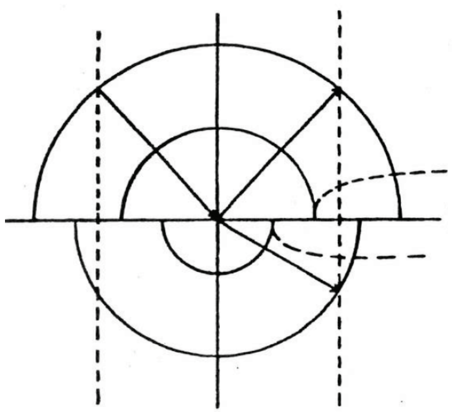


$\sin i < \beta_1/\alpha_2$





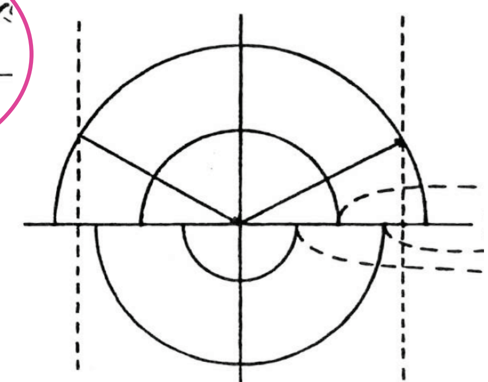
$\sin i > \beta_1/\alpha_2, \quad \sin i < \beta_1/\alpha_1$



$\sin i > \beta_1/\alpha_1, \quad \sin i < \beta_1/\beta_2$

Necessary conditions for this case:

1) Incident S wave
2) $\beta_1 < \alpha_1$, $\beta_1 < \beta_2$, $\beta_1 < \alpha_2$



$\sin i > \beta_1/\beta_2$

Plane waves at a plane interface – isotropic halfspaces (continuation)

Dynamics of the R/T problem – how to find amplitudes of the generated waves? The scalar amplitude coefficients are found from BC.

Divided by F (the same for all the waves), BC lead to the set of equations

$$A^t N_i^t + B^t g_i^{(1)t} + C^t g_i^{(2)t} - A^r N_i^r - B^r g_i^{(1)r} - C^r g_i^{(2)r} = \tilde{D}_i,$$

$$A^t X_i^t + B^t Y_i^t + C^t Z_i^t - A^r X_i^r - B^r Y_i^r - C^r Z_i^r = \tilde{E}_i,$$

$$X_i = \lambda n_i N_k p_k + 2\mu n_j p_j N_i$$

$$Y_i = \mu n_j (g_i^{(1)} p_j + g_j^{(1)} p_i)$$

$$Z_i = \mu n_j (g_i^{(2)} p_j + g_j^{(2)} p_i)$$

Differ according to the type of the incident wave

$$\tilde{D}_i = A N_i \quad , \quad \tilde{E}_i = A X_i$$

$$\tilde{D}_i = B g_i^{(1)} + C g_i^{(2)} \quad , \quad \tilde{E}_i = B Y_i + C Z_i$$

This is a set of six algebraic equations for six unknowns A^r, B^r, C^r, A^t, B^t and C^t. Let us introduce the R/T coefficients R^r_{mn}, R^t_{mn}

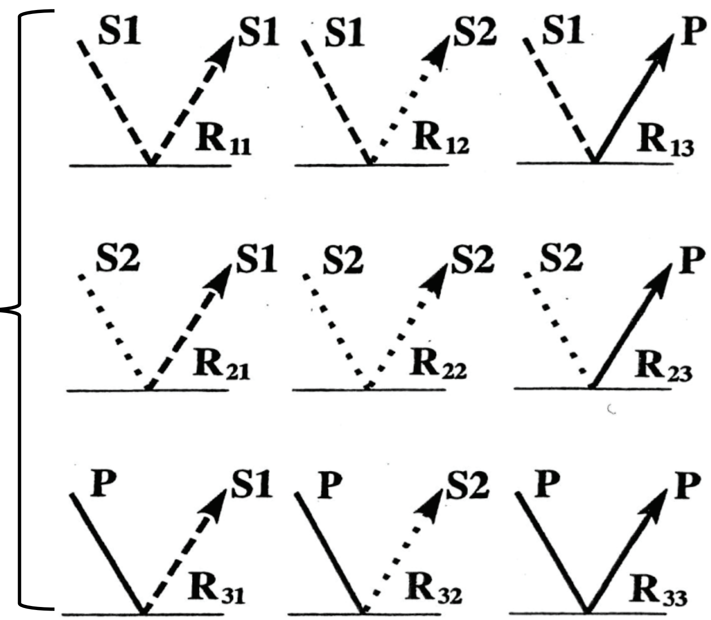
generated w. (m) incident w. (n)

$m, n = 1 \dots S1$
$ 2 \dots S2$
$ 3 \dots P$

(e.g., $R^r_{11} = B^r/B$, $R^r_{31} = B^r/A \dots$)

Altogether we have 9 for reflection and 9 for transmission, e.g.,

$$\boldsymbol{R}^r = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$



Plane waves at a plane interface – isotropic halfspaces (continuation)

In isotropic media, the dynamics of the R/T problem can be simplified if we consider

$\vec{g}^{(2)}, \vec{g}^{(2)r}; \vec{g}^{(2)t} \perp$ plane of incidence. Then:

$\vec{n} \equiv (0,0,1)$, $\vec{N} \equiv (N_1, 0, N_3)$, $\vec{g}^{(1)} \equiv (g_1^{(1)}, 0, g_3^{(1)})$, $\vec{g}^{(2)} \equiv (0,1,0)$ …, etc., and BC

separate into **two independent sets** of equations

P-SV part

$$A^t N_1^t + B^t g_1^{(1)t} - A^r N_1^r - B^r g_1^{(1)r} = \tilde{D}_1 \, ,$$
$$A^t N_3^t + B^t g_3^{(1)t} - A^r N_3^r - B^r g_3^{(1)r} = \tilde{D}_3 \, ,$$
$$A^t X_1^t + B^t Y_1^t - A^r X_1^r - B^r Y_1^r = \tilde{E}_1 \, ,$$
$$A^t X_3^t + B^t Y_3^t - A^r X_3^r - B^r Y_3^r = \tilde{E}_3 \, .$$

SH part

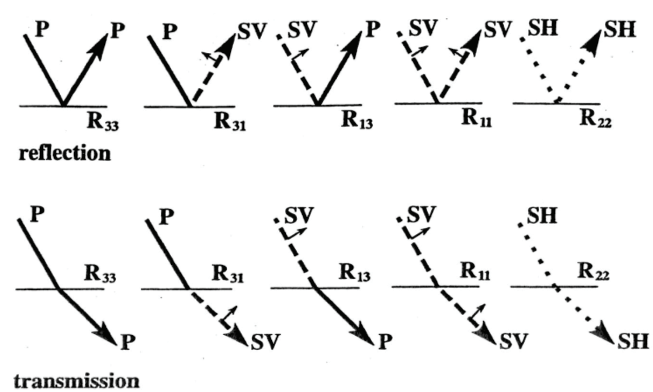
$$C^t - C^r = \tilde{D}_2 \quad ,$$
$$C^t Z_2^t - C^r Z_2^r = \tilde{E}_2$$

(fully analogous to the acoustic case)

$$X_1 = 2\mu p_3 N_1 \, , \qquad X_2 = 0 \, , \qquad X_3 = \lambda(p_1 N_1 + p_3 N_3) + 2\mu p_3 N_3 \, ,$$
$$Y_1 = \mu(g_1^{(1)} p_3 + g_3^{(1)} p_1) \, , \qquad Y_2 = 0 \, , \qquad Y_3 = 2\mu g_3^{(1)} p_3 \, ,$$
$$Z_1 = 0 \, , \qquad Z_2 = \mu p_3 \, , \qquad Z_3 = 0 \, .$$

Thus, we have **only 5 non-zero coefficients for reflection and 5 for transmission**. Analytical formulas can be found for them.

$$\mathbf{R}^r = \begin{pmatrix} R_{11} & 0 & R_{13} \\ 0 & R_{22} & 0 \\ R_{31} & 0 & R_{33} \end{pmatrix}$$



Plane waves at a plane interface – isotropic halfspaces (continuation)

REFLECTION

$$
\begin{aligned}
R_{11} &= D^{-1}[q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(\alpha_1\beta_2P_2P_3 - \beta_1\alpha_2P_1P_4) \\
&- \alpha_1\beta_1P_3P_4Y^2 + \alpha_2\beta_2P_1P_2X^2 - \alpha_1\alpha_2\beta_1\beta_2p^2Z^2] \ , \\
R_{13} &= -2\epsilon\beta_1pP_2D^{-1}(qP_3P_4Y + \alpha_2\beta_2XZ) \ , \\
R_{31} &= 2\epsilon\alpha_1pP_1D^{-1}(qP_3P_4Y + \alpha_2\beta_2XZ) \ , \\
R_{33} &= D^{-1}[q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(\beta_1\alpha_2P_1P_4 - \alpha_1\beta_2P_2P_3) \\
&- \alpha_1\beta_1P_3P_4Y^2 + \alpha_2\beta_2P_1P_2X^2 - \alpha_1\alpha_2\beta_1\beta_2p^2Z^2] \ , \\
R_{22} &= \bar{D}^{-1}(\rho_1\beta_1P_2 - \rho_2\beta_2P_4) \ .
\end{aligned}
$$

