ELECTRICAL IMPEDANCE MAMMOGRAPHY

Forward problem for homogeneous and 3-D sphere

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Outline

Introduction

Electrical resistivity method in geophysics

- standard tool for mineral exploration, CO_2 sequestration monitoring, detection of cavities, etc.

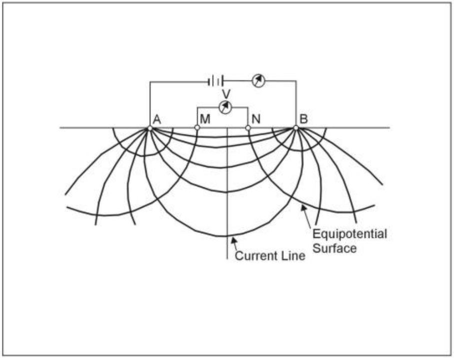


Image courtesy of U.S. Environmental Protection Agency

Introduction

Electrical resistivity method in geophysics

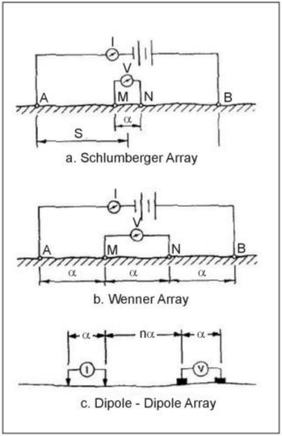


Image courtesy of U.S. Environmental Protection Agency

- DC, square-wave, or low-frequency AC current is injected into the Earth via two electrodes
- voltage is measured between additional electrodes, usually using inline geometry
- lateral separation of electrodes influences the penetration depth

$$\rho_a = \begin{cases} \pi \left(\frac{s^2}{a} - \frac{a}{4} \right) \frac{V}{I} \\ 2\pi\, a \frac{V}{I} \\ \pi\, a\, n\, (n+1)\, (n+2) \frac{V}{I} \end{cases}$$

The risk of a radiation-induced cancer for a woman attending mammographic screening (two views) by the NHSBSP is about 1 in 20 000 per visit

- it is estimated that about 170 cancers are detected by the NHSBSP for every cancer induced
- the mortality benefit of screening exceeds the radiation-induced detriment by about 100:1
- for the very small proportion of women who receive the highest radiation doses, the benefit will exceed the risk by about 20:1.

Review of Radiation Risk in Breast Screening NHSBSP Publication No 54, 2003

Introduction

Applications in Medicine

Some evidence has been found that malignant breast tumors have lower electrical impedance than surrounding normal tissues. Although the separation of malignant tumors from benign lesions based on impedance measurements needs further investigation, electrical impedance could be used as an indicator for breast cancer detection.

Zou, Y., & Gou, Z., 2003. A review of electrical impedance techniques for breast cancer detection, *Medical Engineering & Physics,* **25**, 79–90.

Maxwell's equations in DC limit

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \cdot \boldsymbol{D} &= \rho
\end{aligned}
$$

$$\boldsymbol{j} = \sigma \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

Maxwell's equations in DC limit

$$
\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{j} + \mathrm{i}\omega \boldsymbol{D} \\
\nabla \times \boldsymbol{E} &= 0 - \mathrm{i}\omega \boldsymbol{B} \\
\nabla \cdot \boldsymbol{B} &= 0 \\
\nabla \cdot \boldsymbol{D} &= \rho \\
\boldsymbol{j} = \sigma \boldsymbol{E} & \\
\boldsymbol{B} = \mu \boldsymbol{H} & \\
\boldsymbol{D} = \varepsilon \boldsymbol{E} &
\end{aligned}
$$

Maxwell's equations in DC limit

$$
\begin{aligned}
10^4\,\mathrm{Hz} \approx \omega \quad &\overset{?}{\ll} \quad \frac{\sigma}{\varepsilon_r \varepsilon_0} \approx \frac{10^0\,\mathrm{S/m}}{\mathbf{10^6}\,10^{-11}\,\mathrm{F/m}} \approx 10^5\,\mathrm{Hz} \\
10^4\,\mathrm{Hz} \approx \omega \quad &\ll \quad \frac{1}{\sigma \mu_r \mu_0 L^2} \approx \frac{1}{10^0\,\mathrm{S/m}\,10^0\,10^{-6}\,\mathrm{H/m}\,10^{-2}\,\mathrm{m}^2} \approx 10^8\,\mathrm{Hz} \\
\nabla \times \boldsymbol{H} \quad &= \quad \boldsymbol{j} + \mathrm{i}\omega \boldsymbol{D} \\
\nabla \times \boldsymbol{E} \quad &= \quad 0 - \cancel{\mathrm{i}\omega \boldsymbol{B}} \\
\nabla \cdot (\boldsymbol{j} + \mathrm{i}\omega \boldsymbol{D}) \quad &= \quad 0 \\
\boldsymbol{E} \quad &= \quad \nabla U \\
\boldsymbol{D} \quad &= \quad \varepsilon \boldsymbol{E} \\
\tilde{\sigma} \quad &= \quad \sigma + \mathrm{i}\omega\varepsilon \\
\nabla \cdot (\tilde{\sigma} \nabla U) \quad &= \quad 0
\end{aligned}
$$

Formulation of the forward problem

- Let $(r;\vartheta,\varphi) = (r;\Omega)$ be the spherical coordinate system (radius, colatitude, longitude).
- Let G be a sphere with radius a, boundary Γ, and electrical conductivity distribution $\tilde{\sigma} = \tilde{\sigma}(r;\vartheta,\varphi)$; $\sigma \in C_1(G)$, $\Re(\tilde{\sigma}) > 0$.
- Let $j(\Omega) \in C_1(\Gamma)$, such that $\int_\Gamma j(\Omega)\,\mathrm{d}S = 0$, be the imposed current density.
- Find $U(r;\Omega) \in C_2(G)$ such that

$$\begin{aligned} \nabla \cdot (\tilde{\sigma}\,\nabla U) &= 0 \quad \text{in } G, \\ \tilde{\sigma}\,\boldsymbol{e}_r \cdot \nabla U &= j \quad \text{on } \Gamma. \end{aligned}$$

Imposed currents

- In case of discrete point electrodes, the imposed currents can be written as:

$$j(\Omega) = \sum_{i=1}^{N_\mathrm{I}} I_i \, \delta\left(\Omega - \Omega_i\right),$$

where $\Omega_i = (\vartheta_i, \varphi_i)$ and I_i are the positions and imposed currents applied at individual electrodes ($N_\mathrm{I} \geq 2$), and

$$\begin{aligned} \int_\Gamma \delta(\Omega)\,\mathrm{d}S &= 1, \\ \sum_{i=1}^{N_\mathrm{I}} I_i &= 0. \end{aligned}$$

- Note that $j(\Omega) \notin C_1(\Gamma)$.

Imposed currents

- Alternatively, we can use a smoothed current density

$$\tilde{j}(\Omega) = \sum_{i=1}^{N_\mathrm{I}} I_i \, F\left(\Omega - \Omega_i\right),$$

  where

$$F(\Omega - \Omega_i) = \frac{1}{2\pi\, a^2} \overline{N}\left(\cos\Theta(\Omega, \Omega_i); 1, s, -1, 1\right).$$

  Here $\Theta(\Omega, \Omega_i)$ is the angular distance between points Ω, Ω_i along the main circle, and $\overline{N}(x; \mu, s, \alpha, \beta)$ is truncated normal distribution on interval (α, β).
- Note that $\tilde{j}(\Omega) \in C_1(\Gamma)$, and

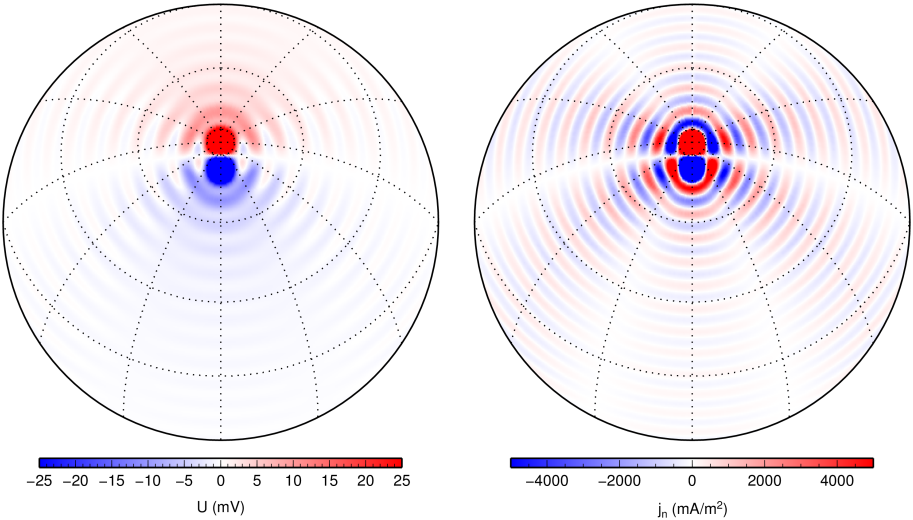
$$\begin{aligned} \int_\Gamma F(\Omega)\, \mathrm{d}S &= 1, \\ \int_\Gamma \tilde{j}(\Omega)\, \mathrm{d}S &= 0. \end{aligned}$$

Analytical solution for a homogeneous sphere

- Let $\tilde{\sigma}(r;\Omega) = \tilde{\sigma}_0 = \mathrm{const}$. Then
$$\nabla^2 U = 0 \quad \text{in } G$$
has a nondiverging analytical solution
$$U(r;\Omega) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} U_{nm} \left(\frac{r}{a}\right)^n Y_{nm}(\Omega)$$
(made unique by leaving U_{00}), where $Y_{nm}(\Omega)$ are orthonormal complex spherical harmonics.
- The boundary condition reads as
$$\tilde{\sigma}_0 \, \boldsymbol{e}_r \cdot \nabla U\big|_{r=a} = \frac{\sigma_0}{a} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} n \, U_{nm} \, Y_{nm}(\Omega) = j(\Omega).$$
- Using the orthonormality, we get
$$U_{nm} = \frac{1}{n \, \tilde{\sigma}_0 \, a} \sum_{i=1}^{N_\mathrm{I}} I_i \, Y_{nm}(\Omega_i) \cong \frac{1}{n \, \tilde{\sigma}_0 \, a} \sum_{i=1}^{N_\mathrm{I}} I_i \int_\Gamma \overline{Y}_{nm}(\Omega) F(\Omega - \Omega_i) \, dS.$$

Analytical solution for a homogeneous sphere

$a = 5\,\mathrm{cm}$, $\tilde{\sigma} = 1\,{}^{\mathrm{S}}/_{\mathrm{m}}$, $N = 40$, point electrodes
$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA})$, $(\vartheta_2, \varphi_1, I_2) = (10°, 0°, -1\,\mathrm{mA})$



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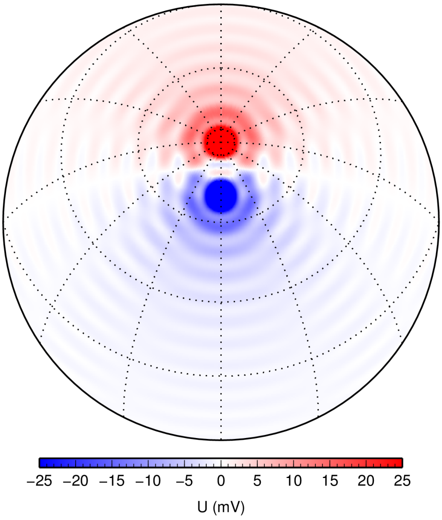

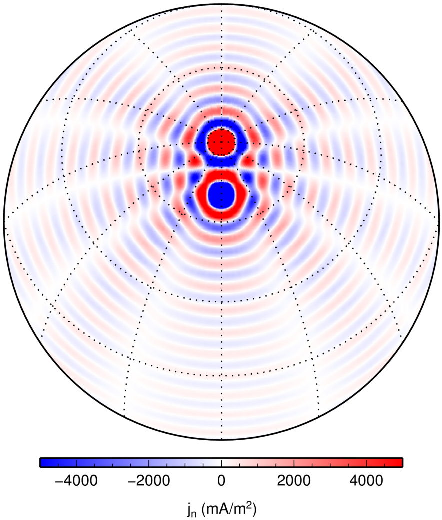

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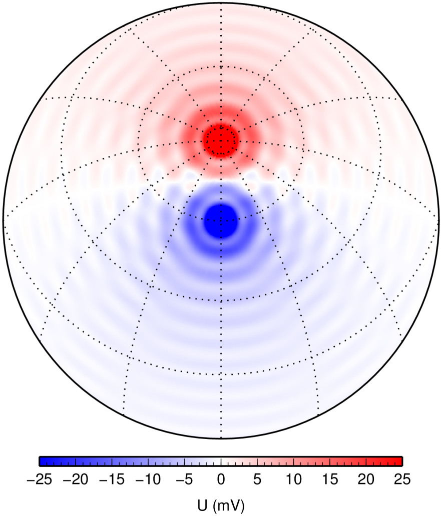

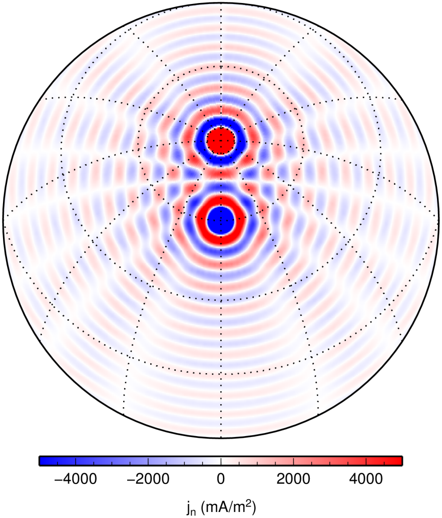

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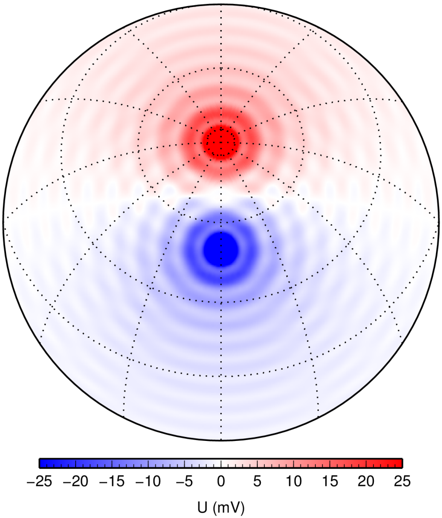

