A model of rupture of lithospheric faults with re-occurring earthquakes

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reflecting collaboration with

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Content of the talk:

1. Model formulation
   - Main phenomena to capture, state of art
   - Model in the bulk
   - Model on the fault, the complete model
2. Analysis
   - Semi-implicit time discretization
   - A-priori estimates
   - Convergence towards weak solutions
3. Computational experiments with 1-DOF slider

Main phenomena to capture:

+ short time scales ($\ll$1 mil. yrs) — small-strain concept suffices,
+ tectonic earthquakes: sudden activated damage,
+ recovery after earthquakes: healing/rebonding,
+ no memory of previous configurations before a last earthquake,
+ fluidic-like aseismic response under slow motions ($\sim 1\mu$m/yr): Maxwell-type rheology (also only small attenuation of seismic waves: the ratio $\frac{\text{dissipated energy per period}}{\text{stored energy}} =: \frac{2\pi}{Q}$ is small;
  Q = the "quality factor", its typical values in Earth $\sim 10^{3\pm1}$,
+ emission and propagation of seismic waves: inertia needed.

Neglected phenomena (e.g.):

– temperature variations (in particular, no volcanic earthquakes),
– "multi-Maxwell" rheology,
– possible opening of the fault (Signorini contact) $\Leftarrow$ big geostatic pressure,
– etc. etc.

State of art in seismic modelling: huge modelling activity during decades worldwide

- multiscale problem in time
  (slow motion between earthquakes vs fast earthquakes)
  but often different time-scalles modelled separately
- multiscale problem in space (large bulk vs narrow faults):
  but models in bulk does not relate with models on the faults
- healing towards (nearly) original configuration – not desired
  but typically either no plastic strain or damage-determined plastic strain
  – validity for only short times
- friction concept on the faults (Dieterich-Ruina model):
  but friction coefficient (depending on "ageing") often allowed negative
- numerical schemes without guarancy of stability or convergence
- no mathematical analysis of continuous models
- no energetics traced numerically, and mostly nor theoretically.

Inelastic ("plastic-like") model occasionally used in geo-physics:

the goal: to record some irreversible strain:

irreversible strain ε_{ij}^{v},
damage $1-\alpha$
($\alpha = 0$ =no damage)
($\alpha = 1$ =complete damage)

Usual damage dynamics:
$\dot{\alpha} = -c\,\mathcal{E}'_\alpha$
with $\mathcal{E}$ the stored energy,
sometimes even
$\dot{\alpha} = -c\left(\mathcal{E} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}\right)'_\alpha$
with $\mathcal{R}$ the viscous dissipation.

Comparison between theoretical predictions and the observed deformation and acoustic emissions from laboratory experiments in granites and sandstones led Hamiel *et al.* (2004a) to incorporate gradual accumulation of a damage-related non-reversible deformation. This irreversible (inelastic) strain, ε_{ij}^{v}, starts to accumulate with the onset of acoustic emission and the rate of its accumulation is suggested to be proportional to the rate of damage increase:

$$\frac{d\varepsilon_{ij}^{v}}{dt} = \begin{cases} C_v \frac{d\alpha}{dt}\sigma_{ij}^{d} & \frac{d\alpha}{dt} > 0 \\ 0 & \frac{d\alpha}{dt} \le 0 \end{cases}, \qquad (7)$$

where C_v is suggested to be a material constant and σ_{ij}^{d} is the deviatoric stress tensor. The effective fluidity or inverse of viscosity $(C_v d\alpha/dt)$ relates the deviatoric stress to the rate of irreversible strain accumulation. Following Maxwell viscoelastic rheology model the total strain tensor, $\varepsilon_{ij}^{\text{tot}}$, is assumed to be a sum of the elastic strain tensor and the irreversible viscous component of deformation, that is, $\varepsilon_{ij}^{\text{tot}} = \varepsilon_{ij} + \varepsilon_{ij}^{v}$. This model assumption means that the total irreversible strain accumulated during the loading should be proportional to the overall damage increase in the tested rock sample.