TRANSMISSION

$$
\begin{aligned}
R_{11} &= 2\beta_1\rho_1P_2D^{-1}(\alpha_1P_3Y + \alpha_2P_1X) \ , \\
R_{13} &= 2\epsilon\beta_1\rho_1pP_2D^{-1}(qP_1P_4 - \alpha_1\beta_2Z) \ , \\
R_{31} &= -2\epsilon\alpha_1\rho_1pP_1D^{-1}(qP_2P_3 - \beta_1\alpha_2Z) \ , \\
R_{33} &= 2\alpha_1\rho_1P_1D^{-1}(\beta_2P_2X + \beta_1P_4Y) \ , \\
R_{22} &= 2\rho_1\beta_1P_2\bar{D}^{-1} \ .
\end{aligned}
$$

$$
\begin{aligned}
D &= q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(\beta_1\alpha_2P_1P_4 + \alpha_1\beta_2P_2P_3) \\
&+ \alpha_1\beta_1P_3P_4Y^2 + \alpha_2\beta_2P_1P_2X^2 + \alpha_1\alpha_2\beta_1\beta_2p^2Z^2 \ , \\
\bar{D} &= \rho_1\beta_1P_2 + \rho_2\beta_2P_4 \ ,
\end{aligned}
$$

$$p = \frac{\sin i}{v}, \quad v = \alpha_1 \mathrm{or} \beta_1$$

$$\epsilon = \mathrm{sgn}(\mathbf{p}\cdot\mathbf{n})$$

$$
\begin{aligned}
q &= 2(\rho_2\beta_2^2 - \rho_1\beta_1^2) \ , \quad X = \rho_2 - qp^2 \ , \quad Y = \rho_1 + qp^2 \ , \quad Z = \rho_2 - \rho_1 - qp^2 \\
P_1 &= (1-\alpha_1^2p^2)^{1/2} \ , \quad P_2 = (1-\beta_1^2p^2)^{1/2} \ , \quad P_3 = (1-\alpha_2^2p^2)^{1/2} \ , \quad P_4 = (1-\beta_2^2p^2)^{1/2}
\end{aligned}
$$

Square roots P_i, $i = 1,2,3,4$, may be imaginary. … complex-valued R/T coefficients

Plane waves at a plane interface – isotropic halfspaces (continuation)

REFLECTION

$$
\begin{aligned}
R_{11} &= D^{-1}[q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(\alpha_1\beta_2P_2P_3 - \beta_1\alpha_2P_1P_4) \\
&- \alpha_1\beta_1P_3P_4Y^2 + \alpha_2\beta_2P_1P_2X^2 - \alpha_1\alpha_2\beta_1\beta_2p^2Z^2] , \\
R_{13} &= -2\epsilon\beta_1pP_2D^{-1}(qP_3P_4Y + \alpha_2\beta_2XZ) , \\
R_{31} &= 2\epsilon\alpha_1pP_1D^{-1}(qP_3P_4Y + \alpha_2\beta_2XZ) , \\
R_{33} &= D^{-1}[q^2p^2P_1P_2P_3P_4 + \rho_1\rho_2(\beta_1\alpha_2P_1P_4 - \alpha_1\beta_2P_2P_3) \\
&- \alpha_1\beta_1P_3P_4Y^2 + \alpha_2\beta_2P_1P_2X^2 - \alpha_1\alpha_2\beta_1\beta_2p^2Z^2] , \\
R_{22} &= \bar{D}^{-1}(\rho_1\beta_1P_2 - \rho_2\beta_2P_4) .
\end{aligned}
$$

TRANSMISSION

$$
\begin{aligned}
R_{11} &= 2\beta_1\rho_1P_2D^{-1}(\alpha_1P_3Y + \alpha_2P_1X) , \\
R_{13} &= 2\epsilon\beta_1\rho_1pP_2D^{-1}(qP_1P_4 - \alpha_1\beta_2Z) , \\
R_{31} &= -2\epsilon\alpha_1\rho_1pP_1D^{-1}(qP_2P_3 - \beta_1\alpha_2Z) , \\
R_{33} &= 2\alpha_1\rho_1P_1D^{-1}(\beta_2P_2X + \beta_1P_4Y) , \\
R_{22} &= 2\rho_1\beta_1P_2\bar{D}^{-1} .
\end{aligned}
$$

The scalar amplitude factors of the generated waves may become complex-valued, which has important consequences for:

1) S-wave polarization (elliptical instead of linear)
2) Seismogram waveform (shape of the R/T signal)

As only the real part of *F* has the physical meaning of displacement (or pressure), e.g., for PP reflection we get

$$\Re\{u_i\} = \Re\{R_{33}AN_iF\} = \begin{cases} R_{33}AN_i\Re\{F\} & \text{...for the real-valued coefficient} \\ AN_i(\Re\{F\}\Re\{R_{33}\} - \Im\{F\}\Im\{R_{33}\}) & \text{...for the complel-valued coefficient} \end{cases}$$

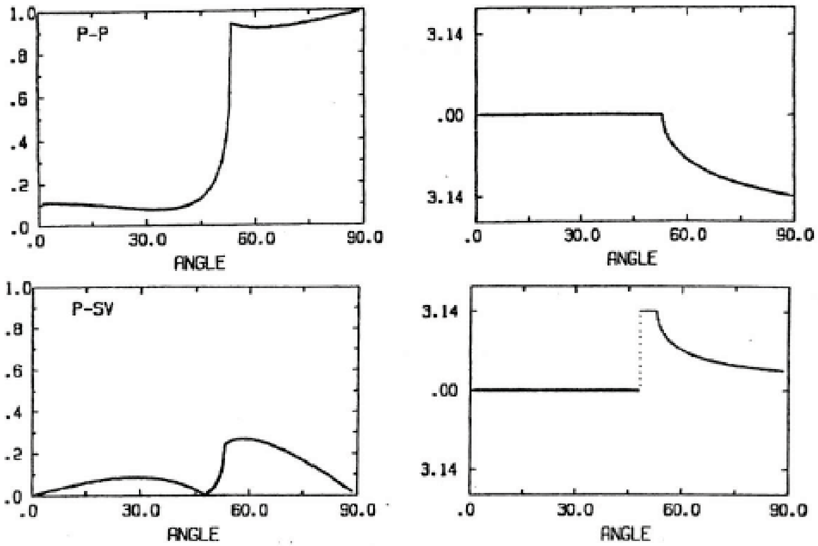
Square roots P_i, $i = 1, 2, 3, 4$, may be imaginary. … complex-valued R/T coefficients

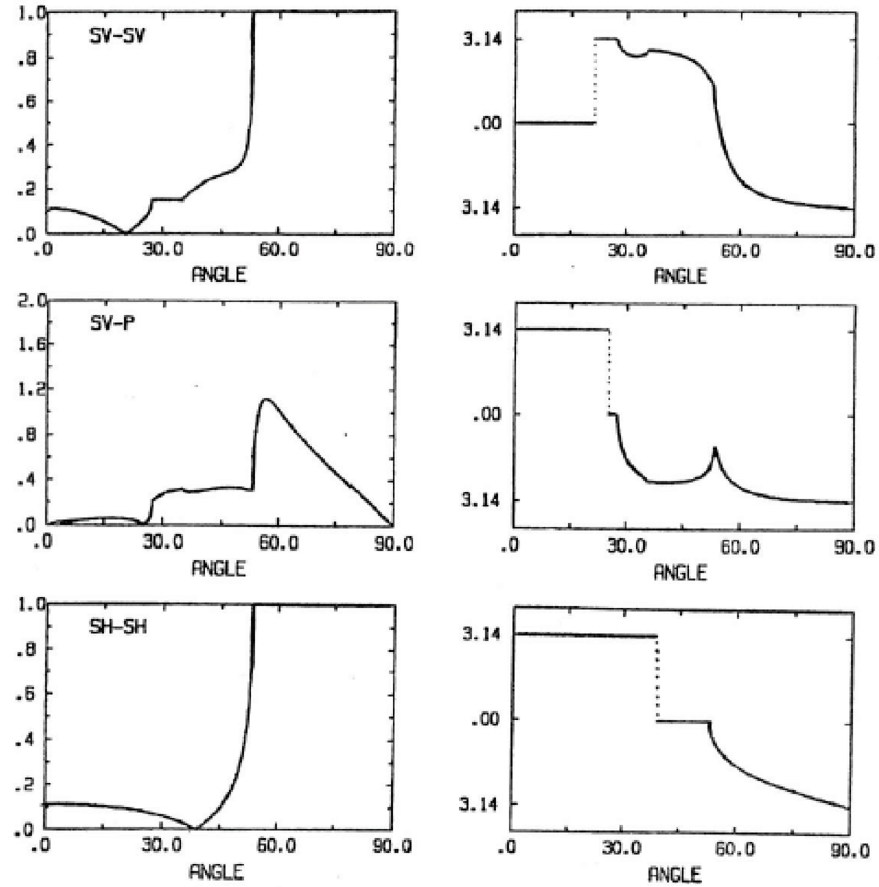
Plane waves at a plane interface – isotropic halfspaces (continuation)