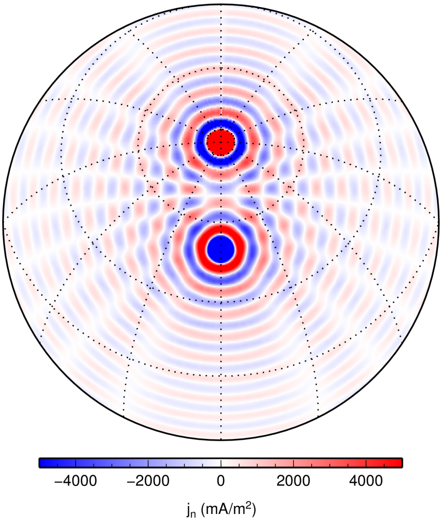

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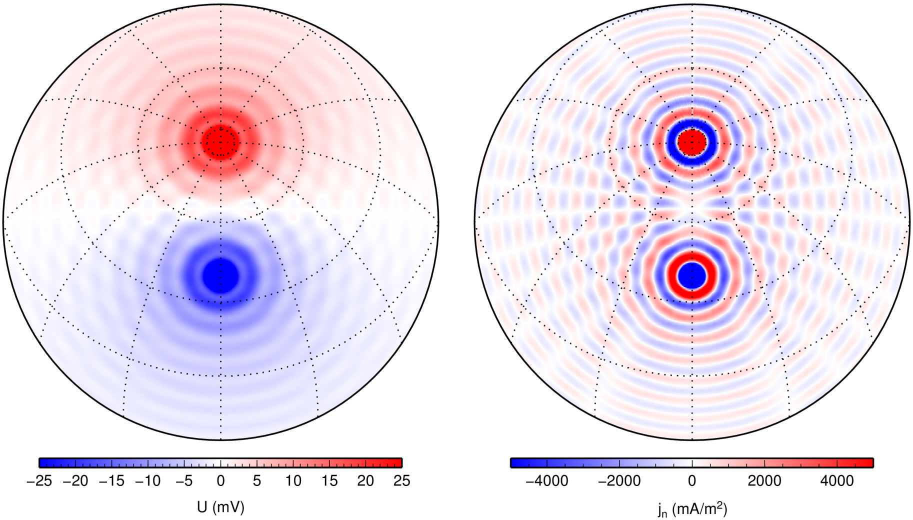

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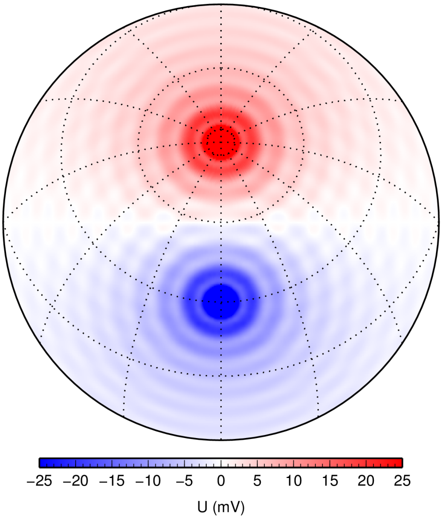

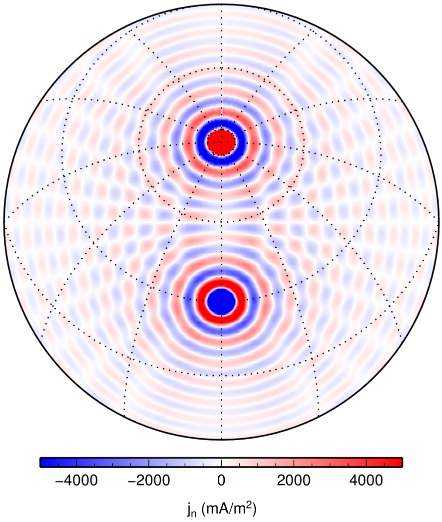

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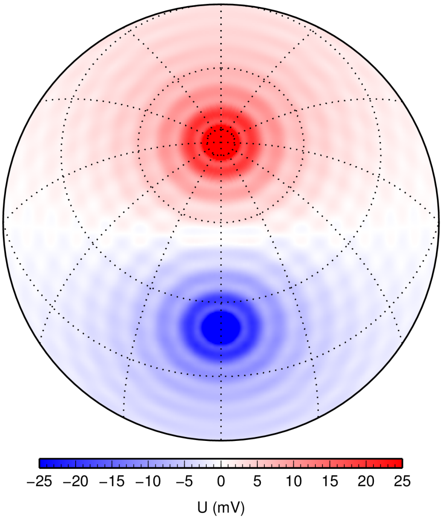

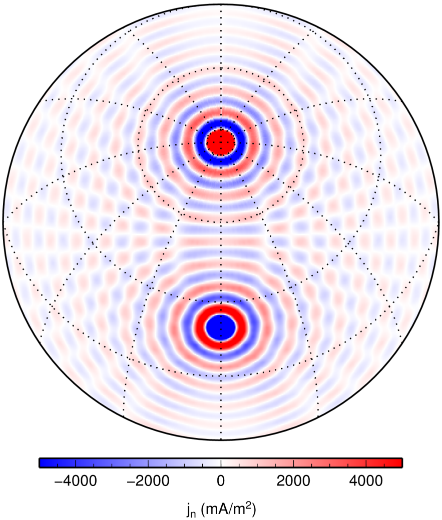

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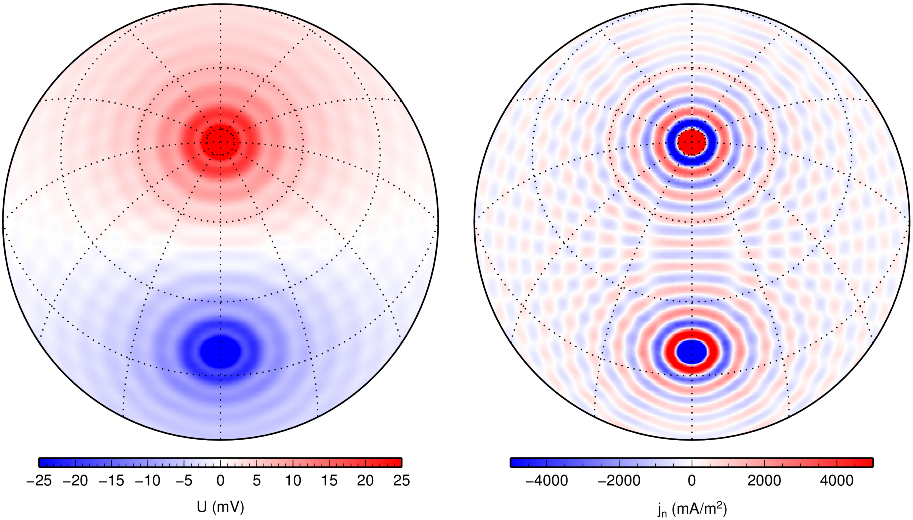

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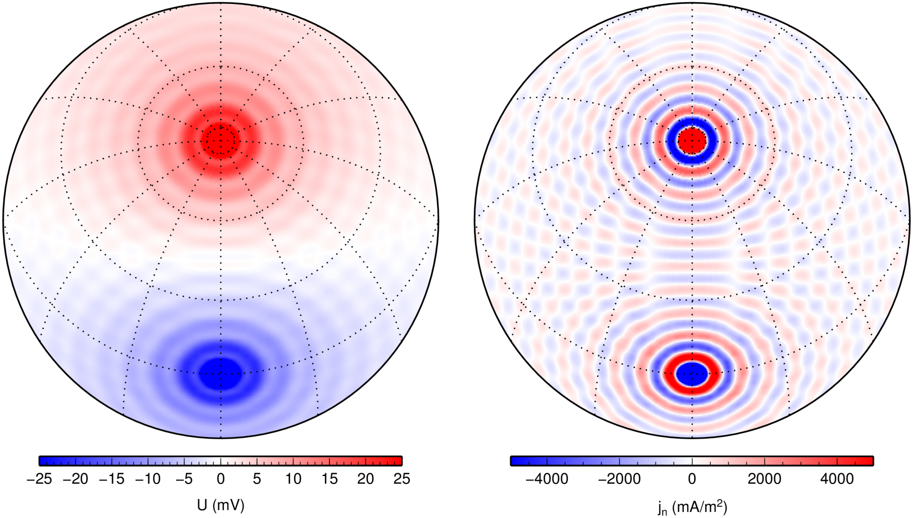

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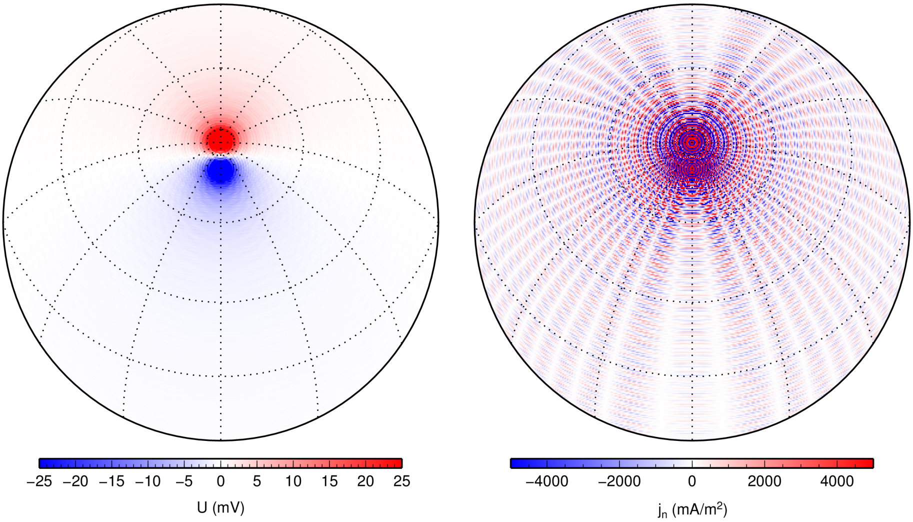

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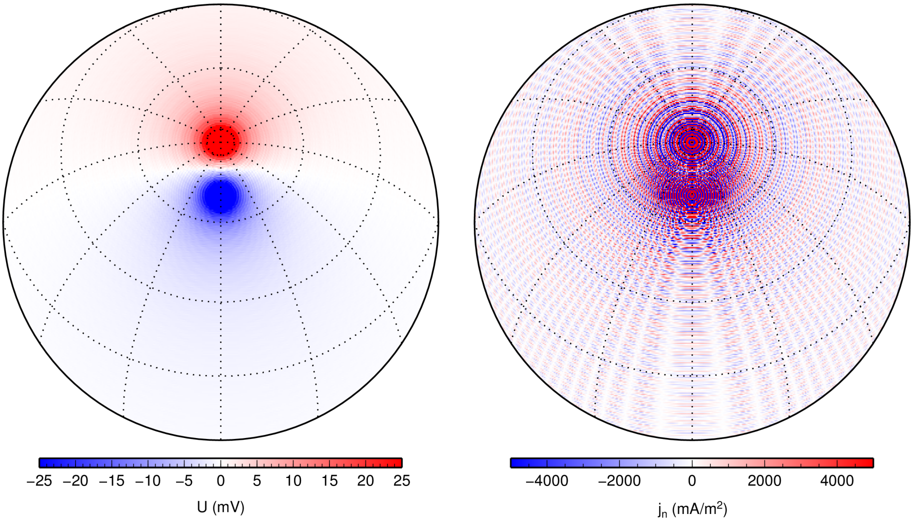

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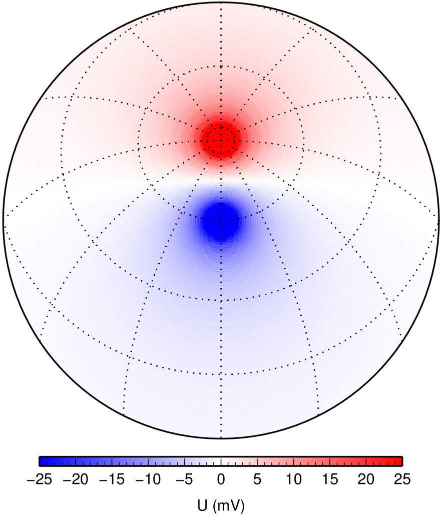

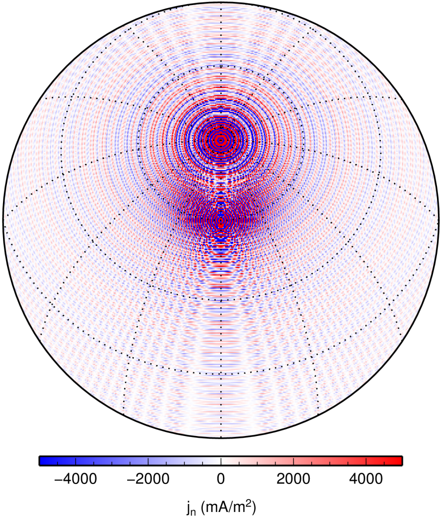

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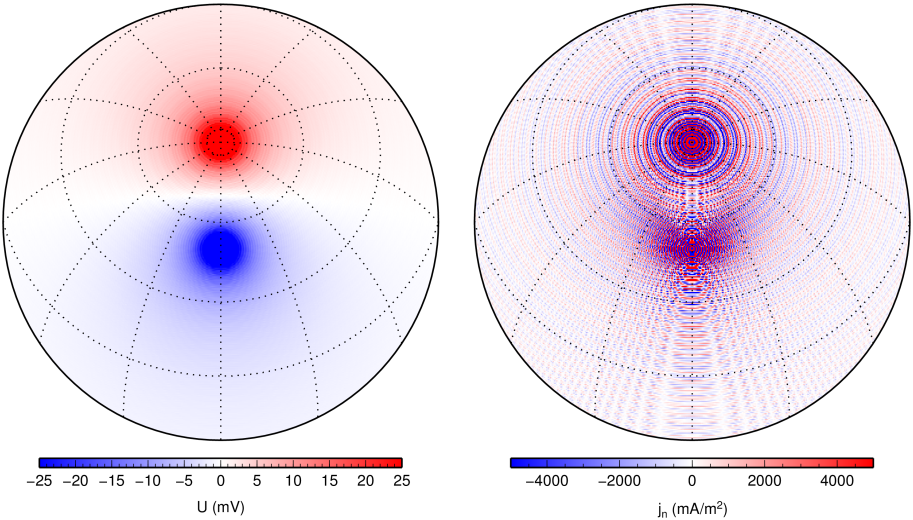