Y.Hamiel, O.Katz, V.Lyakhovsky, Z.Reches, Y.Fialko, *Geophys. J. Int.*, 2006:
also e.g. in Y.Hamiel, V.Lyakhovsky, S.Stachits, G.Dresen, Y.Ben-Zion *Geophys. J. Int.*,2009

Dieterich-Ruina's model:

the most popular friction-type model:

friction coefficient τ,
depending on rate $\dot{\delta}$ (=slip speed)
and ageing θ.

Usual ageing ODE dynamics:

$$\dot{\theta} = 1 - \theta\Big(c_1\dot{\delta} + c_2\frac{\dot{\sigma}}{\sigma}\Big)$$

with σ the stress
(Linker, Dieterich, 1992)

J.H. Dieterich, a survey chapter, 2007:

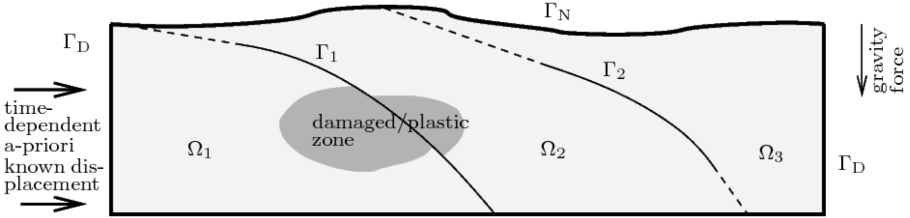
Several closely related rate- and state-dependent formulations have been used to study sliding phenomena and earthquake processes (Dieterich, 1979, 1981; Ruina, 1983; Rice, 1983). The Ruina (1983) simplification of the Dieterich (1981) formulation for sliding resistance is widely used and may be written as

$$\tau = \sigma\left[\mu_0 + a\ln\left(\frac{\dot{\delta}}{\dot{\delta}^*}\right) + b\ln\left(\frac{\theta}{\theta^*}\right)\right] \qquad [1]$$

where τ and σ are shear and effective normal stress (compression positive), respectively; μ_0, a, and b are experimentally determined constants; $\dot{\delta}$ is sliding speed; θ is a state variable (discussed below) that evolves with slip- and normal stress-history; and $\dot{\delta}^*$ and θ^* are normalizing constants. The constant θ^* in eqn [1] is often replaced by $\theta^* = D_c/\dot{\delta}^*$ where D_c is a characteristic slip parameter described below. The nominal coefficient of friction, μ_0, is defined at a reference slip rate and state ($\dot{\delta} = \dot{\delta}^*, \theta = \theta^*$ and generally has values of 0.6–0.7. For silicates at room temperature, a and b have roughly similar values in the range 0.005–0.015 (Dieterich, 1981; Ruina, 1983; Tullis and Weeks, 1986; Linker and Dieterich, 1992; Kilgore *et al.*, 1993; Marone and Kilgore, 1993; Marone, 1998).

As $\dot{\delta}$ or θ approach zero, eqn [1] yields unacceptably small (or negative) values of sliding resistance. To limit the minimum value of μ, some studies (Dieterich, 1986, 1987; Okubo and Dieterich, 1986; Okubo, 1989; Shibazaki and Iio, 2003) use a modified

Spatially multiscale problem: Schematic geometry:



Notation: $\Gamma_C = \Gamma_1 \cup \Gamma_2 \cup ....$ for pre-existing faults.

Philosophy of the model:

- concept of internal parameters systematically used,
- energy-governed evolution, rational mechanics,
- damage with healing (rate dependent) + plasticity without hardening (here rate indedendent),
- analogously on faults (⇒ adhesive contact + interface plasticity).

Variables in the bulk:

u displacement, $e(u)$ small-strain tensor,
π plastic strain,
ε Maxwell strain,
ζ damage.

Governing equations in the bulk:
momentum equilibrium:

$$\varrho\,\ddot{u} - \mathrm{div}\,\sigma = f \quad = \text{a bulk force (here just gravity)}\ ,$$

with ϱ mass density, and σ the stress:

$$\sigma = \mathbb{D}_0(\zeta)e(\dot{u}) + \mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon) \quad \text{with}$$
$$\mathbb{D}(\zeta)\dot{\varepsilon} = \mathbb{C}(\zeta)(e(u){-}\pi{-}\varepsilon)\ ,$$

where $\mathbb{C}$ is a tensor of elastic moduli (dependent on damage ζ) $\sim$ 10GPa
$\mathbb{D}_0$ is a tensor of Kelvin-Voigt-viscosity moduli (dependent on damage ζ),
$\mathbb{D}$ is a tensor of Maxwell-viscosity moduli (dependent on damage ζ),
presumably large to pronounce such aseismic fluidic-like behavior only for medium or very large time scales

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(typical values in Earth mantle are $\sim 10^{22\pm 2}$ Pa s).