Examples of R/T coefficients

WEAK INTERFACE - **REFLECTION**

$\alpha_1/\alpha_2 = 0.8 \quad \rho_2/\rho_1 = 1 \quad \alpha/\beta = \sqrt{3}$ (Poisson solid)





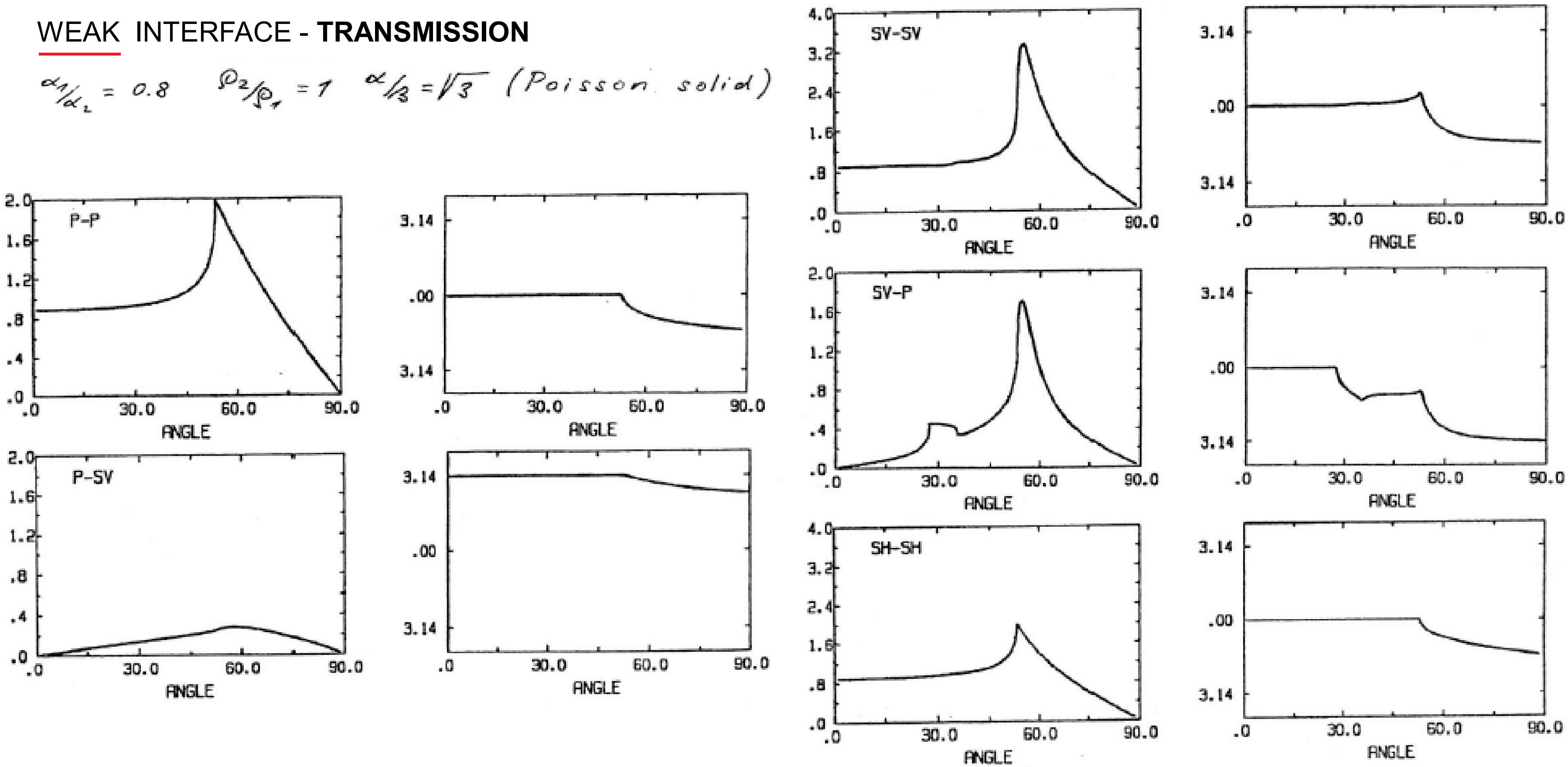
$$i_1^* = \arcsin\left(\frac{\beta_1}{\alpha_2}\right) = 27.51^\circ; \quad i_2^* = \arcsin\left(\frac{\beta_1}{\alpha_1}\right) = 35.26^\circ; \quad i_3^* = \arcsin\left(\frac{\alpha_1}{\alpha_2}\right) = \arcsin\left(\frac{\beta_1}{\beta_2}\right) = 53.13^\circ$$

Plane waves at a plane interface – isotropic halfspaces (continuation)

Examples of R/T coefficients

WEAK INTERFACE - **TRANSMISSION**

$\alpha_1/\alpha_2 = 0.8 \quad \rho_2/\rho_1 = 1 \quad \alpha/\beta = \sqrt{3}$ (Poisson solid)



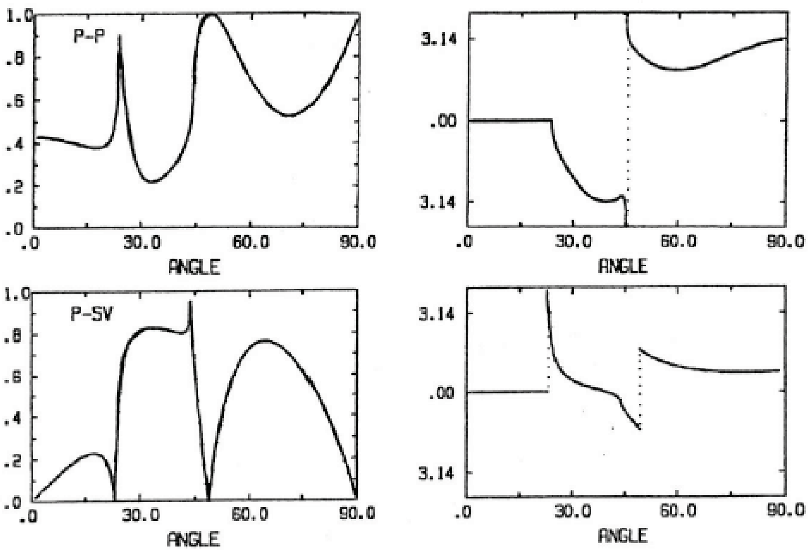
$$i_1^* = \arcsin\left(\frac{\beta_1}{\alpha_2}\right) = 27.51^\circ; \quad i_2^* = \arcsin\left(\frac{\beta_1}{\alpha_1}\right) = 35.26^\circ; \quad i_3^* = \arcsin\left(\frac{\alpha_1}{\alpha_2}\right) = \arcsin\left(\frac{\beta_1}{\beta_2}\right) = 53.13^\circ$$

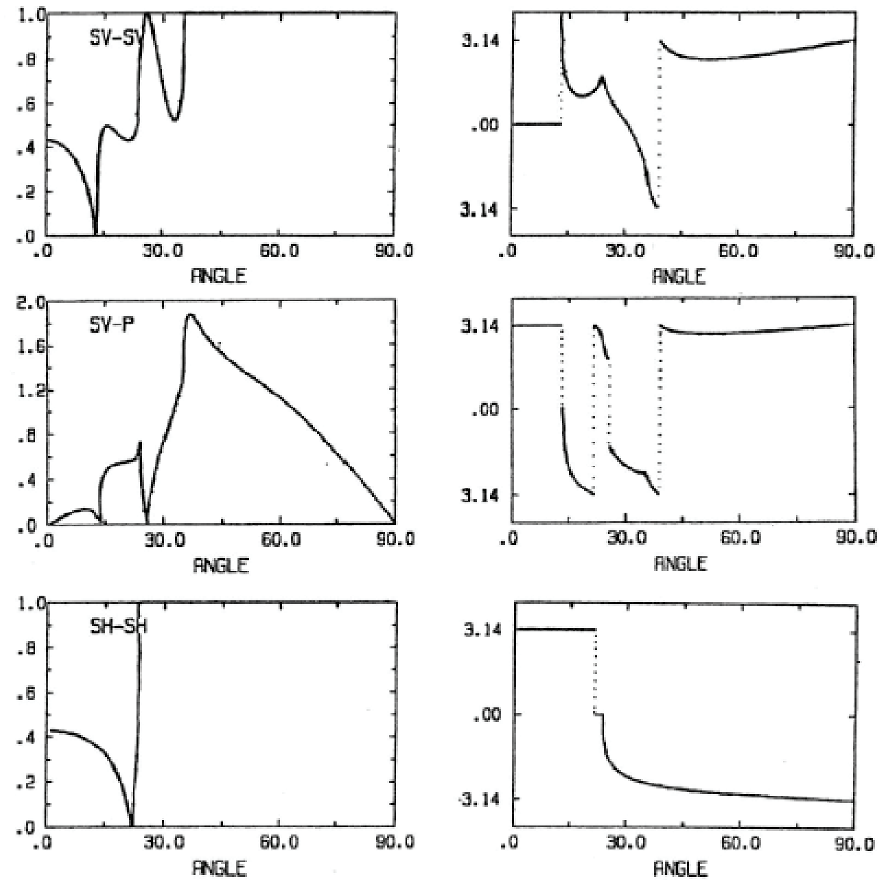
Plane waves at a plane interface – isotropic halfspaces (continuation)