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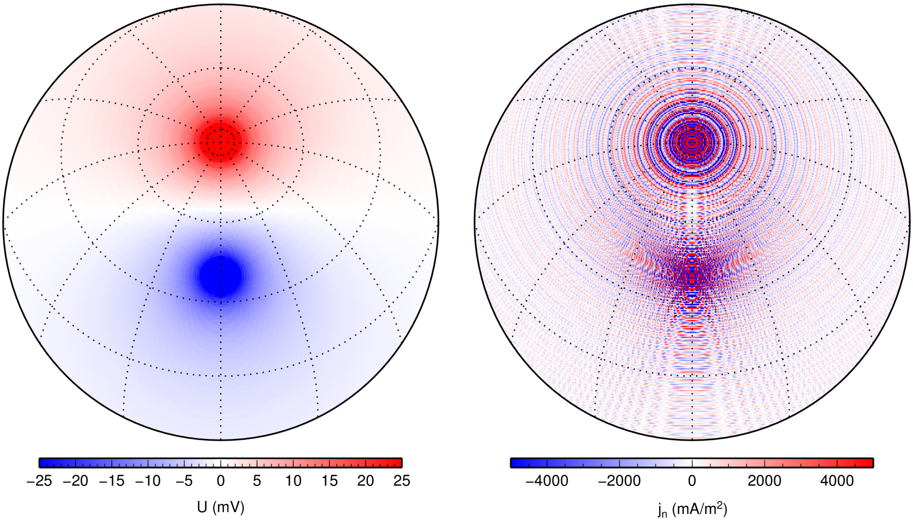

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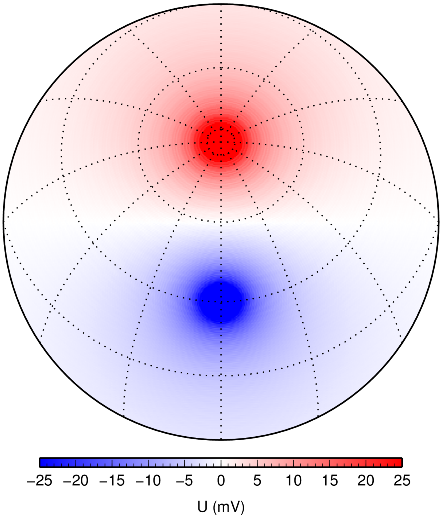

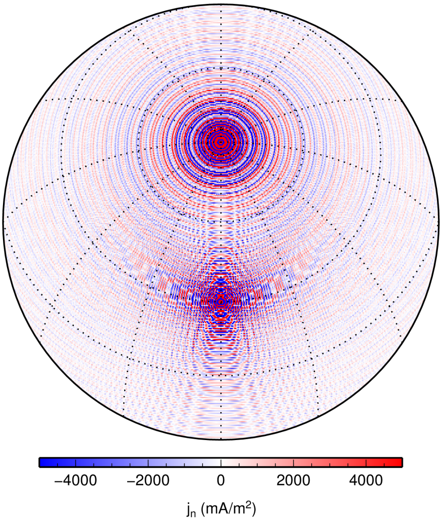

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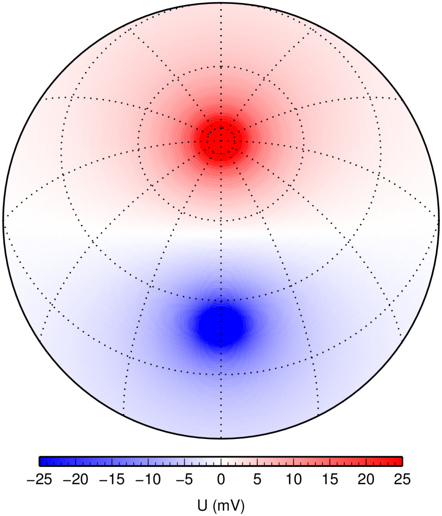

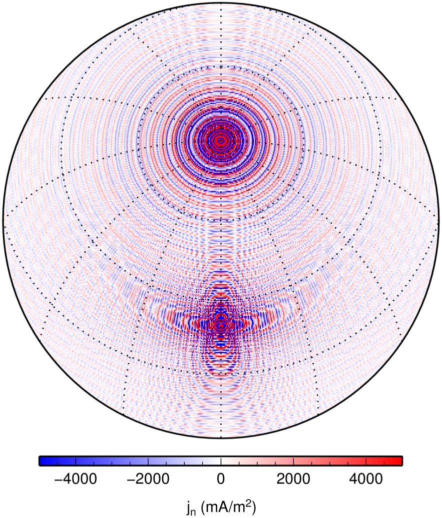

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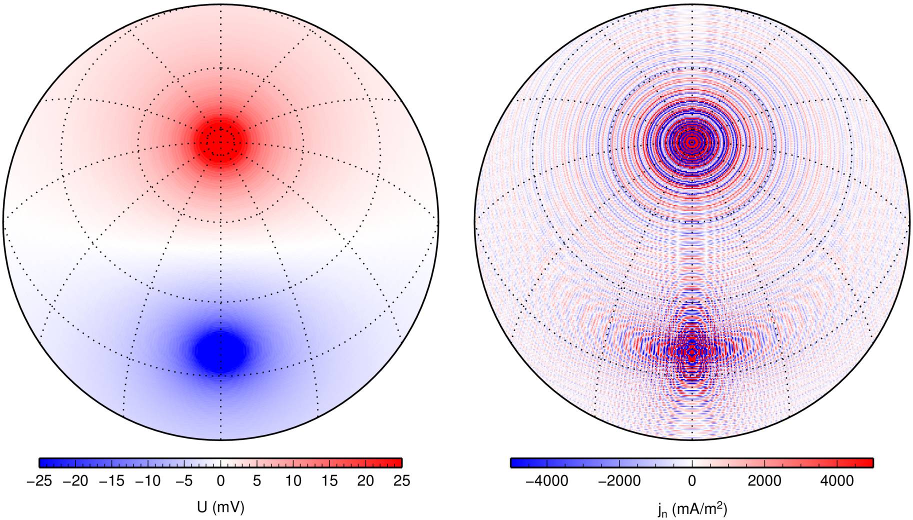

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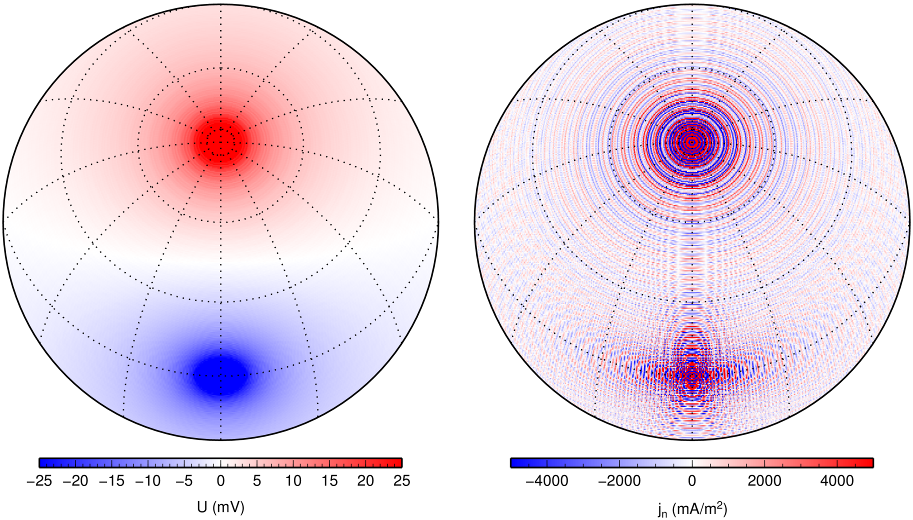

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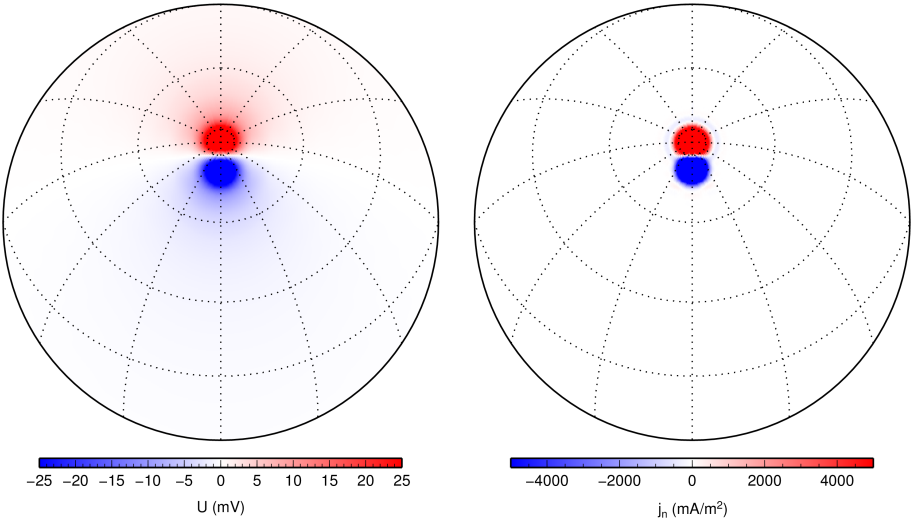

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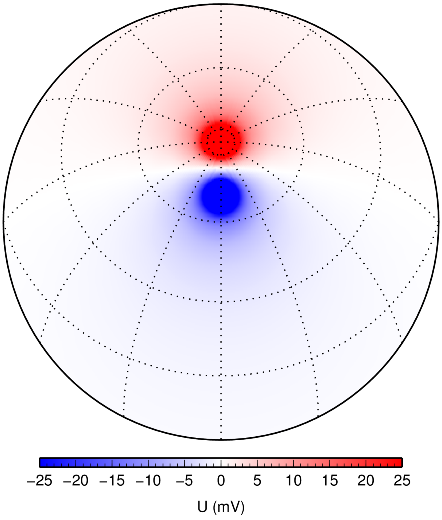

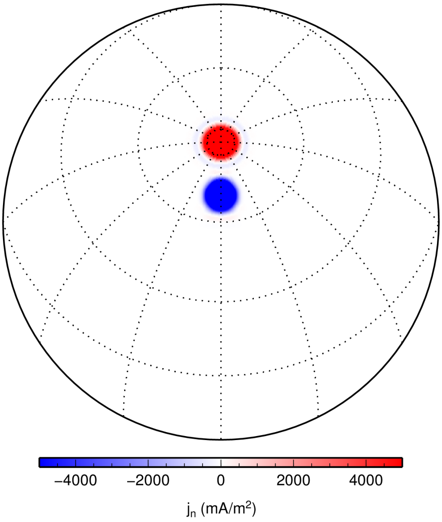

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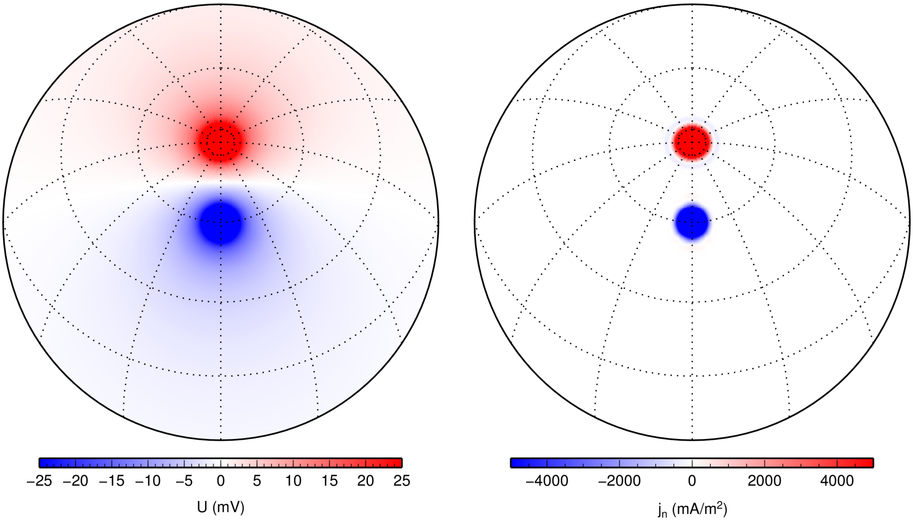

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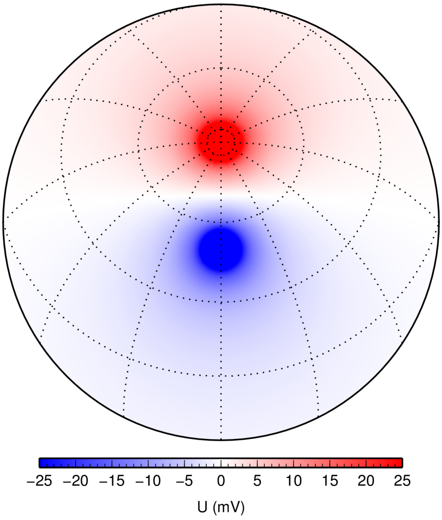

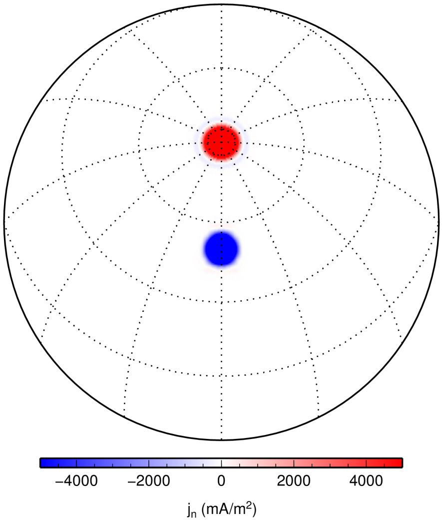

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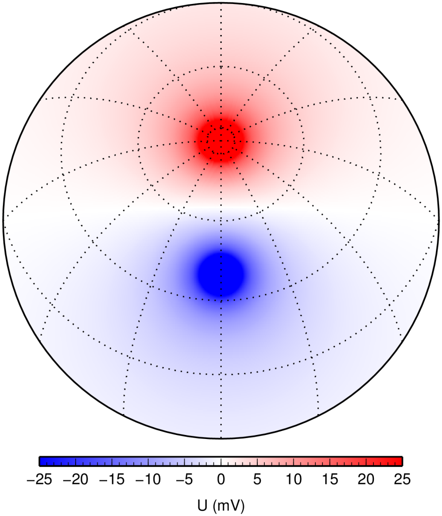
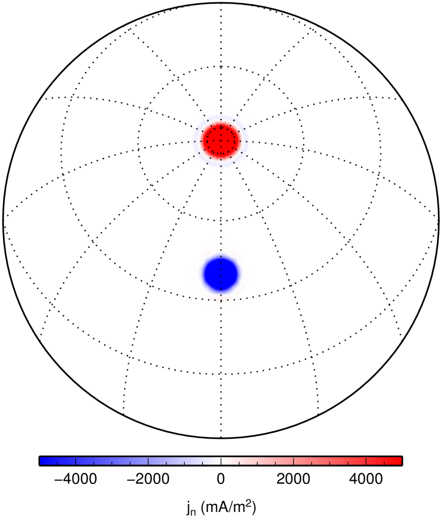
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (50°, 0°, -1\,\mathrm{mA})$$