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(typical values in Earth mantle are $\sim$ .3x$10^{\pm 2}$ GPa mil.yrs).

A plastic flow rule (single-threshold linearized gradient plasticity, no hardening)

$$\dot{\pi} \in N_{\alpha(\zeta)P}\Big(\mathrm{dev}\big(\mathbb{C}(\zeta)(e(u)-\pi-\varepsilon) - \kappa\Delta\pi\big)\Big) = N_P\Big(\mathrm{dev}\Big(\frac{\mathbb{D}(\zeta)\dot{\varepsilon}-\kappa\Delta\pi}{\alpha(\zeta)}\Big)\Big)$$

with $\alpha : [0,1] \to [0,1]$ monotone with $\alpha(1) = 1$ and N_P =the normal cone to the convex set P whose surface determines the *plastic yield stress* in undamaged material, and the flow rule for a scalar gradient damage

$$\mathfrak{f}(\dot{\zeta}) - \mathfrak{c}'(\zeta) \ni -\frac{1}{2}\mathbb{C}'(\zeta)\big(e(u)-\pi-\varepsilon\big):\big(e(u)-\pi-\varepsilon\big) + \mathrm{div}\Big(\kappa_0\nabla\zeta + \kappa_1|\nabla\dot{\zeta}|^{r-2}\nabla\dot{\zeta}\Big),$$

$$\text{with} \quad \mathfrak{f}\big(\dot{\zeta}\big) = \begin{cases} \mathfrak{a}\dot{\zeta} & \text{if } \dot{\zeta} > 0, \\ [-\mathfrak{d}, 0] & \text{if } \dot{\zeta} = 0, \\ \mathfrak{b}\dot{\zeta} - \mathfrak{d} & \text{if } \dot{\zeta} < 0, \end{cases}$$

with $\mathfrak{c} = \mathfrak{c}(\zeta)$ the stored energy for bulk damage,
$\mathfrak{d}$ is the dissipation energy for bulk damage,
$\kappa_0, \kappa_1 > 0$ (small) coefficients for length-scale of damage profiles.

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$$\text{with } \mathfrak{f}=\partial\mathfrak{F}, \qquad \mathfrak{F}(\dot{\zeta}) = \frac{\mathfrak{a}}{2}|\dot{\zeta}^+|^2 + \frac{\mathfrak{b}}{2}|\dot{\zeta}^-|^2 - \mathfrak{d}\dot{\zeta}^-,$$

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The modelling assumptions:

$\mathbb{C}(\cdot)$ and $\mathfrak{c}(\cdot)$ are constant on $(-\infty, 0]$ and on $[1, \infty)$, respectively:

$\Longrightarrow$ the desired contraints $0 \le \zeta(\cdot) \le 1$ kept and only one set-valued mapping in the ζ-flow rule (...math works).

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The modelling assumptions:

$\alpha(\zeta)$ decays for $\zeta\to 0+$ sufficiently fast w.r.t. $\mathbb{C}(\zeta)$

$\Longrightarrow$ when undergoing damage, stress decays but the plastic yield stress decays faster so π may start evolving.

The rheological model used in the bulk:

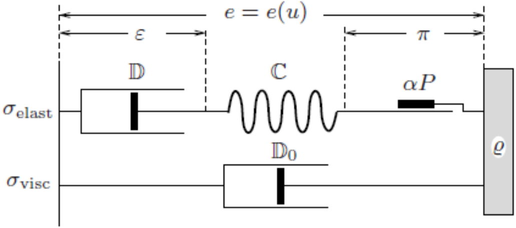

Fig. 1 *Schematic 4-parameter rheological model used in (2.1a-c,e): Maxwell material* $(\mathbb{C}, \mathbb{D})$ *in series with perfectly plastic element* P *and parallel with a damper* $\mathbb{D}_0$. *Damage* ζ *influencing* $\mathbb{C}$, $\mathbb{D}$, $\mathbb{D}_0$, *and* α *is not depicted.*

Combination of Maxwell + Kelvin-Voigt = Jeffrey
(+plasticity without hardening)
The Maxwell attenuation $\mathbb{D}$ large (but physically justified).
The Kelvin-Voigt attenuation $\mathbb{D}_0$ only expectedly very small (saving math).