Examples of R/T coefficients

STRONG INTERFACE - **REFLECTION**

$\alpha_1/\alpha_2 = 0.4 \quad \rho_2/\rho_1 = 1 \quad \alpha/\beta = \sqrt{3}$ (Poisson solid)





$$i_1^* = \arcsin\left(\frac{\beta_1}{\alpha_2}\right) = 13.35^\circ;\ i_2^* = \arcsin\left(\frac{\alpha_1}{\alpha_2}\right) = 23.59^\circ;\ i_3^* = \arcsin\left(\frac{\beta_1}{\alpha_1}\right) = 35.26^\circ;\ i_4^* = \arcsin\left(\frac{\alpha_1}{\beta_2}\right) = 43.84^\circ$$

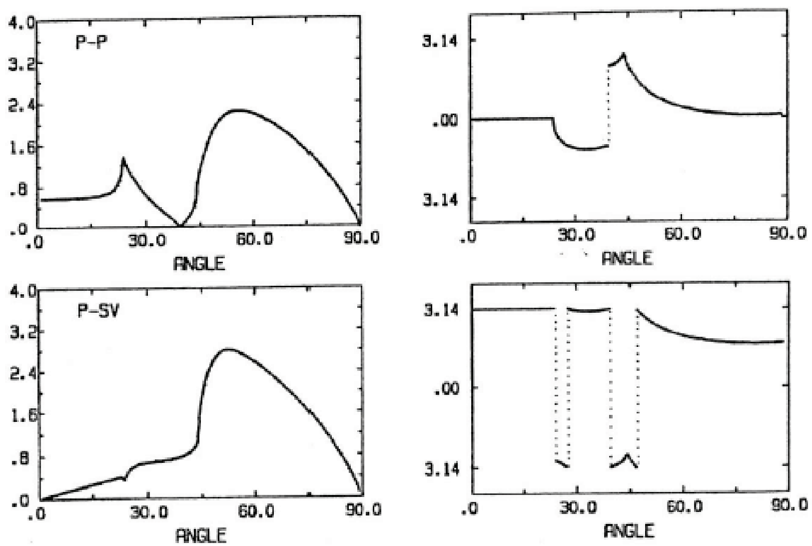
$$\left(= \arcsin\left(\frac{\beta_1}{\beta_2}\right) \right)$$

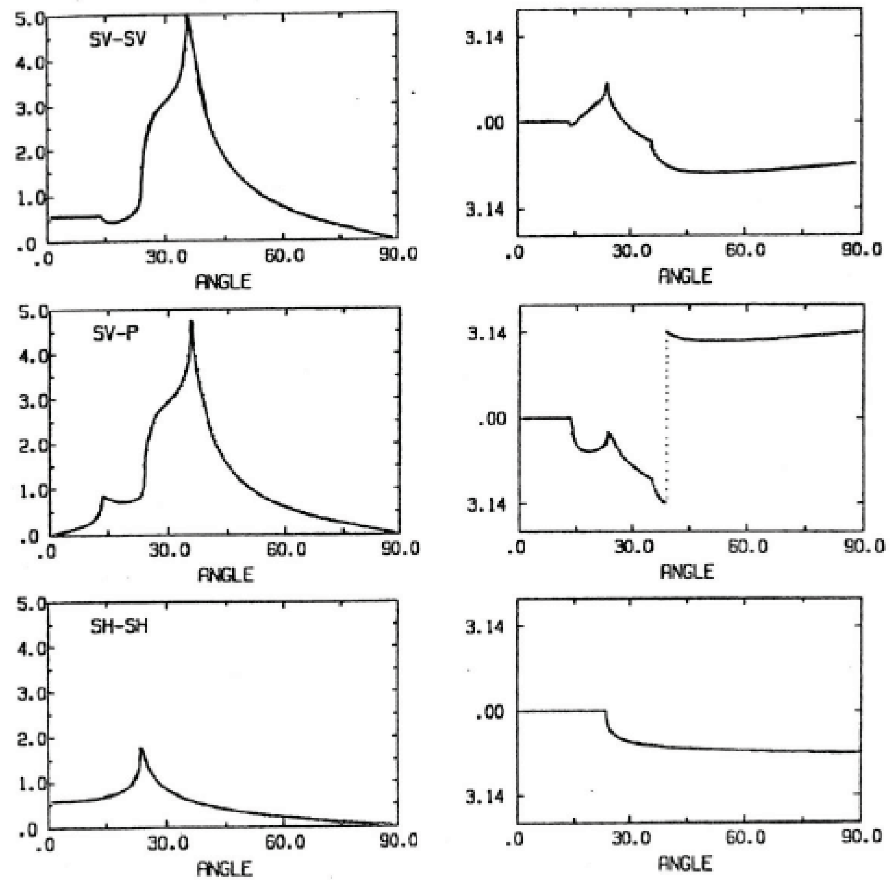
Plane waves at a plane interface – isotropic halfspaces (continuation)

Examples of R/T coefficients

STRONG INTERFACE - **TRANSMISSION**

$\alpha_1/\alpha_2 = 0.4 \quad \rho_2/\rho_1 = 1 \quad \alpha/\beta = \sqrt{3}$ (Poisson solid)





$$i_1^* = \arcsin\left(\frac{\beta_1}{\alpha_2}\right) = 13.35^\circ;\ i_2^* = \arcsin\left(\frac{\alpha_1}{\alpha_2}\right) = 23.57^\circ;\ i_3^* = \arcsin\left(\frac{\beta_1}{\alpha_1}\right) = 35.26^\circ;\ i_4^* = \arcsin\left(\frac{\alpha_1}{\beta_2}\right) = 43.84^\circ$$

$$\left(= \arcsin\left(\frac{\beta_1}{\beta_2}\right) \right)$$

Spherical wave … another simple solution of the eq. of motion in homogeneous media (closer to reality than the plane waves)

We will introduce it for the acoustic case (isotropic and anisotropic cases are analogous). First, we transform the equation of motion **for homogeneous medium** ($\rho = const$)

$$\frac{\partial^2 p}{\partial x_i^2} - c^2 \frac{\partial^2 p}{\partial t^2} = F$$

to general curvilinear coordinates γ_1, γ_2 and γ_3:

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \gamma_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial p}{\partial \gamma_1}\right) + \frac{\partial}{\partial \gamma_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial p}{\partial \gamma_2}\right) + \frac{\partial}{\partial \gamma_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial p}{\partial \gamma_3}\right)\right] - c^{-2}\frac{\partial^2 p}{\partial t^2} = F$$

Scale factors

Body force (usually set to zero)

For example, for spherical coordinates

$$x_1 = r\sin\theta\cos\varphi \qquad h_r = 1$$
$$x_2 = r\sin\theta\sin\varphi \qquad h_\theta = r$$
$$x_3 = r\cos\theta \qquad h_\varphi = r\sin\theta$$

we get

$$\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial p}{\partial \varphi}\right)\right] - c^{-2}(r,\theta,\varphi)\frac{\partial^2 p}{\partial t^2} = \bar{F}$$

$$\bar{F} = \bar{F}(r,\theta,\varphi,t)$$

$$\Downarrow$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial \varphi^2} - c^{-2}(r,\theta,\varphi)\frac{\partial^2 p}{\partial t^2} = \bar{F}$$

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$$\bar{F} = \bar{F}(r,\theta,\varphi,t)$$

$$\Downarrow$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\varphi^2}-c^{-2}(r,\theta,\varphi)\frac{\partial^2 p}{\partial t^2}=\bar{F}$$

We seek for a **spherically symmetric solution, i.e. independent on θ and ϕ.**

Spherical wave

We get the following equation of motion

$$\frac{2}{r}\frac{\partial p(r,t)}{\partial r} + \frac{\partial^2 p(r,t)}{\partial r^2} - c^{-2}\frac{\partial^2 p(r,t)}{\partial t^2} = 0 \quad ,$$

which can be formally rewritten for rp instead of p itself: $\frac{\partial^2 rp(r,t)}{\partial r^2} - c^{-2}\frac{\partial^2 rp(r,t)}{\partial t^2} = 0$.