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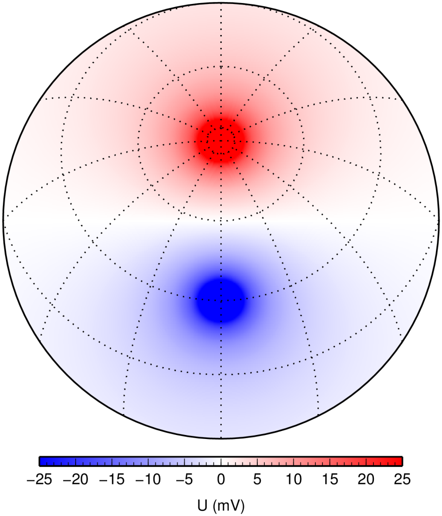

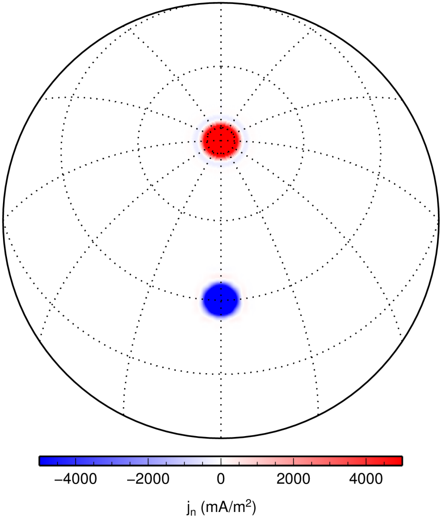

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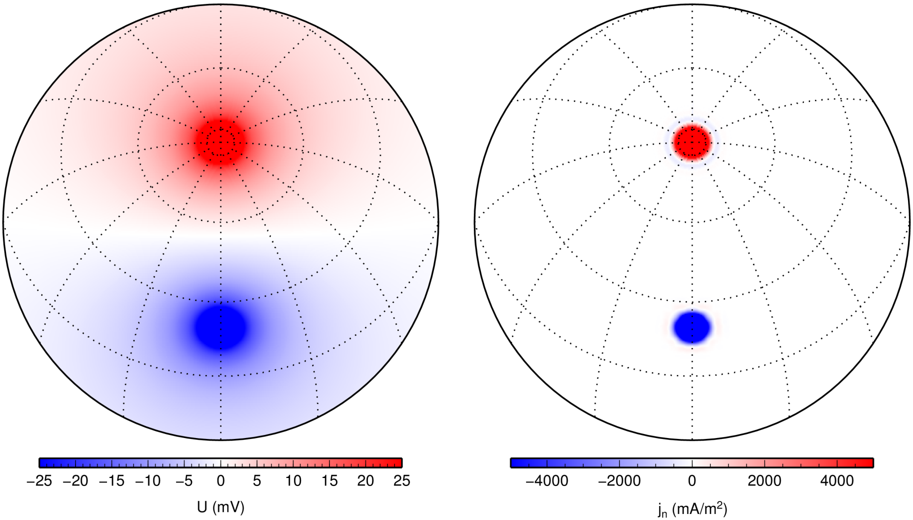

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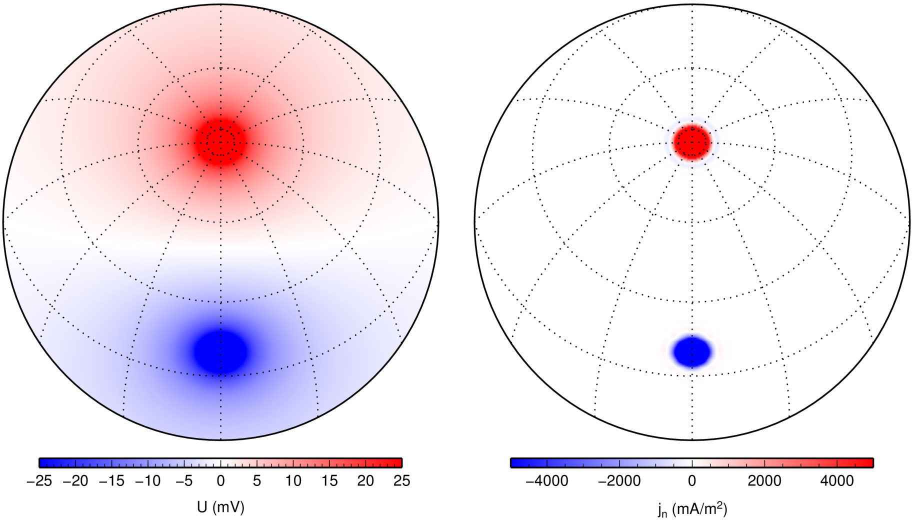

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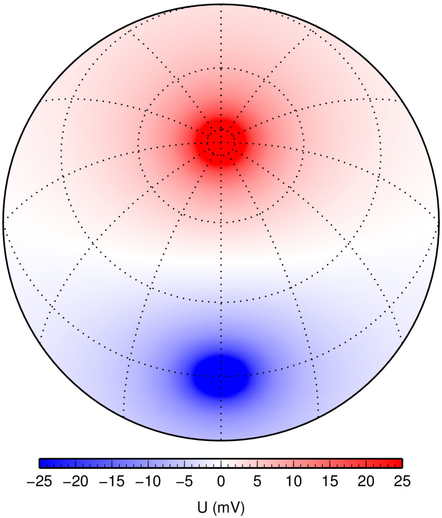

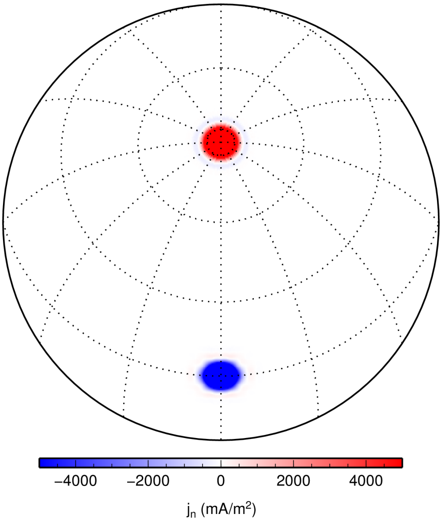

Analytical solution for a homogeneous sphere

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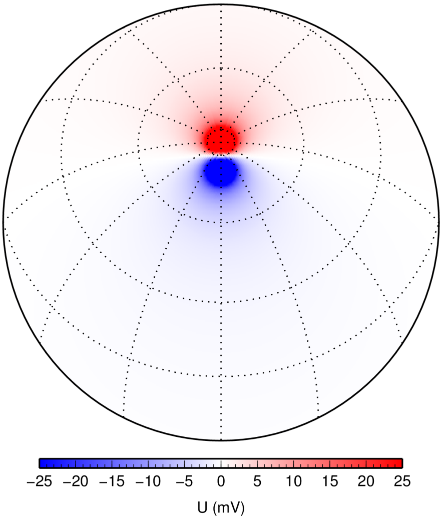

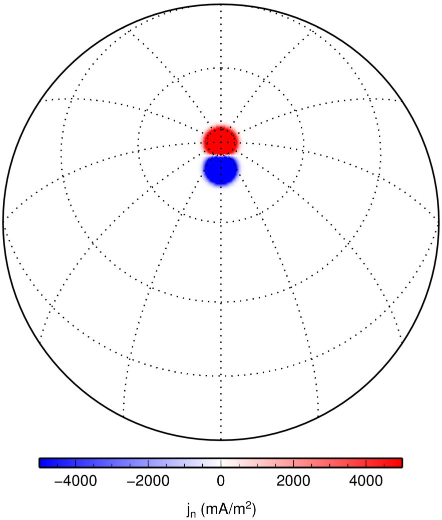

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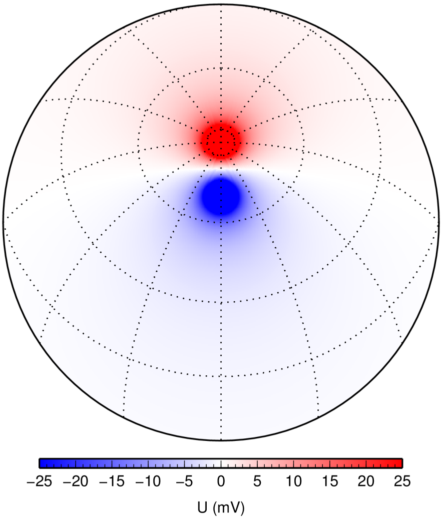

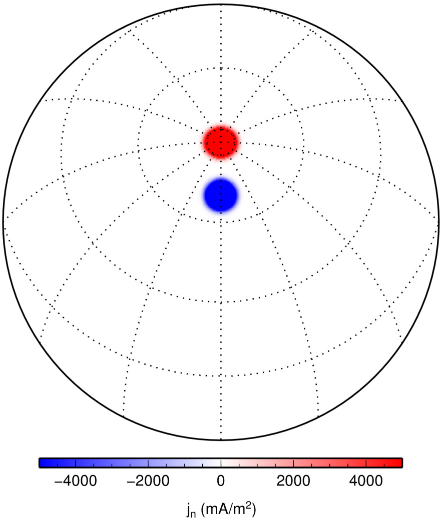

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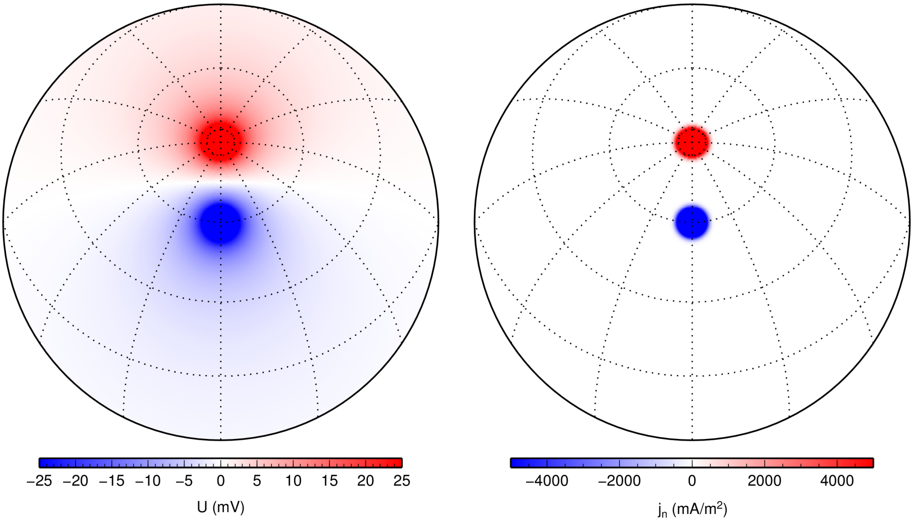

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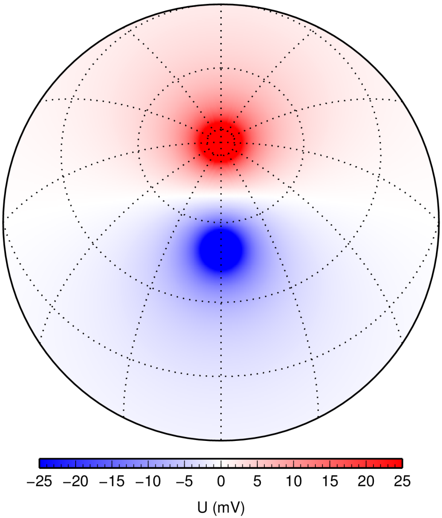

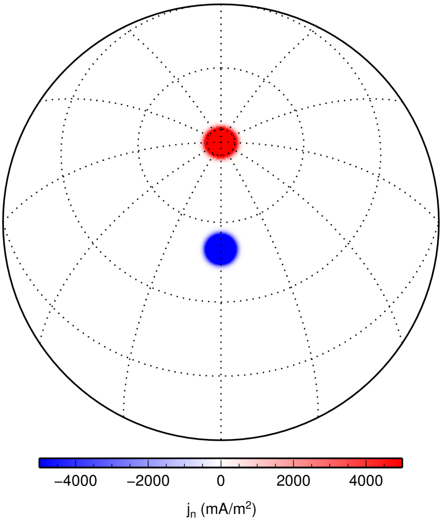

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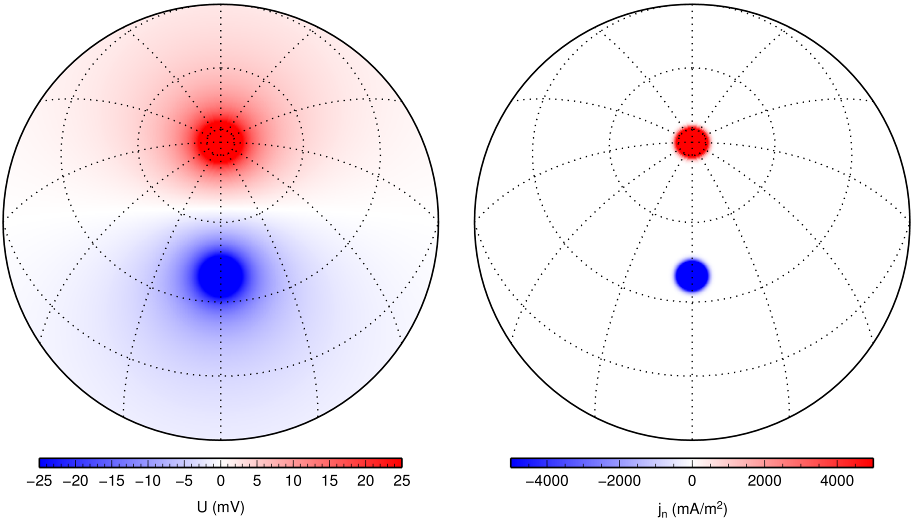

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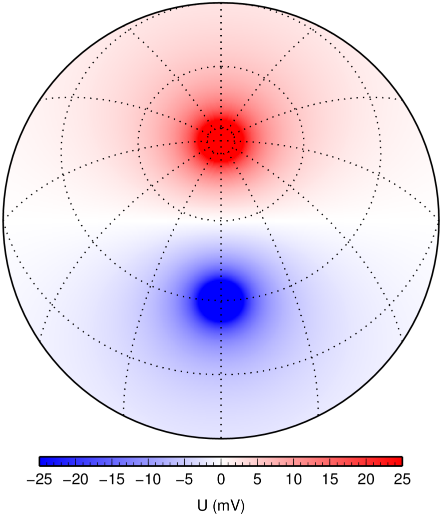