Flow rule for damage:

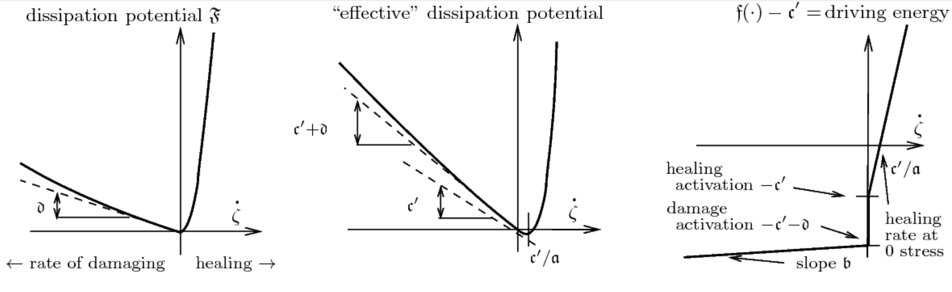

Fig. 2 *Schematic illustration of damage/healling driven by "effective" dissipation potential, its shift by a contribution coming from the stored energy if* $\mathfrak{c}(\cdot)$ *were affine (middle) and the maximal monotone graph (=its gradient) occurring in the left-hand side of the flow rule* (3.10b) *(right).*

Energetics in the bulk:

the *bulk* contribution to the *stored energy*:

$$\mathcal{E}_{\rm bulk}(t,u,\zeta,\pi,\varepsilon)=\begin{cases}\displaystyle\int_{\Omega\setminus\Gamma_{\rm C}}\frac12\mathbb{C}(\zeta)\big(e(u){-}\pi{-}\varepsilon\big){:}\big(e(u){-}\pi{-}\varepsilon\big)-\mathfrak{c}(\zeta)\\ \displaystyle\quad -f{\cdot}u+\frac{\kappa_0}{2}|\nabla\zeta|^2+\frac{\kappa}{2}|\nabla\pi|^2\,{\rm d}x & \text{if } [\![u]\!]_{\rm n}=0 \text{ a.e. on } \Gamma_{\rm C}, \text{ and}\\ & u|_{\Gamma_{\rm D}}=u_{\rm Dir}(t) \text{ a.e. on } \Gamma_{\rm D},\\ +\infty & \text{elsewhere,}\end{cases}$$

where $[\![u]\!]_{\rm n}=$ the normal component of the differences of the traces on $\Gamma_{\rm C}$,
the (pseudo)*potential of dissipative forces*:

$$\mathcal{R}_{\rm bulk}(\zeta;\dot u,\dot\zeta,\dot\pi,\dot\varepsilon)=\int_\Omega\frac12\mathbb{D}_0(\zeta)e(\dot u){:}e(\dot u)+\mathfrak{F}(\dot\zeta)$$
$$+\frac{\kappa_1}{r}|\nabla\dot\zeta|^r+\alpha(\zeta)\delta_P^*(\dot\pi)+\frac12\mathbb{D}(\zeta)\dot\varepsilon{:}\dot\varepsilon\,{\rm d}x$$

where δ_P^*= Fenchel-Legendre' conjugate to the indicator function δ_P of a convex set P, and the *kinetic energy* is

$$\mathcal{M}(\dot u)=\int_\Omega\frac{\varrho}{2}|\dot u|^2\,{\rm d}x.$$

The idea: to "translate" the model from the d-dimensional bulk to the fault considered as a $(d{-}1)$-dimensional surface.

Variables on the fault(s) $\Gamma_{\rm C}$:

- $e_{\rm i}$ interfatial "strain" $= [\![u]\!] =$ jump of displacements,
- $\pi_{\rm i}$ interfatial plastic slip,
- $\varepsilon_{\rm i}$ interfatial Maxwell slip,
- $\zeta_{\rm i}$ interfatial damage.