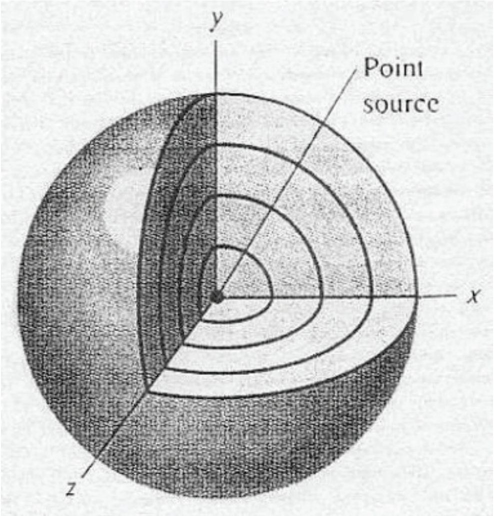
This is 1D wave equation, the well known solution of which is the plane wave:

$$rp(r,t) = P \cdot F\left(t - \frac{r}{c}\right)$$

Thus $$p(r,t) = \frac{P}{r} F\left(t - \frac{r}{c}\right) .$$ … a spherical wave

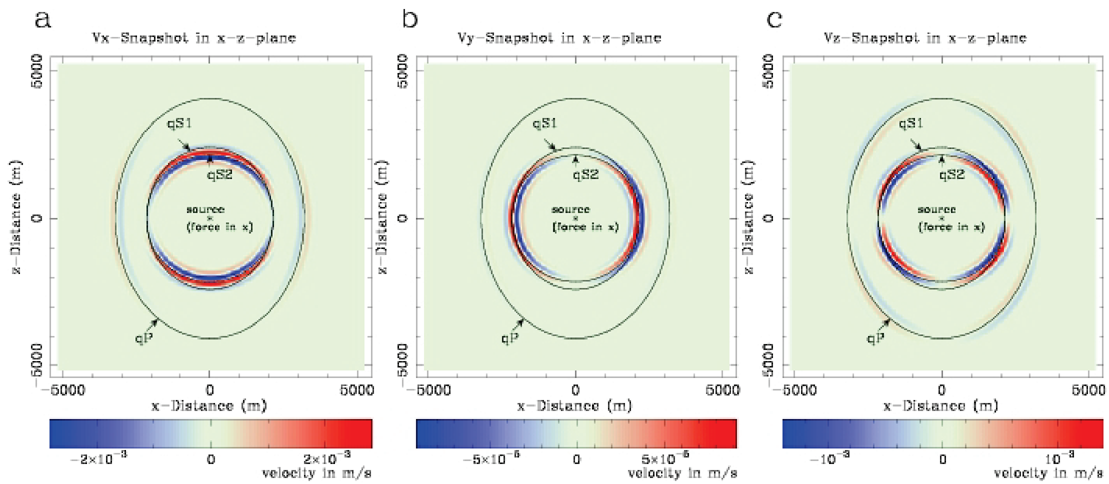
Amplitude is not constant
- it decays with distance as ~ *1/r* (geometrical spreading)

Surfaces of constant phase (*t=r/c*) are spheres (wavefronts) originating at the coordinate origin (point source) with the radius *r*



Spherical waves has analogous properties to plane waves in terms of:

1) Number of waves in acoustic (1), isotropic (2) and anisotropic (3) media.



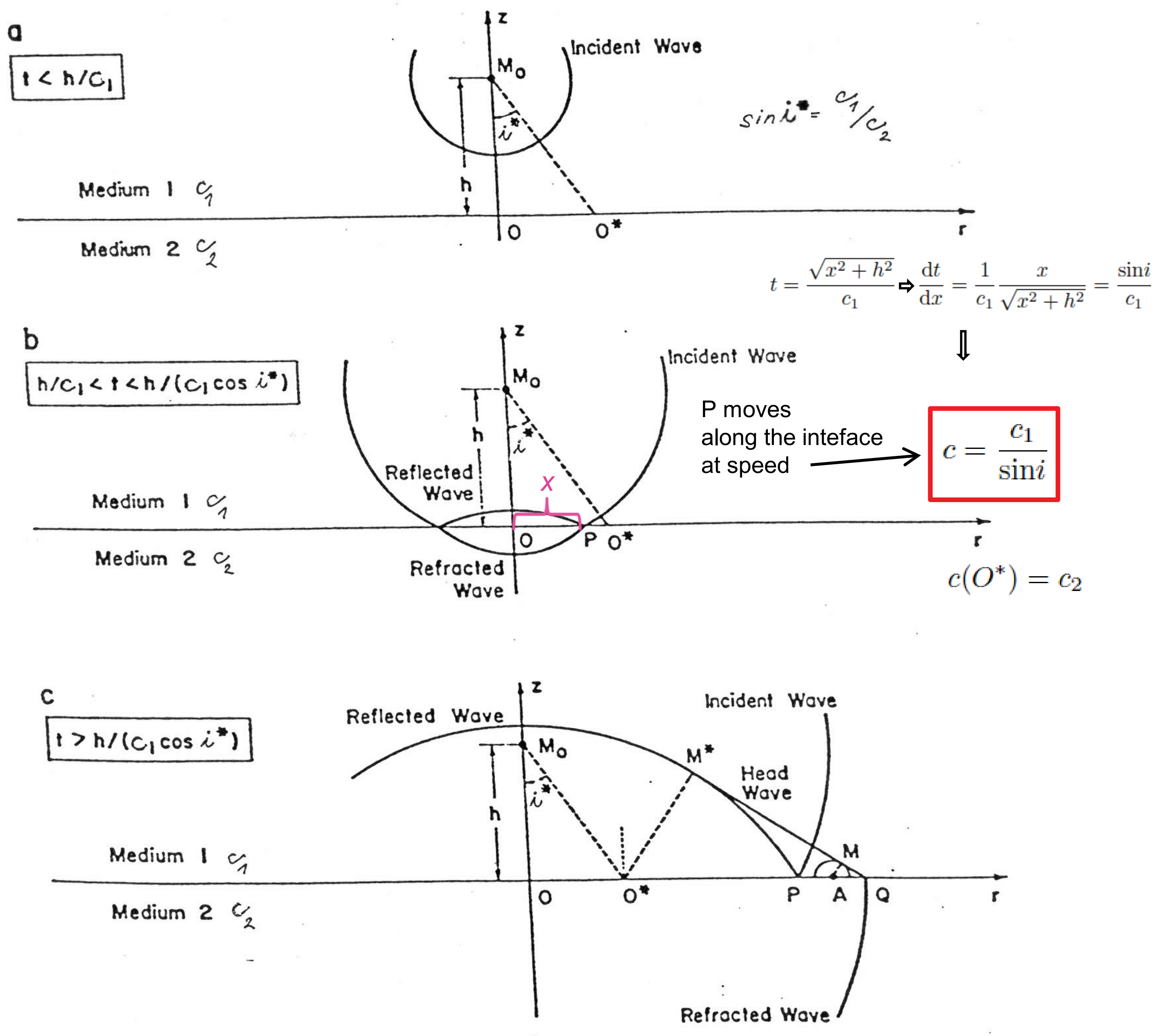
… an example of anisotropic wavefronts
(from Brietzke, Diploma Thesis)

2) Energy flux and group velocity direction (perpendicular to wavefronts only in acoustic and isotropic media; in anisotropic media it can be perpendicular only for certain symmetries of the medium in certain directions)

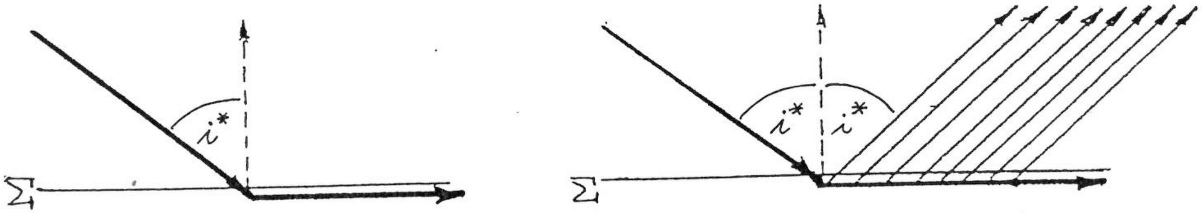
Any spherical wave can be decomposed to plane waves (Weyl integral). The R/T problem can be solved for the elemental plane waves and the final solution is obtained by their integral superposition.