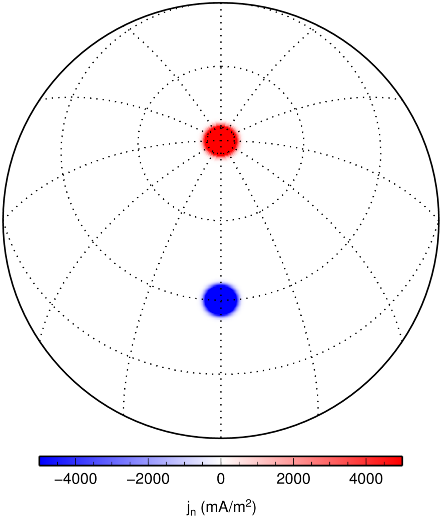

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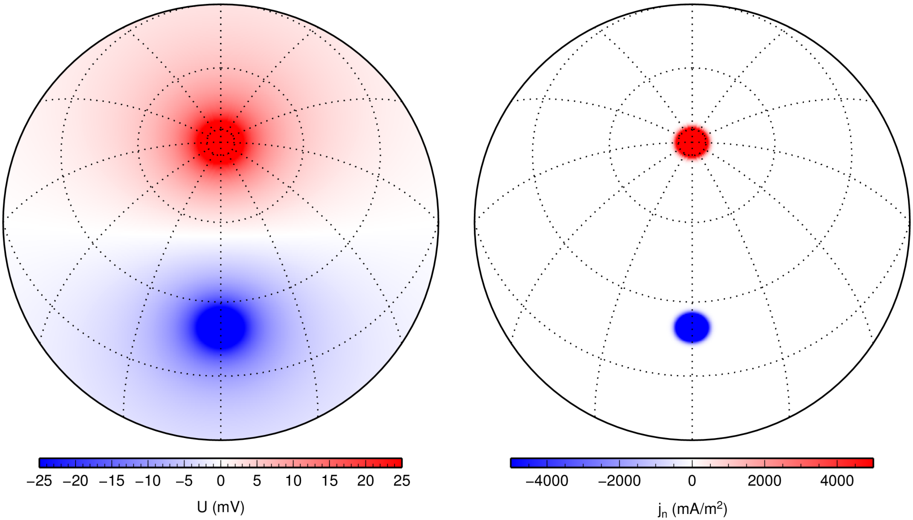

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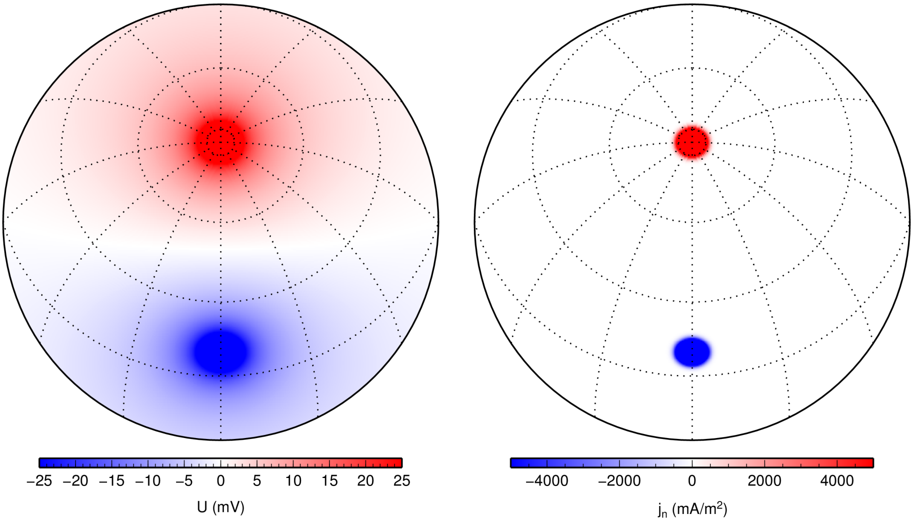

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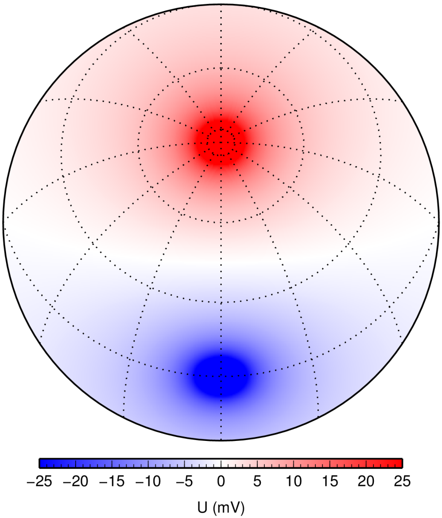
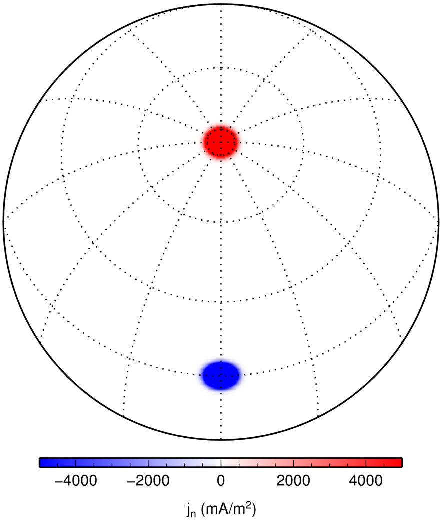




Weak formulation for a 3-D sphere

- Let $\tilde{\sigma} \in L_2(G)$, $\Re(\tilde{\sigma}) > 0$.
- Let $j(\Omega) \in W_1^{1/2}(\Gamma)$, such that $\int_\Gamma j(\Omega)\,\mathrm{d}S = 0$, be the imposed current density.
- Find $U(r;\Omega) \in W_1^2(G)$ such that $\forall \delta U(r;\Omega) \in W_1^2(G)$ holds

$$\int_G \tilde{\sigma}\,\nabla\overline{\delta U} \cdot \nabla U\,\mathrm{d}V \quad = \quad \int_\Gamma \overline{\delta U}\,j\,\mathrm{d}S.$$

Galerkin method and SH-FE discretization

$$\begin{pmatrix} U \\ \delta U \end{pmatrix}(r;\Omega) = \sum_{k=1}^{K+1} \sum_{n=1}^{N} \sum_{m=-n}^{n} \begin{pmatrix} U_{k,nm} \\ \delta U_{k,nm} \end{pmatrix} \psi_k(r)\, Y_{nm}(\Omega)$$

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

$$\mathbf{A}_{\iota\iota'} = \int_G \tilde{\sigma} \nabla \left(\psi_k \overline{Y}_{nm}\right) \cdot \nabla \left(\psi_{k'} Y_{n'm'}\right) \mathrm{d}V$$

$$\mathbf{x}_{\iota'} = \left\{ \left[\left(U_{k',n'm'}\right)_{m'=-n'}^{n'} \right]_{n'=1}^{N} \right\}_{k'=1}^{K+1}$$

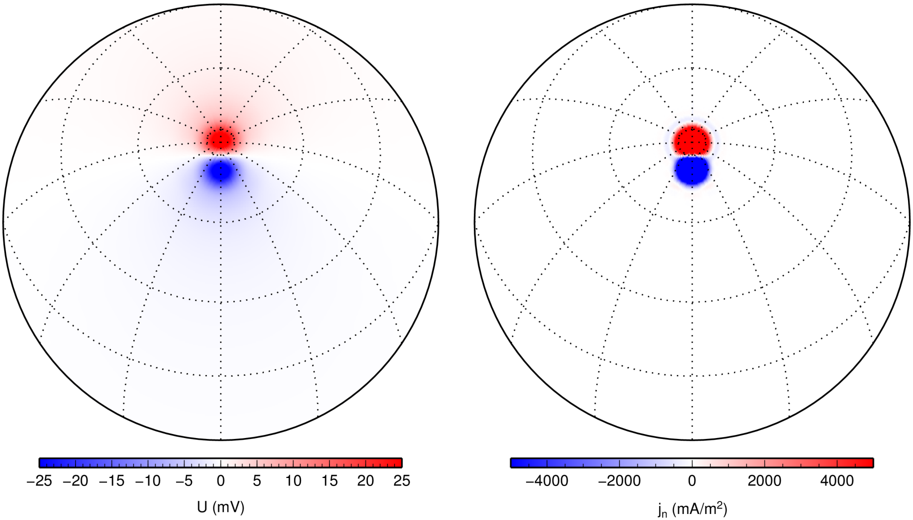
$$\mathbf{b}_{\iota} = \sum_{i=1}^{N_{\mathrm{I}}} I_i \, \overline{Y}_{nm}(\Omega_i) \cong \int_\Gamma \overline{Y}_{nm}(\Omega) \sum_{i=1}^{N_{\mathrm{I}}} I_i \, F(\Omega - \Omega_i) \, \mathrm{d}S$$

Galerkin method and SH-FE discretization

- $\mathbf{A}$ is a block-tridiagonal matrix with block size $N(N+2)$, symmetric and positive-definite (?) if $\Im(\tilde{\sigma}) = 0$.
- It can be quickly assembled using FFT-based routines.
- Tested algorithms: Thomas algorithm (for real case), partial LU decomposition of blocks (for complex case).
- Cyclic reduction?
- For iterative solvers (Krylov type), routines for fast matrix-vector product based on FFT are available. In that case, the matrix even doesn't have to be stored.
- $\mathbf{A}$ depends on $\tilde{\sigma}$, but remains unchanged for different electrode setups.
- $\mathbf{b}$ depends only on the positions of electrodes and imposed currents.
- $\mathbf{b}$ is non-zero only in the last $N(N+2)$ lines. This further simplifies the Thomas algorithm.

Galerkin method and SH-FE discretization

$$a = 5\,\mathrm{cm},\ \tilde{\sigma} = 1\,\mathrm{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\mathrm{mm}/a)$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (10^\circ, 0^\circ, -1\,\mathrm{mA})$$



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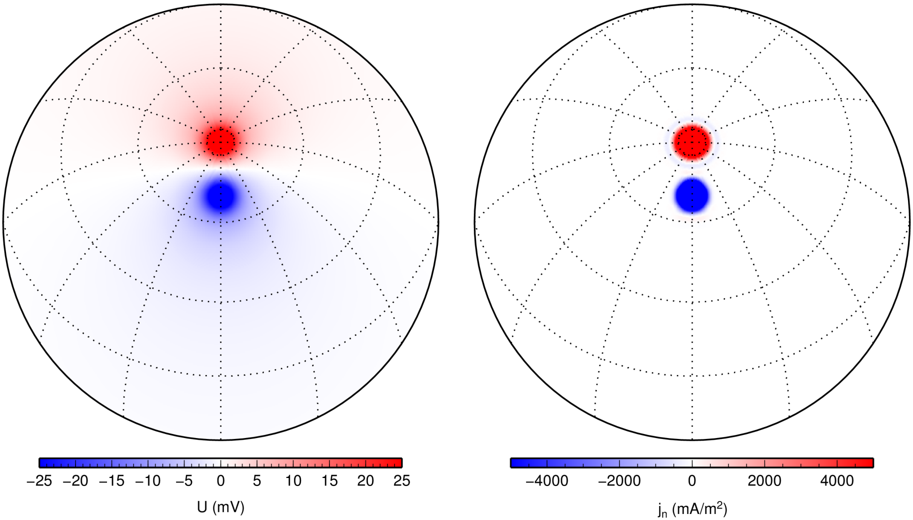

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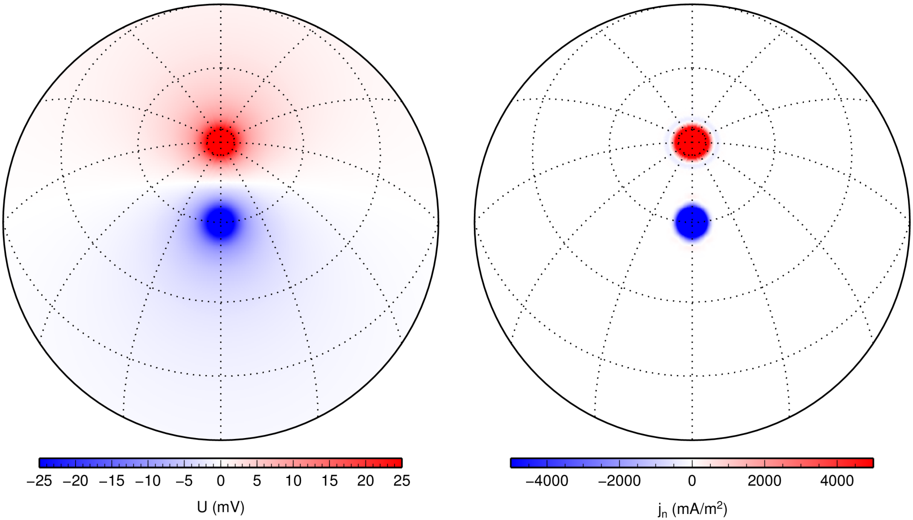

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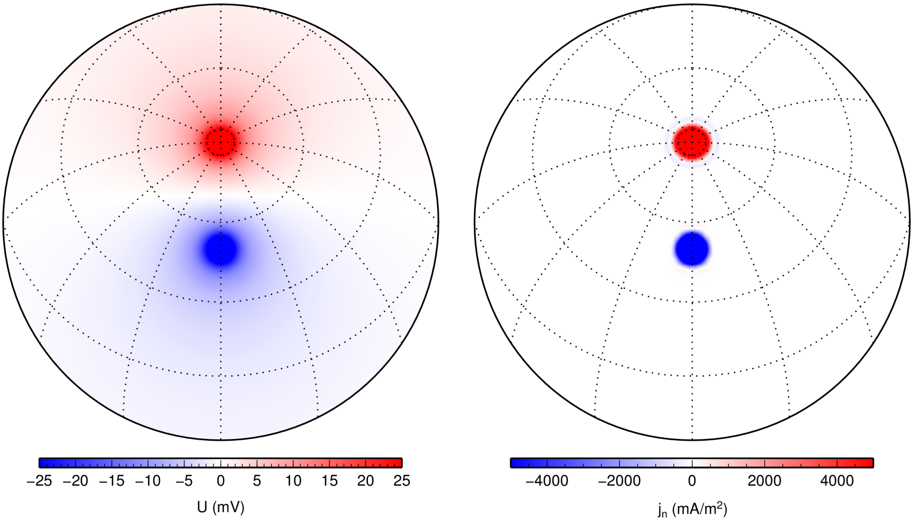