The interfacial stored energy:

$$\mathcal{E}_{\rm fault}(e_{\rm i},\zeta_{\rm i},\pi_{\rm i},\varepsilon_{\rm i}) := \int_{\Gamma_{\rm C}} \frac{1}{2}\mathbb{C}_{\rm i}(\zeta_{\rm i})\big(e_{\rm i}-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big)\cdot\big(e_{\rm i}-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big) - \mathfrak{c}_{\rm i}(\zeta_{\rm i}) + \frac{\kappa_{0\rm i}}{2}|\nabla_{\rm S}\zeta_{\rm i}|^2 + \frac{\kappa_{1\rm i}}{2}|\nabla_{\rm S}\pi_{\rm i}|^2 {\rm d}S$$

and the interfacial potential of dissipative forces:

$$\mathcal{R}_{\rm fault}(\zeta_{\rm i};\dot\zeta_{\rm i},\dot\pi_{\rm i},\dot\varepsilon_{\rm i}) = \int_{\Gamma_{\rm C}} \mathfrak{F}_{\rm i}(\dot\zeta_{\rm i}) + \frac{\kappa_{1\rm i}}{r_{\rm i}}|\nabla_{\rm S}\dot\zeta_{\rm i}|^{r_{\rm i}} + \alpha_{\rm i}(\zeta_{\rm i})\delta^*_{P_{\rm i}}(\dot\pi_{\rm i}) + \frac{1}{2}\mathbb{D}_{\rm i}(\zeta_{\rm i})\dot\varepsilon_{\rm i}{:}\dot\varepsilon_{\rm i}\,{\rm d}S,$$

where $\nabla_{\rm S}$ denotes the "surface gradient" (i.e. the tangential derivative defined as $\nabla_{\rm S} v = \nabla v - (\nabla v{\cdot}\nu)\nu$ for v defined around $\Gamma_{\rm C}$).

Merging the bulk and the fault models: the state as the 7-tuple

$$\mathbf{q} = (u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) \quad \text{with} \;\; \boldsymbol{\zeta} = (\zeta, \zeta_{\rm i}),\; \boldsymbol{\pi} = (\pi, \pi_{\rm i}),\; \boldsymbol{\varepsilon} = (\varepsilon, \varepsilon_{\rm i}).$$

Then the overall stored energy $\mathcal{E} = \mathcal{E}(t, \mathbf{q})$:

$$\mathcal{E}(t, \mathbf{q}) = \mathcal{E}(t, u, \boldsymbol{\zeta}, \boldsymbol{\pi}, \boldsymbol{\varepsilon}) = \mathcal{E}_{\rm bulk}(t, u, \zeta, \pi, \varepsilon) + \mathcal{E}_{\rm fault}(\llbracket u \rrbracket, \zeta_{\rm i}, \pi_{\rm i}, \varepsilon_{\rm i}),$$

the overall (pseudo)potential of dissipative forces $\mathcal{R} = \mathcal{R}(\mathbf{q}; \dot{\mathbf{q}})$:

$$\mathcal{R}(\mathbf{q}; \dot{\mathbf{q}}) = \mathcal{R}(\boldsymbol{\zeta}; \dot{u}, \dot{\boldsymbol{\zeta}}, \dot{\boldsymbol{\pi}}, \dot{\boldsymbol{\varepsilon}}) = \mathcal{R}_{\rm bulk}(\zeta; \dot{u}, \dot{\zeta}, \dot{\pi}, \dot{\varepsilon}) + \mathcal{R}_{\rm fault}(\zeta_{\rm i}; \dot{\zeta}_{\rm i}, \dot{\pi}_{\rm i}, \dot{\varepsilon}_{\rm i}),$$

and the kinetic energy as before

$$\mathcal{M}(\dot{\mathbf{q}}) = \mathcal{M}(\dot{u}).$$

The evolution to be governed formally by:

$$\mathcal{M}' \ddot{u} + \partial_{\dot{\mathbf{q}}} \mathcal{R}(\boldsymbol{\zeta}; \dot{\mathbf{q}}) + \mathcal{E}'_{\mathbf{q}}(t, \mathbf{q}) \ni 0.$$