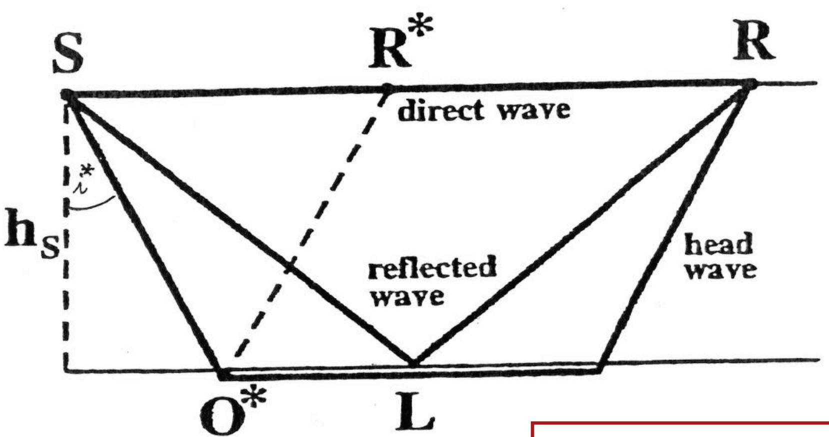
HEAD WAVES

$c_2 > c_1$

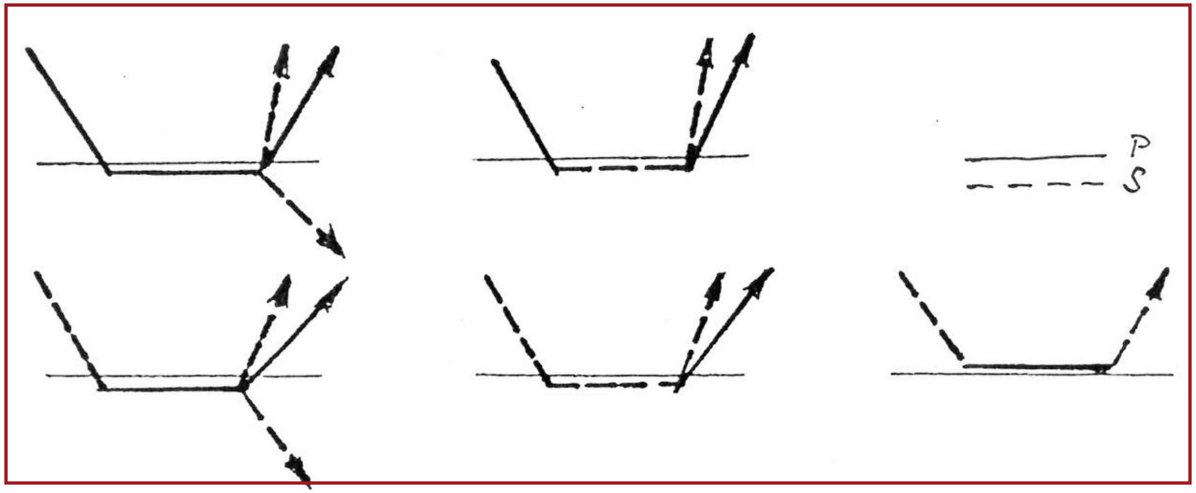


HEAD WAVES

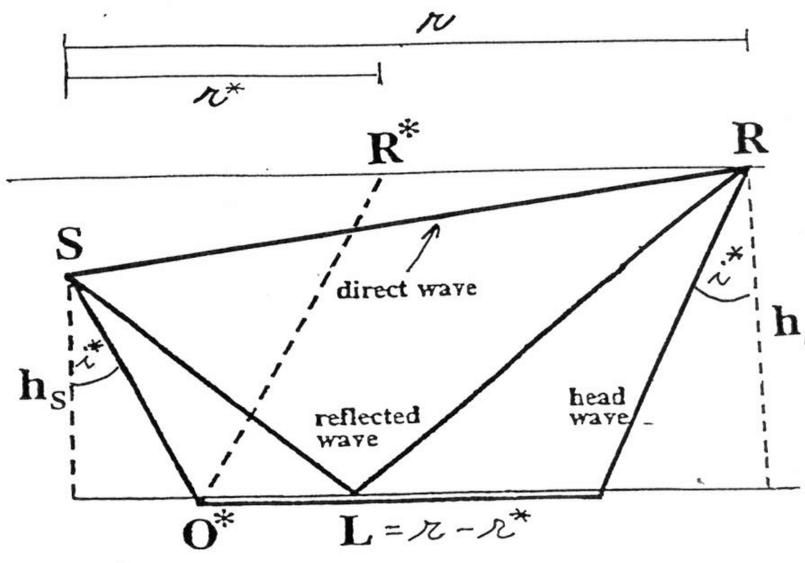


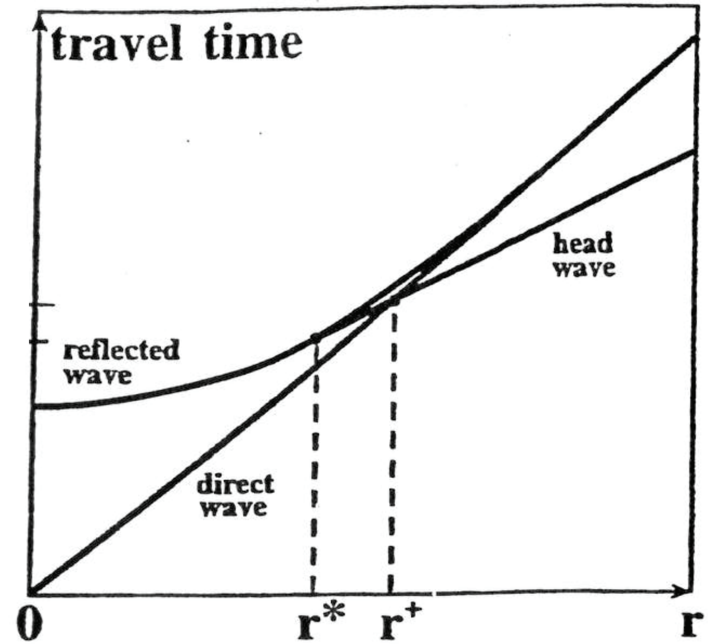


In isotropic media
various types of head waves
can exist



HEAD WAVES - traveltimes





Direct wave travel time

$$T^d(r) = [r^2 + (h_R - h_S)^2]^{1/2}/c_1$$

… for $h_R = h_S$ linearly depends on r

Reflected wave travel time

$$T^r(r) = [r^2 + (h_R + h_S)^2]^{1/2}/c_1$$

Head wave travel time

$$T^h(r) = T^r(r^*) + \frac{r - r^*}{c_2} = \frac{h_S + h_R}{c_1\sqrt{1-n^2}} + \frac{r - r^*}{c_2}$$

$$= \frac{h_S + h_R}{c_1}\sqrt{1-n^2} + \frac{r}{c_2}$$

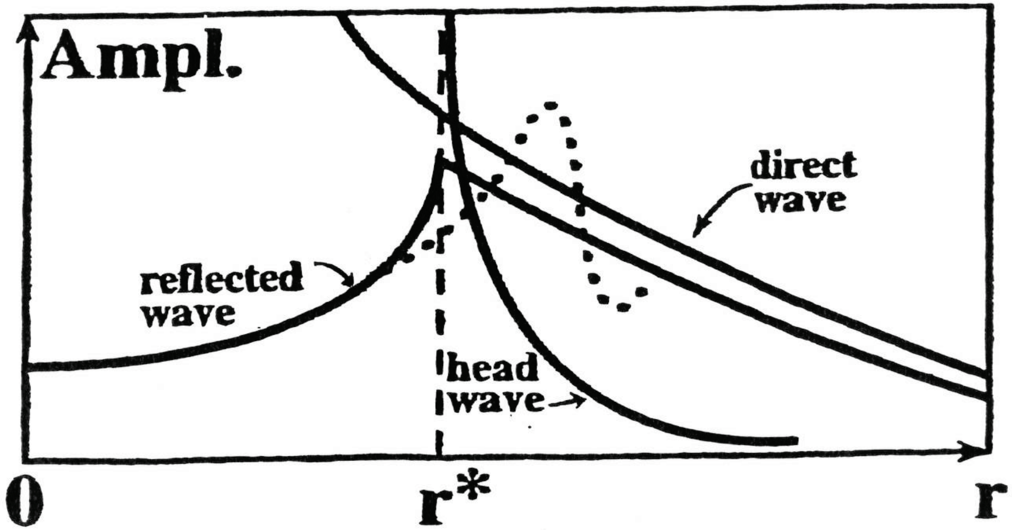
… always linearly depends on r

HEAD WAVES - amplitudes

Asymptotic expression for head wave

$$p^h(R,t) = \frac{2\rho_1 c_1 n}{\rho_2(1-n^2)r^{1/2}L^{3/2}} F^{(1)}(t - T^h(R,S))$$

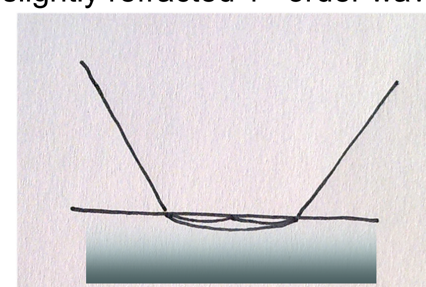
$\hookrightarrow F^{(1)}(\xi) = \int_{-\infty}^{\xi} F(\xi')d\xi'$ → analytic signal of the incident wave

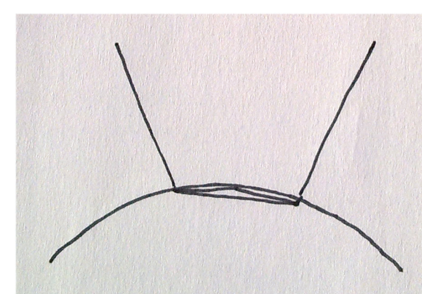


$\sim r^{-1/2}\underbrace{(r - r^*)}_{L}{}^{-3/2}$

for $r \gg r^* \ldots \sim r^{-2}$ a 2[nd] order wave

Kinematically analogous waves - slightly refracted 1[st] order waves:





$\sim r^{-1}$

“whispering gallery” wave

Green's function – acoustic case

Assume the scalar acoustic equation $(\rho^{-1}p_{,i})_{,i} + f^p = \kappa\ddot{p}$ with a very special source term.