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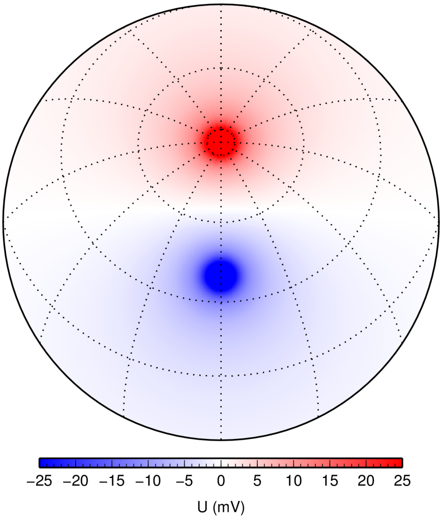

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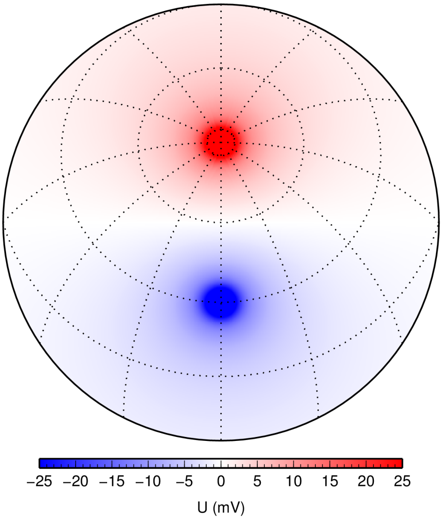

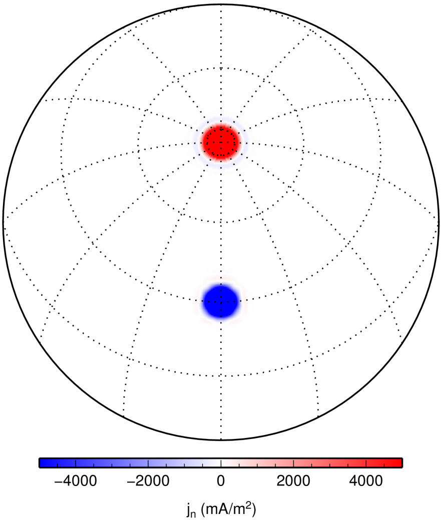

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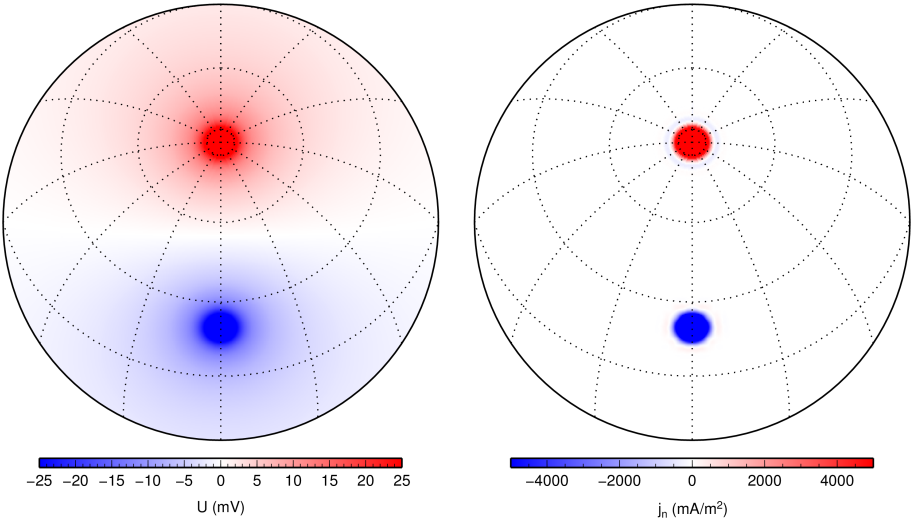

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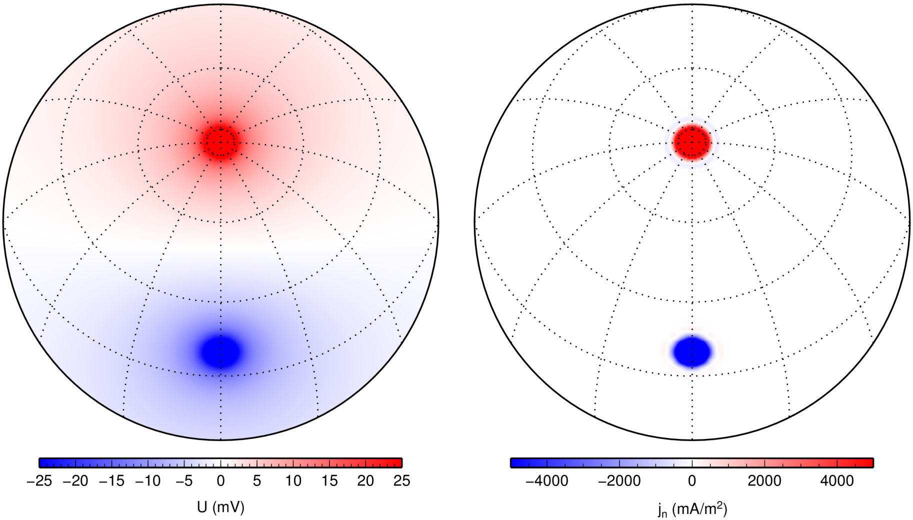

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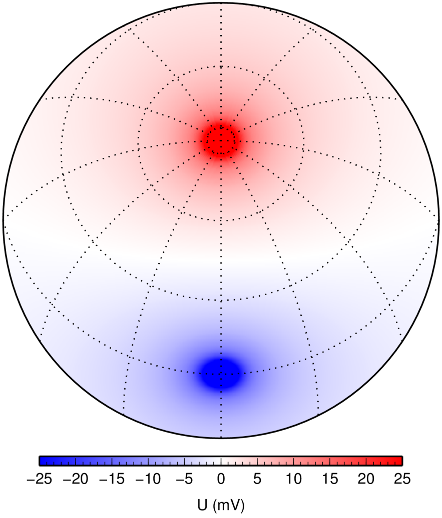

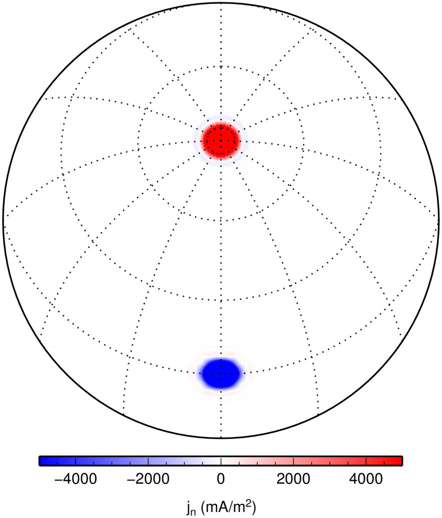

Galerkin method and SH-FE discretization

$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = 1\,\mathrm{S/m},\ \tilde{\sigma}_1 = 10\,\mathrm{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\mathrm{mm}/a)$$
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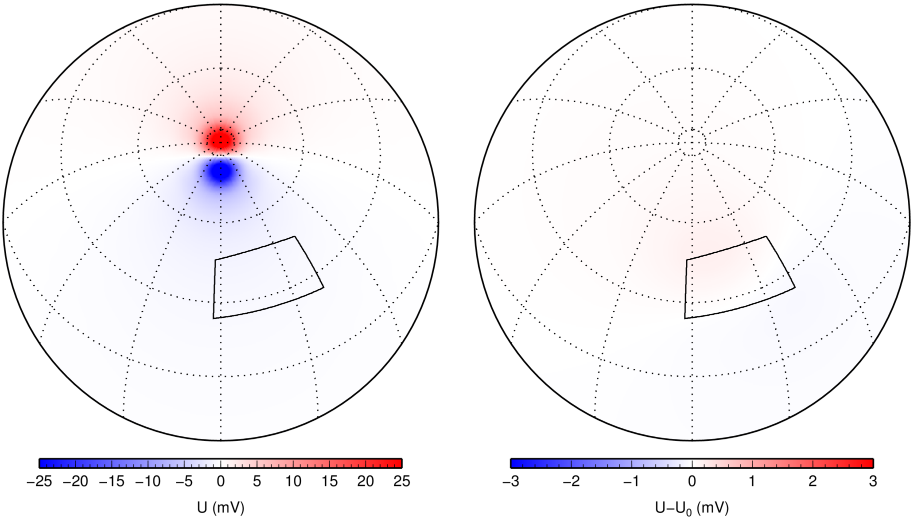

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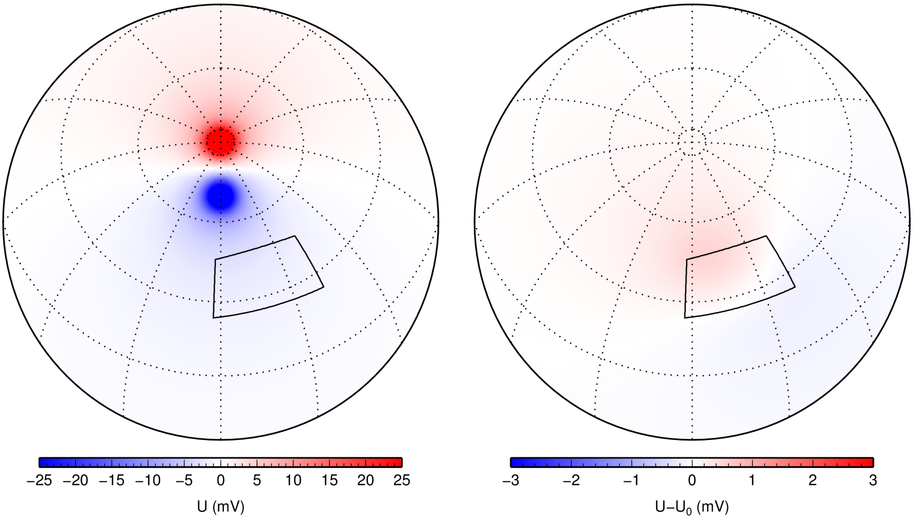

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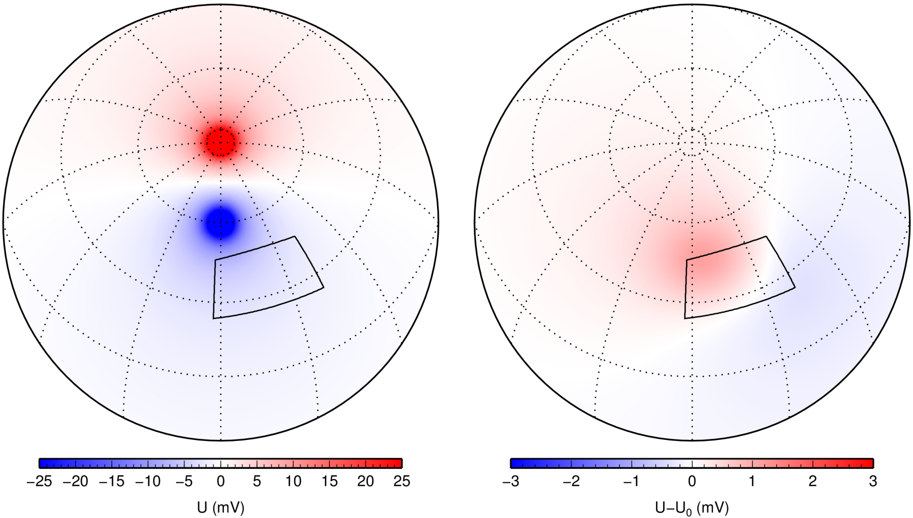

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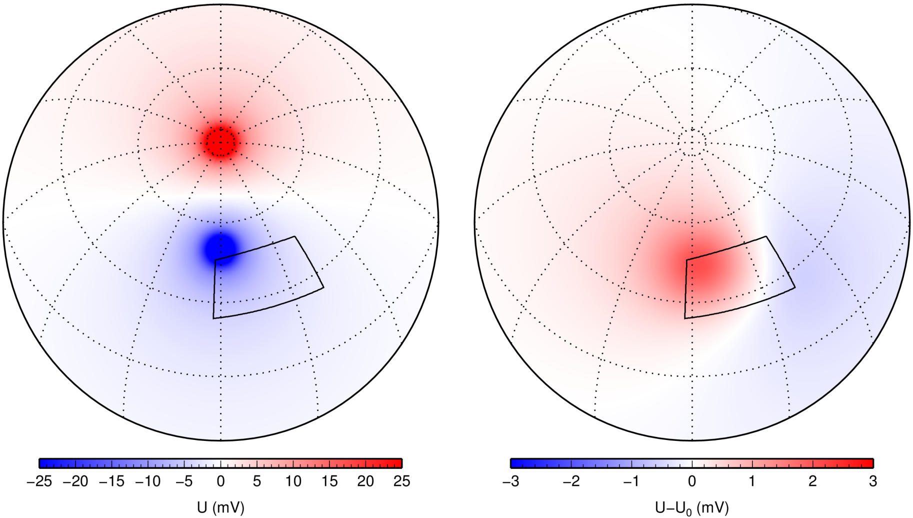

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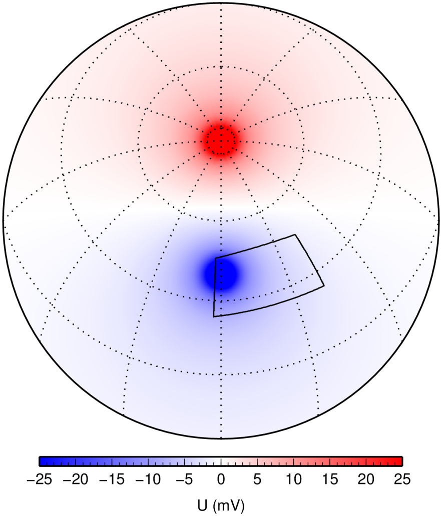
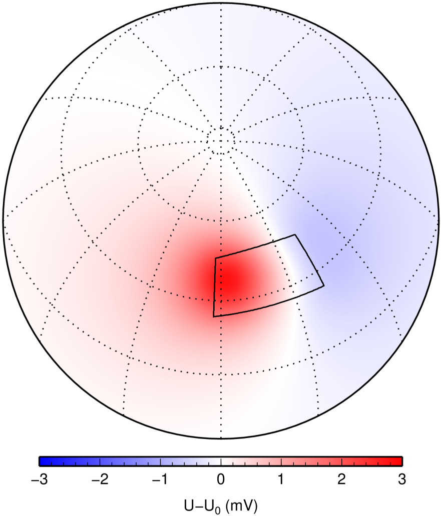
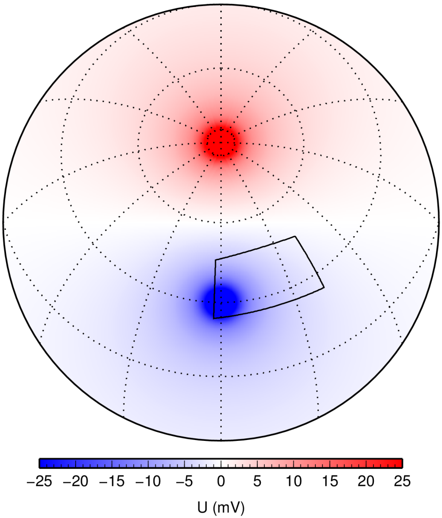
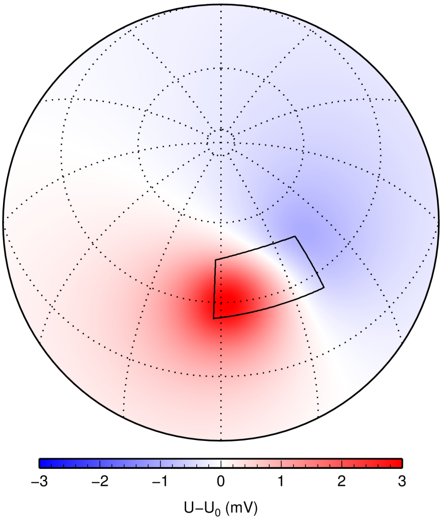




Galerkin method and SH-FE discretization

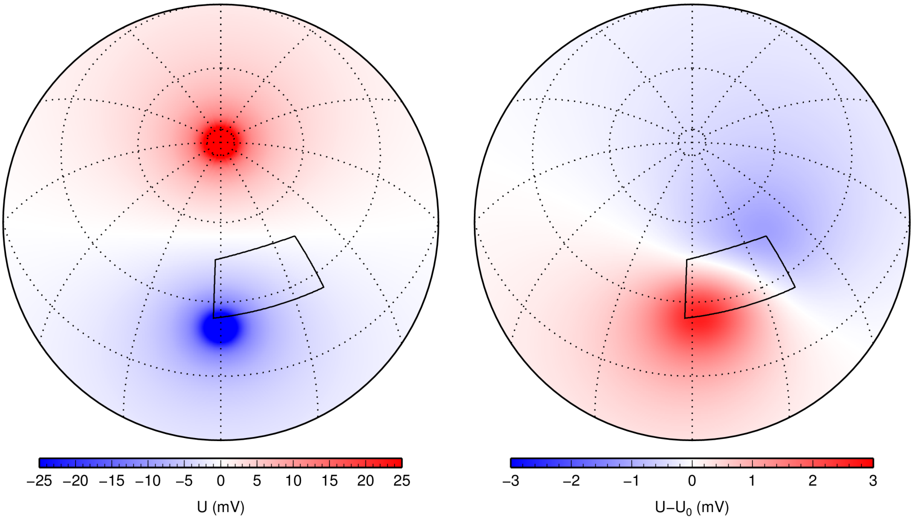
$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = 1\,\text{S/m},\ \tilde{\sigma}_1 = 10\,\text{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\text{mm}/a)$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (60^\circ, 0^\circ, -1\,\text{mA})$$





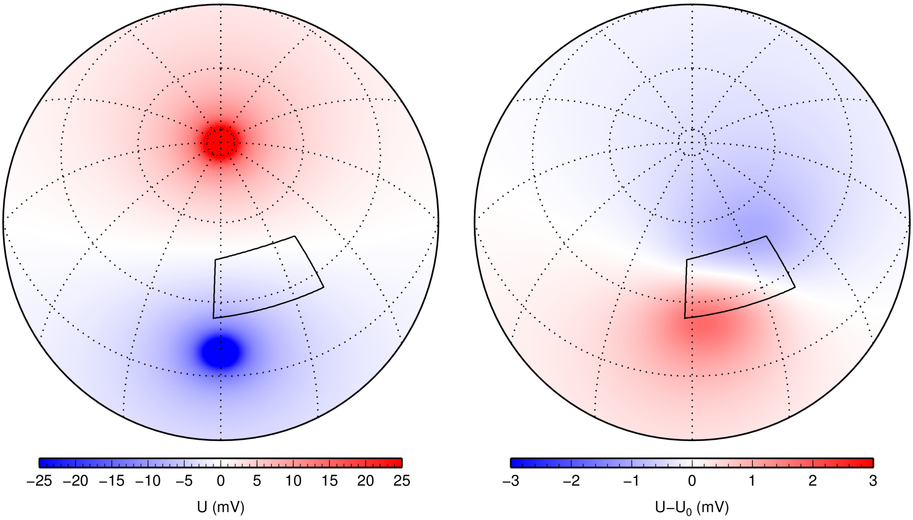
Galerkin method and SH-FE discretization

$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = 1\,\text{S/m},\ \tilde{\sigma}_1 = 10\,\text{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\text{mm}/a)$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (70°, 0°, -1\,\text{mA})$$



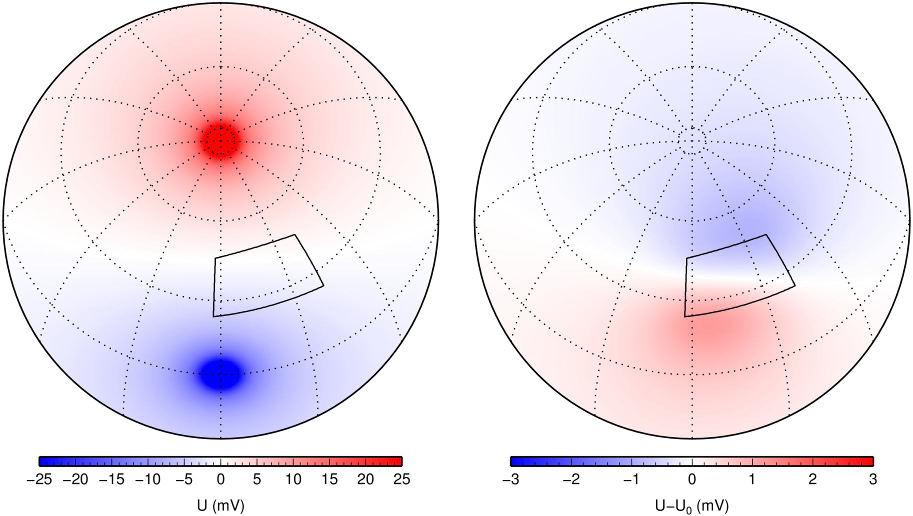
Galerkin method and SH-FE discretization

$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = 1\,\mathrm{S/m},\ \tilde{\sigma}_1 = 10\,\mathrm{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\mathrm{mm}/a)$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (80°, 0°, -1\,\mathrm{mA})$$



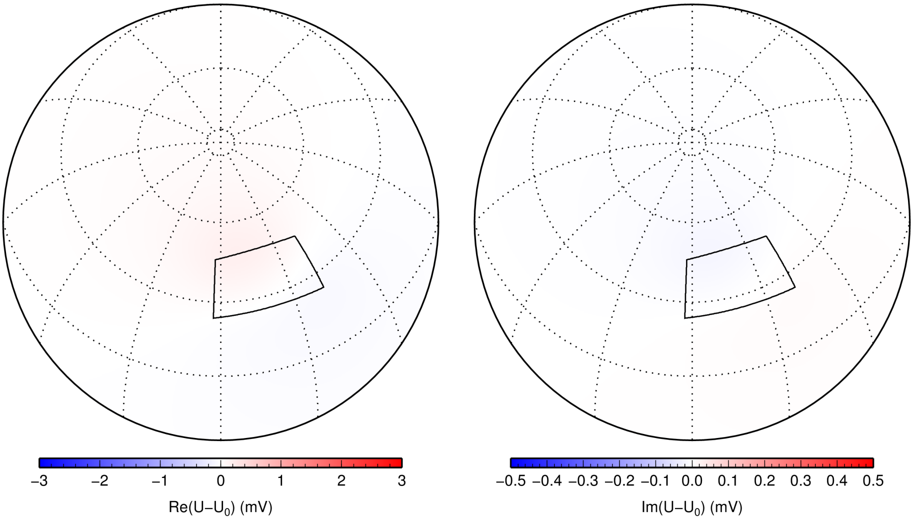
Galerkin method and SH-FE discretization

$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = 1\,\mathrm{S/m},\ \tilde{\sigma}_1 = 10\,\mathrm{S/m},\ N = 40,\ K = 20,\ s = 1 - \cos(4\,\mathrm{mm}/a)$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (90°, 0°, -1\,\mathrm{mA})$$



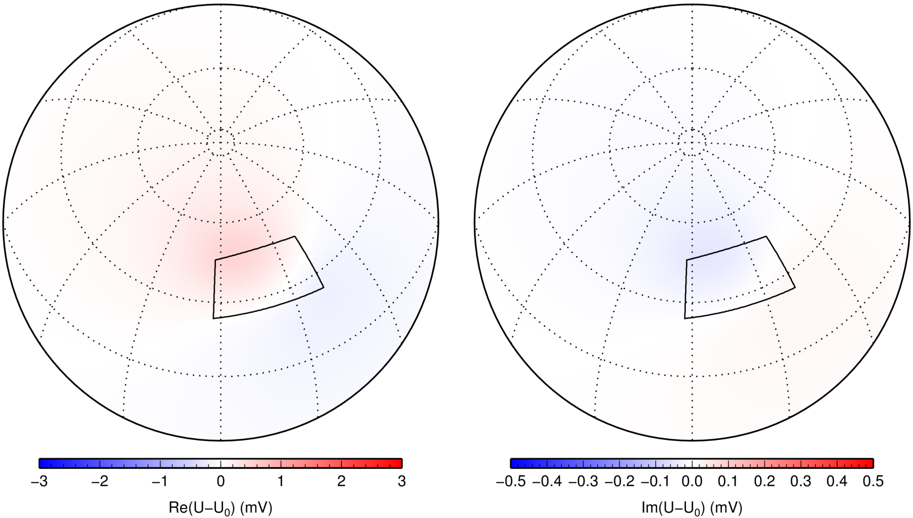
Galerkin method and SH-FE discretization

$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,{}^{\text{S}}/_{\text{m}},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,{}^{\text{S}}/_{\text{m}},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (10^\circ, 0^\circ, -1\,\text{mA})$$



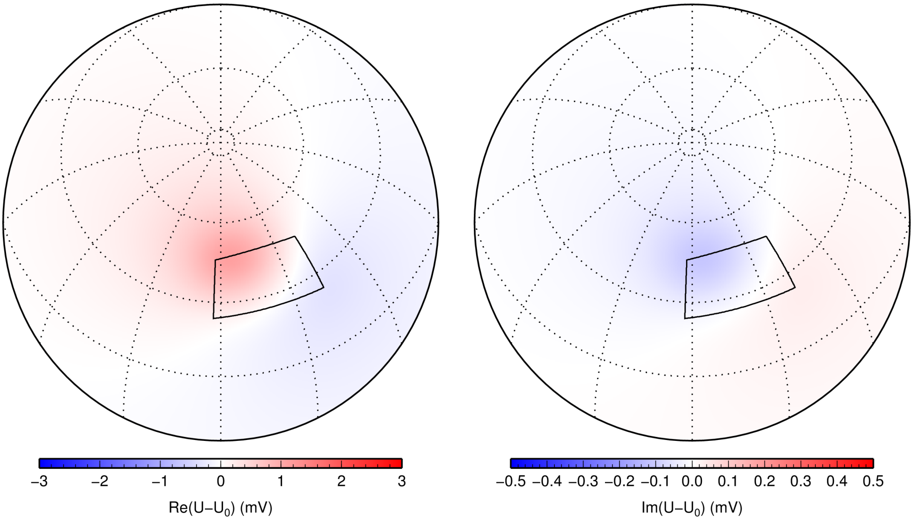
Galerkin method and SH-FE discretization

$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = (1 + 0.1\mathrm{i})\,\mathrm{S/m},\ \tilde{\sigma}_1 = (10 + 1\mathrm{i})\,\mathrm{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (20°, 0°, -1\,\mathrm{mA})$$



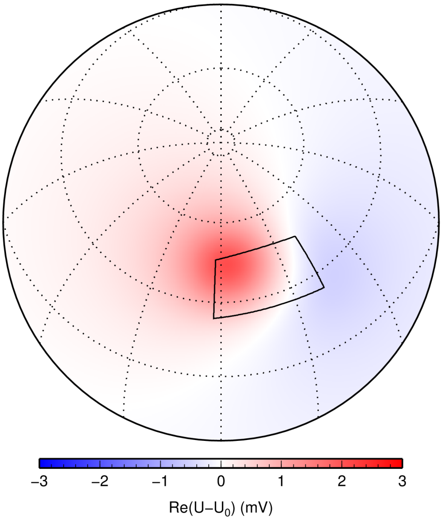
Galerkin method and SH-FE discretization

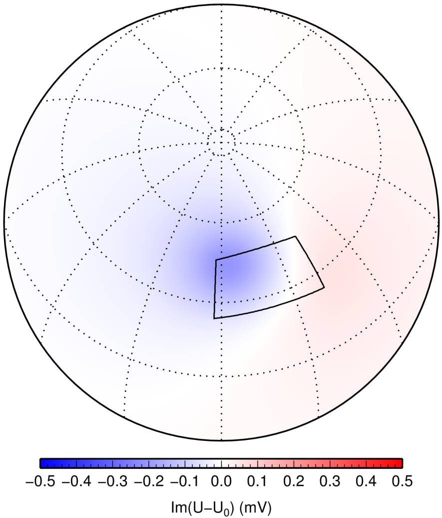
$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = (1 + 0.1\mathrm{i})\,{}^{\mathrm{S}}\!/\!_{\mathrm{m}},\ \tilde{\sigma}_1 = (10 + 1\mathrm{i})\,{}^{\mathrm{S}}\!/\!_{\mathrm{m}},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (30°, 0°, -1\,\mathrm{mA})$$



Galerkin method and SH-FE discretization

$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,\text{S/m},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,\text{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (40°, 0°, -1\,\text{mA})$$