$f^p \rightarrow -\delta(t-t_0)\,\delta(\vec{x}-\vec{x}_0)$

…. a unit pulse in space and time (the source acts at the time t_0 and at the point $\boldsymbol{x_0}$)

The solution for such a special source term is called "Green's function" . Let us denote it *G*. It is a solution of

$$\left(\frac{1}{\rho}G_{,i}\right)_{,i} - \kappa\ddot{G} = -\delta(t-t_0)\delta(\mathbf{x}-\mathbf{x}_0) \; .$$

The solution depends on the 'source' position and time as well as on the 'receiver' position and time:

$$G = G(\underbrace{x_m, t}_{\text{receiver}};\underbrace{x_{om}, t_o}_{\text{source}})$$

receiver *source*

For general inhomogeneous medium, the equation must be solved numerically. In **homogeneous media** (ρ=*const*), the solution can be found analytically. It has the form of a spherical wave:

$$G(x_m, t; x_{om}, t_o) = \frac{\rho}{4\pi r}\delta\left(t - t_o - \frac{r}{c}\right)$$

$r = |\mathbf{x} - \mathbf{x}_0|$

source time *propagation time*

In homogeneous media, the GF does not depend on $\boldsymbol{x}$ and $\boldsymbol{x_0}$ but only on the source-receiver distance.

Green's function in acoustic media (continuation)

What is the Green's function good for ?

It allows us to find easily the solution for a more general source (distributed generally in space and time).

1) A point source at $\boldsymbol{x_0}$, general source time function $f(t\text{-}t_0)$

$$(\rho^{-1}p_{,i})_{,i} - \kappa\ddot{p} = -\delta(\mathbf{x}-\mathbf{x_0})f(t-t_0)\ , \quad f(t-t_0) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau_0-\tau)\mathrm{d}\tau$$

$$GF = \frac{\rho}{4\pi r}\delta\left(t-t_0-\tau-\frac{r}{c}\right)$$

$$p(x_m,t) = \frac{\rho}{4\pi r}\int_{-\infty}^{\infty} f(\tau)\delta\left(t-t_0-\tau-\frac{r}{c}\right)\mathrm{d}t = \frac{\rho}{4\pi r}f\left(t-t_0-\frac{r}{c}\right)$$

2) A finite-extent source (generally distributed in space) with general source time function $f(t\text{-}t_0)$

$$(\rho^{-1}p_{,i})_{,i} - \kappa\ddot{p} = -f(x_m,t-t_0)$$

$$\iiint_V \delta(x_m - x'_m)f(t-t_0)\mathrm{d}V'$$

⇩

$$p(x_m,t) = \frac{\rho}{4\pi}\iiint_V \frac{f(x'_m,t-t_0-\frac{r'}{c})}{r'}\mathrm{d}V'$$

Green's function – elastodynamic case (anisotropic, isotropic)

In elastodynamic case the GF is a second-rank tensor $G_{in}(x_m, t, x_{0m}, t_0)$.

receiver (x_m, t) — source (x_{0m}, t_0)

Its physical meaning is the following: it represents the i-th component of displacement at a point $\boldsymbol{x}$ and time t induced by the unit pulse body force acting at a point $\boldsymbol{x_0}$ and time t_0 in the direction of the n-th coordinate axis.

$$f_i = \delta_{in}\delta(x_m - x_{om})\delta(t - t_o)$$

It is the solution of the equation:

$$(c_{ijkl}G_{kn,l})_{,j} - \rho G_{in,tt} = -\delta_{in}\delta(x_m - x_{om})\delta(t - t_o)$$

…must be solved numerically

Analytical solution is known only for homogeneous, isotropic, unbounded medium (Stokes, 1849):

$$G_{in}(x_m, t; x_{om}, t_o) = \frac{3N_iN_n - \delta_{in}}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t - t_o - \tau)d\tau + \frac{N_iN_n\delta(t - t_o - \frac{r}{\alpha})}{4\pi\rho\alpha^2 r} - \frac{(N_iN_n - \delta_{in})\delta(t - t_o - \frac{r}{\beta})}{4\pi\rho\beta^2 r}$$

In homogeneous media, the GF does not depend on $\boldsymbol{x}$ and $\boldsymbol{x_0}$ but only on the source-receiver distance.
Moreover, the result will not change when interchanging source and receiver,

Green's function – elastodynamic case (anisotropic, isotropic)

In elastodynamic case the GF is a second-rank tensor $G_{in}(x_m, t, x_{0m}, t_0)$.

receiver (i; x_m, t) source (x_{0m}, t_0; n)

Its physical meaning is the following: it represents the i-th component of displacement at a point $\boldsymbol{x}$ and time t induced by the unit pulse body force acting at a point $\boldsymbol{x_0}$ and time t_0 in the direction of the n-th coordinate axis.

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$$(c_{ijkl}G_{kn,l})_{,j} - \rho G_{in,tt} = -\delta_{in}\delta(x_m - x_{om})\delta(t - t_o)$$

…must be solved numerically

Analytical solution is known only for homogeneous, isotropic, unbounded medium (Stokes, 1849):

near-field part

$$G_{in}(x_m, t; x_{om}, t_o) = \frac{3N_iN_n - \delta_{in}}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t - t_o - \tau)d\tau$$
$$+ \frac{N_iN_n\delta(t - t_o - \frac{r}{\alpha})}{4\pi\rho\alpha^2 r} - \frac{(N_iN_n - \delta_{in})\delta(t - t_o - \frac{r}{\beta})}{4\pi\rho\beta^2 r}$$

far-field terms

$$G_{in}(x_m, t; x_{om}, t_o) = G_{ni}(x_{om}, t; x_m, t_o)$$

Green's function – isotropic case (continuation)

Discussion of the Stokes solution:

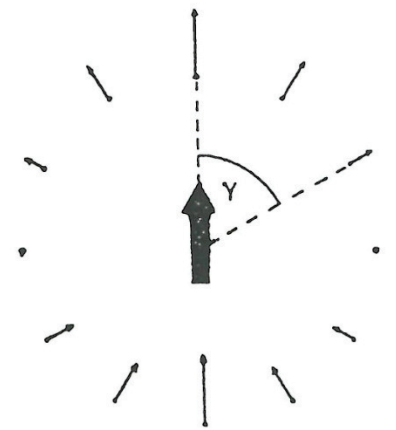
in terms of wavelength

Near-field part decays more rapidly with distance compared to the far field terms ⟹ far-field terms dominate at greater distances from the source.

(far-field/near-field ratio is $\frac{\omega r}{\alpha}$ for P and $\frac{\omega r}{\beta}$ for S wave).

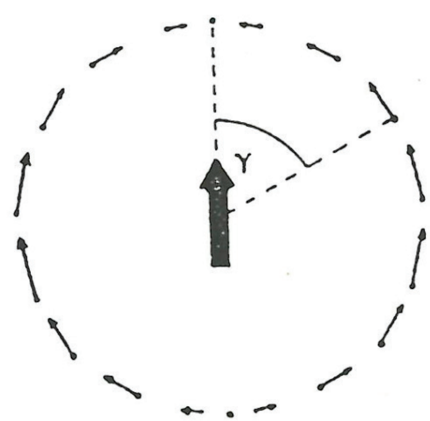
2π x number of wavelengths between source and receiver

Far-field P $$\frac{1}{4\pi\rho\alpha^2 r} N_i N_n \delta\left(t - t_o - \frac{r}{\alpha}\right)$$



… a spherical wave propagating at the speed α from the source point;
amplitude decays as *1/r;*
the wave is radially polarized (vector product with ***N*** vanishes);
amplitude not constant along the wavefront
~ cos γ (angle between the force and radius vector)

Far-field S $$\frac{1}{4\pi\rho\beta^2 r}(\delta_{in} - N_i N_n)\delta\left(t - t_o - \frac{r}{\beta}\right)$$



… a spherical wave propagating at the speed β from the source point;
amplitude decays as *1/r;*
the wave is tranversally polarized (scalar product with ***N*** vanishes);
amplitude not constant along the wavefront
~ sin γ (angle between the force and radius vector)

Ratio of maximum amplitudes of S and P waves is $\alpha^2/\beta^2 \sim 3$ (for Poisson solid)

Green's function – discussion of the Stokes solution (continuation)

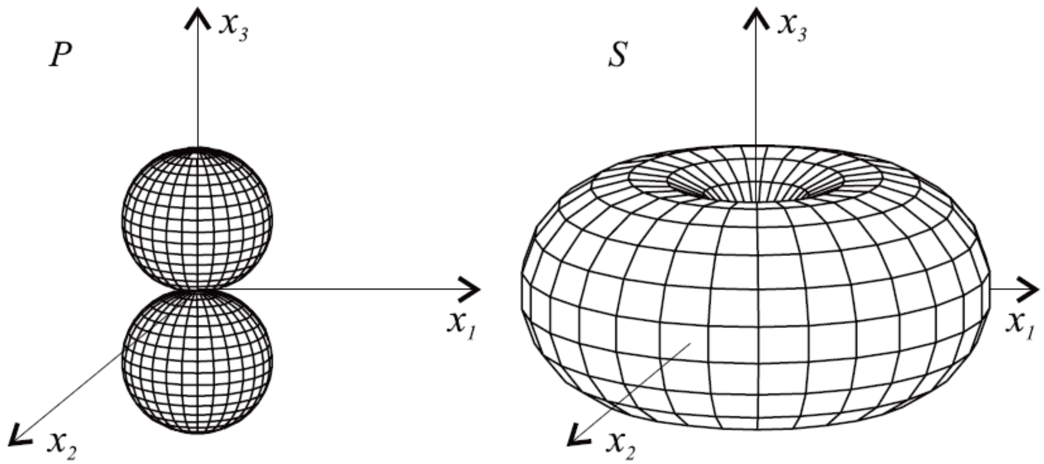


Figure 19: Radiation functions of P- (a) and S waves (b) due to a vertical point force in an isotropic medium.