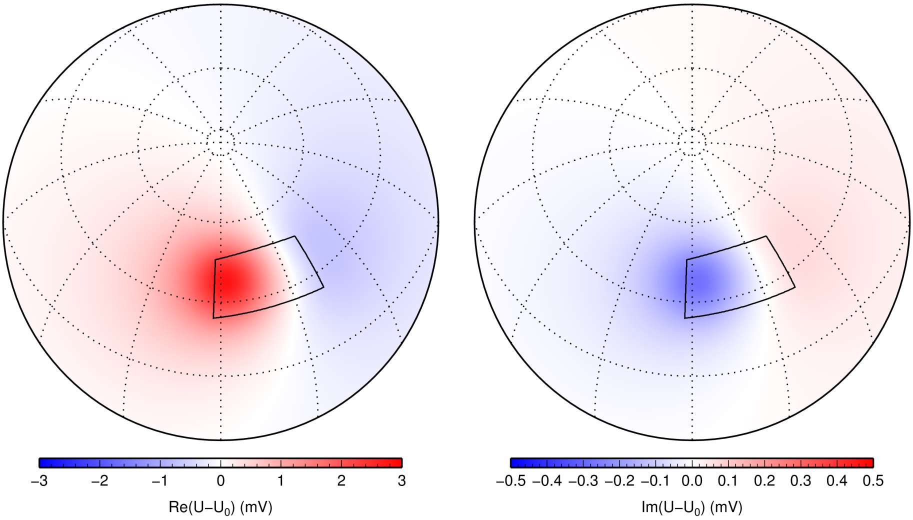




Galerkin method and SH-FE discretization

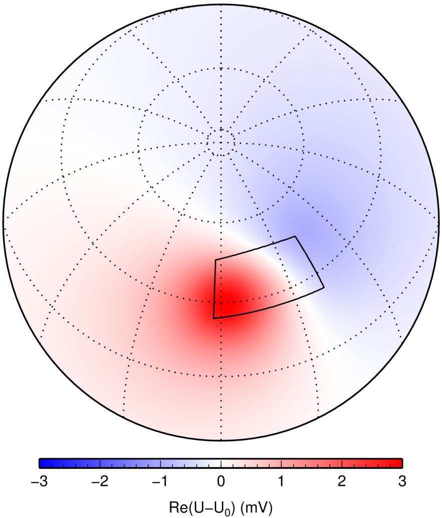
$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,\text{S/m},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,\text{S/m},\ N = 40,\ K = 20$$

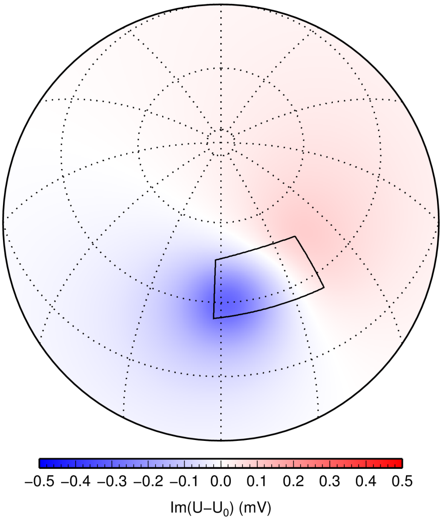
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (50^\circ, 0^\circ, -1\,\text{mA})$$



Galerkin method and SH-FE discretization

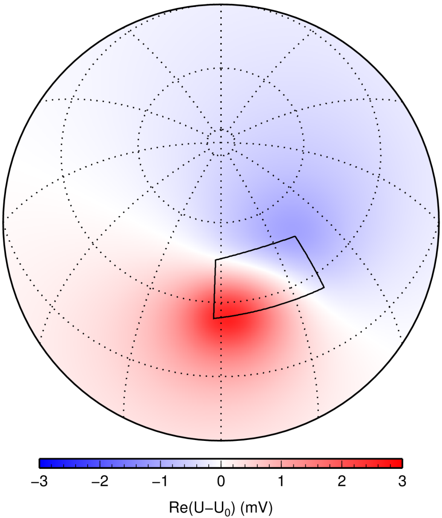
$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,\text{S/m},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,\text{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (60^\circ, 0^\circ, -1\,\text{mA})$$

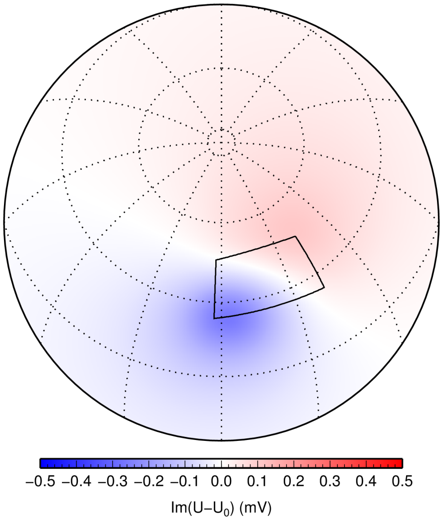




Galerkin method and SH-FE discretization

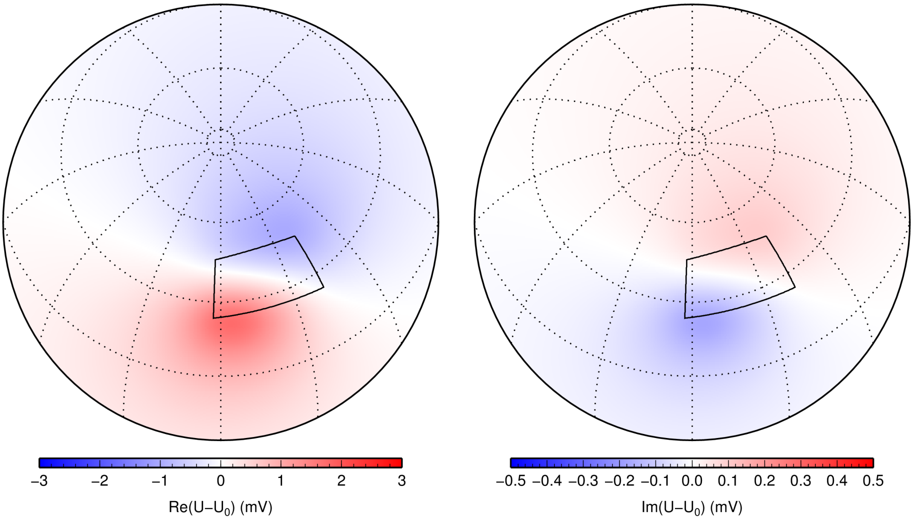
$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,\text{S/m},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,\text{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (70^\circ, 0^\circ, -1\,\text{mA})$$





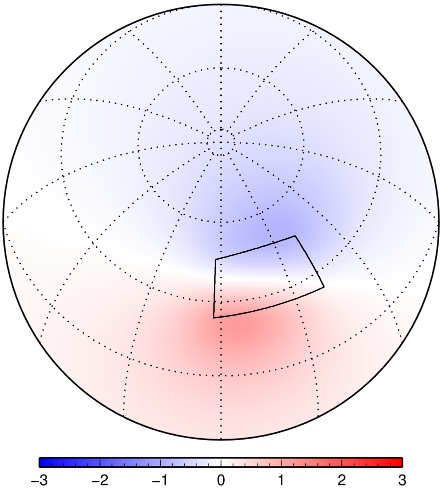
Galerkin method and SH-FE discretization

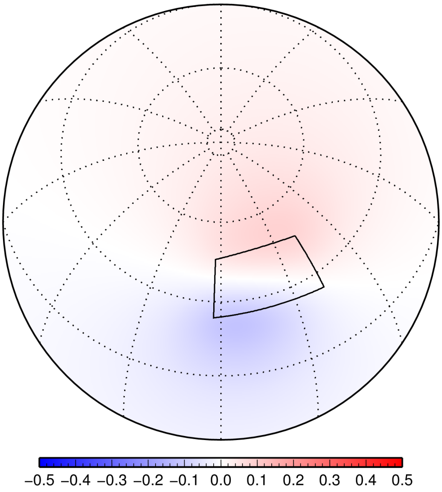
$$a = 5\,\mathrm{cm},\ \tilde{\sigma}_0 = (1 + 0.1\mathrm{i})\,\mathrm{S/m},\ \tilde{\sigma}_1 = (10 + 1\mathrm{i})\,\mathrm{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0°, 0°, 1\,\mathrm{mA}),\ (\vartheta_2, \varphi_1, I_2) = (80°, 0°, -1\,\mathrm{mA})$$



Galerkin method and SH-FE discretization

$$a = 5\,\text{cm},\ \tilde{\sigma}_0 = (1 + 0.1\text{i})\,\text{S/m},\ \tilde{\sigma}_1 = (10 + 1\text{i})\,\text{S/m},\ N = 40,\ K = 20$$
$$(\vartheta_1, \varphi_1, I_1) = (0^\circ, 0^\circ, 1\,\text{mA}),\ (\vartheta_2, \varphi_1, I_2) = (90^\circ, 0^\circ, -1\,\text{mA})$$





Galerkin method and SH-FE discretization

- Runtimes for $N = 40$, $K = 20$, $N_\mathrm{I} = 2$
- Intel Core i5-4300U CPU @ 1.90GHz, 2 cores, auto-parallelization
- Intel Fortran Compiler 13.0.1, Intel Math Kernel Library 11.0

Operation	Real case	Complex case
Matrix assembly	22 s	45 s
Matrix factorization	40 s	202 s
Right-hand-side assembly	$<$1 s for 9 runs	
Solution	$<$1 s	2 s

Remarks on the inverse problem

- Model vector, residua, data misfit and regularization:

$$\begin{aligned}
\mathbf{m} &= \left\{\log_{10}\left(\sigma^{k,ij}/\sigma_0\right), \log_{10}\varepsilon_r^{k,ij}\right\} \\
\Delta U_{kl}^{p} &= \Big[U\left(\mathbf{m}; j_p; \Omega_k\right) - U\left(\mathbf{m}; j_p; \Omega_l\right)\Big] - \Big[U_k^{(\mathrm{o}),p} - U_l^{(\mathrm{o}),p}\Big] \\
&= \int\limits_{\Gamma} U\left(\Omega\right)\left[\delta(\Omega-\Omega_k) - \delta(\Omega-\Omega_l)\right]\,\mathrm{d}S - \Big[U_k^{(\mathrm{o}),p} - U_l^{(\mathrm{o}),p}\Big] \\
\chi_p^2(\mathbf{m}) &= \frac{1}{2}\sum_{k=1}^{N_\mathrm{U}}\sum_{l=k+1}^{N_\mathrm{U}} \frac{\left|\Delta U_{kl}^{p}\right|^2}{\Delta_{kl}^2} \\
\tilde{\mathbf{m}}(\lambda) &= \min_{\mathbf{m}}\left[\sum_{p=1}^{N_\mathrm{E}}\chi_p^2(\mathbf{m}) + \lambda R^2(\mathbf{m})\right]
\end{aligned}$$

- Gradient of misfit in model space:

$$D_{\mathbf{m}}\chi_p^2(\mathbf{m}) = \Re\sum_{kl}\int\limits_{\Gamma}\frac{\overline{\Delta U_{kl}^{p}}}{\Delta_{kl}^2}\left[\delta(\Omega-\Omega_k) - \delta(\Omega-\Omega_l)\right] D_{\mathbf{m}}U\,\mathrm{d}S$$

Remarks on the inverse problem

- Derivation of the adjoint problem for $\hat{U}(\Omega;\hat{j}_p)$

$$
\begin{aligned}
0 &= \int_G \hat{U}\, D_{\mathbf{m}} \nabla \cdot (\tilde{\sigma}\, \nabla U)\, \mathrm{d}V \\
&= -\int_G \nabla \hat{U} \cdot D_{\mathbf{m}} \left(\tilde{\sigma} \nabla U\right)\, \mathrm{d}V + \int_\Gamma \hat{U} D_{\mathbf{m}} \left(\tilde{\sigma} \boldsymbol{e}_r \cdot \nabla U\right)\, \mathrm{d}S \\
&= -\int_G D_{\mathbf{m}} \tilde{\sigma}\, \nabla \hat{U} \cdot \nabla U\, \mathrm{d}V - \int_G \tilde{\sigma}\, \nabla \hat{U} \cdot \nabla D_{\mathbf{m}} U\, \mathrm{d}V + 0 \\
&= -\int_G D_{\mathbf{m}} \tilde{\sigma}\, \nabla \hat{U} \cdot \nabla U\, \mathrm{d}V \\
&\quad + \int_G \nabla \cdot \left(\tilde{\sigma}\, \nabla \hat{U}\right) D_{\mathbf{m}} U\, \mathrm{d}V - \int_\Gamma \tilde{\sigma} \boldsymbol{e}_r \cdot \nabla \hat{U}\, D_{\mathbf{m}} U\, \mathrm{d}S
\end{aligned}
$$

Remarks on the inverse problem

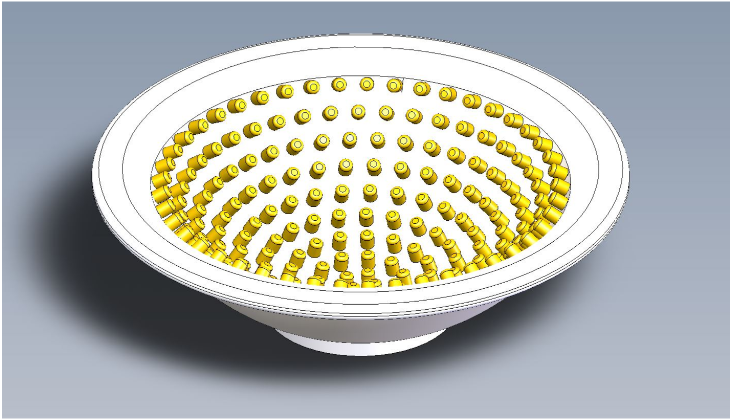
- Derivation of the adjoint problem for $\hat{U}(\Omega;\hat{j}_p)$

$$\begin{aligned}
\nabla \cdot \left(\tilde{\sigma}\, \nabla \hat{U}\right) &= 0 \quad \text{in } G \\
\tilde{\sigma} \boldsymbol{e}_r \cdot \nabla \hat{U} &= \hat{j}_p \quad \text{on } \Gamma \\
&= \sum_{kl} \frac{\overline{\Delta U_{kl}^p}}{\Delta_{kl}^2} \left[\delta(\Omega - \Omega_k) - \delta(\Omega - \Omega_l)\right] \\
D_{\mathbf{m}} \chi_p^2(\mathbf{m}) &= -\Re \int_G D_{\mathbf{m}} \tilde{\sigma}\, \nabla \hat{U} \cdot \nabla U \, \mathrm{d}V
\end{aligned}$$

- Adjoint solution $\hat{U}$ is obtained by solving the same system with right-hand side based on the residua of the forward run
- Likely to replace $\delta(\Omega - \Omega_l)$ by $F(\Omega - \Omega_l)$

Current status

Hardware



- functional prototype of flat sensor head
- spherical sensor head in construction

Current status

Software

- 3-D forward spherical solver working but needs testing and could use some acceleration
- 3-D inverse spherical solver in development (based on limited memory quasi-Newton minimization with adjoint sensitivities)
- alternative approach using 2-D finite-element modelling in circular cross-sections (p. Pšenka) www.eit.org.uk