Near-field term

$$\frac{3N_iN_n - \delta_{in}}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau$$

linear time dependence

… non-zero only between P and S arrivals;

amplitude decays as $1/r^3$,

$$\frac{2N_nN_i}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau = \frac{N_iN_n}{2\pi\rho r^3}(t-t_o)$$

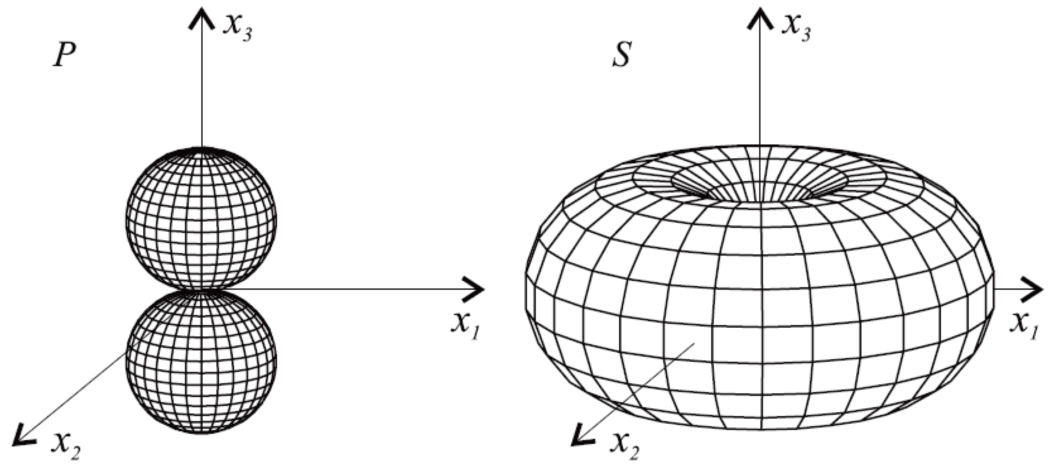
same as P

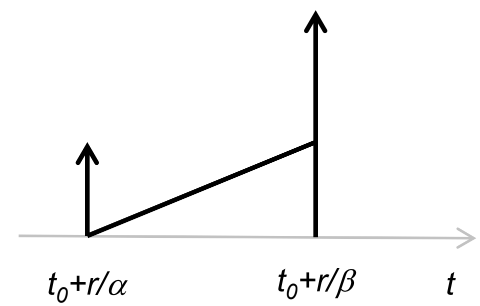
same as S

it can be decomposed into the radial component (taking scalar product with ***N***) and transverse component (subtracting the radial component)

$$\frac{N_iN_n - \delta_{in}}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau = \frac{N_iN_n - \delta_{in}}{4\pi\rho r^3}(t-t_o)$$

Green's function – discussion of the Stokes solution (continuation)





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… non-zero only between P and S arrivals;

amplitude decays as $1/r^3$,
$$\frac{2N_nN_i}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau = \frac{N_iN_n}{2\pi\rho r^3}(t-t_o)$$

linear time dependence

same as P

same as S

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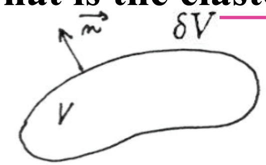
Green's function – elastodynamic case (anisotropic, isotropic)

both analytical and numerical (for inhom. isotropic or anisotropic structure)

What is the elastodynamic Green's function good for ?



It allows us to find the solution for a general source (distributed generally in space and time).

Assume $G_{in}(x_m, t; x_m^{(s)}, t_s)$ corresponding to the source term (force distribution)

$$\delta_{in}\delta(x_m - x_m^{(s)})\delta(t - t_s)$$

the solution we seek for

+ homogeneous initial and boundary conditions. G_{in} is unique (uniqueness theorem).
Assume further a general solution $u_i(x_m,t)$ corresponding to the force distribution $f_i(x_m,t)$
+ given boundary conditions (e.g., stress-less Earth's surface) and homogeneous initial condition ("quiescent past"). It is unique (uniqueness theorem).

THE TWO SOLUTIONS ARE CONNECTED THROUGH THE SO CALLED **REPRESENTATION THEOREM :** (for simplicity assume t_s=0)

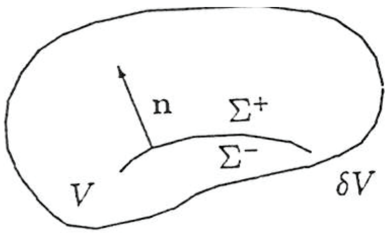
$$u_p(x_m,\tau) = \int_{-\infty}^{+\infty} dt \iiint\limits_V f_i(t)\; G_{pi}(x_m,\tau,x_m^{(s)},t)\; dV$$

$$- \int_{-\infty}^{+\infty} dt \iint\limits_{\delta V} c_{ijkl} n_j u_i(x_m,t)\; \frac{\partial G_{pk}}{\partial x_l}(x_m,\tau,x_m^{(s)},t)\; dS$$

Seismic source representation

How to introduce real seismic source into the representation theorem ? There are two ways:

1) we consider an additional surface (fault) in the model, with predescribed displacement and traction. Assume zero body forces inside the volume V and homogeneous BC for **G** on the outer boundary δV. Then the representation theorem simplifies significantly. The only non-zero contribution is the surface integral over the additional boundary Σ. In the frequency domain (to get rid of the time convolution) it is:



$$u_p(\mathbf{x},\omega) = \int_{\Sigma+} n_j(\mathbf{y}) c_{ijkl}(\mathbf{y}) \{u_{k,l}(\mathbf{y},\omega)\, G_{pi}(\mathbf{x},\mathbf{y},\omega) - G_{pk,l}(\mathbf{x},\mathbf{y},\omega)\, u_i(\mathbf{y},\omega)\} d^2\mathbf{y}$$
$$- \int_{\Sigma-} n_j(\mathbf{y}) c_{ijkl}(\mathbf{y}) \{u_{k,l}(\mathbf{y},\omega)\, G_{pi}(\mathbf{x},\mathbf{y},\omega) - G_{pk,l}(\mathbf{x},\mathbf{y},\omega)\, u_i(\mathbf{y},\omega)\} d^2\mathbf{y}$$

Assuming further (without loss of generality) continuity of **G** and traction across Σ, and introducing the slip function as displacement discontinuity across Σ, $\mathbf{s}(\mathbf{y},\omega) = \mathbf{u}(\mathbf{y},\omega)\big|_{\Sigma-} - \mathbf{u}(\mathbf{y},\omega)\big|_{\Sigma+}$ we get

$$u_p(\mathbf{x},\omega) = \int_{\Sigma} n_j(\mathbf{y}) c_{ijkl}(\mathbf{y})\, G_{pk,l}(\mathbf{x},\mathbf{y},\omega)\, s_i(\mathbf{y},\omega) d^2\mathbf{y} \quad .$$

In the time domain (applying the inverse Fourier transform) it has the form:

$$u_p(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \int_{\Sigma} n_j(\mathbf{y}) c_{ijkl}(\mathbf{y})\, G_{pk,l}(\mathbf{x}, t-\tau, \mathbf{y}, 0)\, s_i(\mathbf{y},\tau) d^2\mathbf{y}$$

Seismic source representation (continuation)

But there is another, alternative possibility how to introduce real seismic source into the representation theorem.

2) Assume that because of the source action, the real stress σ_{ij} differs from the model, linearized, Hookian stress τ_{ij} in the source region. In order to keep the elastodynamic equation valid, we have to introduce an extra body force $f_i = \sigma_{ij,j} - \tau_{ij,j} = -m_{ij,j}$ to balance this difference.

equivalent body force (f_i) — seismic moment tensor density (m_{ij})

Assume the same boundary conditions as in the first case. Then the only remaining term in the representation theorem is:

$$u_p(\mathbf{x},\omega) = -\int_V m_{ij,j}(\mathbf{y},\omega) G_{pi}(\mathbf{x},\mathbf{y},\omega) d^3\mathbf{y}$$

integrating per partes we get

$$= \int_V m_{ij}(\mathbf{y},\omega) G_{pi,j}(\mathbf{x},\mathbf{y},\omega) d^3\mathbf{y}$$

If the equivalent body force is located only along the fault plane Σ, we get:

$$u_p(\mathbf{x},\omega) = \int_\Sigma m_{ij}(\mathbf{y},\omega) G_{pi,j}(\mathbf{x},\mathbf{y},\omega) d^2\mathbf{y}$$

Comparing with the case 1) we obtain $m_{ij} = c_{ijkl} n_k s_l$

In the time domain:

$$u_p(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \int_\Sigma m_{ij}(\mathbf{y},\tau) G_{ip,j}(\mathbf{y},t-\tau,\mathbf{x},0) d^2\mathbf{y}$$