In a bit more details, in terms of the particular components:

dynamics for displacement (momentum equation + boundary conditions):

$$\mathcal{M}'\ddot{u} + \mathcal{R}'_{\dot{u}}(\zeta;\dot{u}) + \mathcal{E}'_u(t,u,\zeta,\pi,\varepsilon) = 0,$$

damage flow rule:

$$\partial_{\dot{\zeta}}\mathcal{R}(\dot{\zeta}) + \mathcal{E}'_\zeta(t,u,\zeta,\pi,\varepsilon) \ni 0,$$

plastic flow rule:

$$\partial_{\dot{\pi}}\mathcal{R}(\zeta;\dot{\pi}) + \mathcal{E}'_\pi(t,u,\zeta,\pi,\varepsilon) \ni 0,$$

dynamics for Maxwellian strain/slip:

$$\mathcal{R}'_{\dot{\varepsilon}}(\zeta;\dot{\varepsilon}) + \mathcal{E}'_\varepsilon(t,u,\zeta,\pi,\varepsilon) = 0,$$

by using that $\partial_{\dot{u}}\mathcal{R} = \partial_{\dot{u}}\mathcal{R}_{\rm bulk}$ is single-valued independent of $\zeta_{\rm i}$, that $\partial_{\dot{\zeta}}\mathcal{R}$ is independent of ζ, and $\mathcal{E}(t,\cdot,\cdot,\cdot)$ is smooth.

The initial conditions:

$$u(0) = u_0, \quad \zeta(0) = \zeta_0, \quad \boldsymbol{\pi}(0) = \boldsymbol{\pi}_0, \quad \varepsilon(0) = \varepsilon_0, \quad \dot{u}(0) = v_0.$$

Energy balance formally:

$$\underbrace{\mathcal{M}(\dot{u}(t)) + \mathcal{E}(t, \mathbf{q}(t))}_{\substack{\text{kinetic + stored energy} \\ \text{at time } t}} + \underbrace{\int_0^t \Xi(\zeta(t); \dot{\mathbf{q}}(t))\,\mathrm{d}t}_{\substack{\text{dissipated energy over} \\ \text{the time interval } [0,t]}} = \underbrace{\mathcal{M}(v_0) + \mathcal{E}(t, \mathbf{q}_0)}_{\substack{\text{kinetic+stored energy} \\ \text{at time } t=0}} + \underbrace{\int_0^t \mathcal{E}_t'(t, \mathbf{q})\,\mathrm{d}t}_{\substack{\text{work done by loading} \\ \text{over time interval } [0,t]}}$$

with $\mathbf{q}_0 = (u_0, \zeta_0, \boldsymbol{\pi}_0, \varepsilon_0)$ and the dissipation rate $\Xi(\zeta; \dot{\mathbf{q}}) = \langle \partial_{\dot{\mathbf{q}}} \mathcal{R}(\zeta; \dot{\mathbf{q}}), \dot{\mathbf{q}} \rangle$.

In fact, a transformation to time-constant Dirichlet by replacing u with $u+u_{\rm D}(t)$ with a suitable extension $u_{\rm D}(t)$ of $u_{\rm Dir}(t)$ is needed to give a sense to $\mathcal{E}_t'$.

The governing equations/inclusions arising on the faults $\Gamma_{\rm C}$:

$$[\![\sigma]\!]_{\rm n} = 0, \qquad [\![\sigma]\!]_{\rm t} = \mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big), \qquad [\![u(t,\cdot)]\!]\cdot\nu = 0,$$

where $[\![\sigma]\!]_{\rm n} = \nu\cdot[\![\mathbb{D}_0(\zeta)e(\dot u(t,\cdot)) + \mathbb{C}(\zeta)(e(u)-\pi-\varepsilon)]\!]\nu$ and
where $[\![\sigma]\!]_{\rm t} = [\![\mathbb{D}_0(\zeta)e(\dot u(t,\cdot)) + \mathbb{C}(\zeta)(e(u)-\pi-\varepsilon)]\!] - [\![\sigma]\!]_{\rm n}\nu$,

$$\mathfrak{f}_{\rm i}(\dot\zeta_{\rm i}) - \mathfrak{c}_{\rm i}'(\zeta_{\rm i}) \ni -\frac{1}{2}\mathbb{C}_{\rm i}{}'(\zeta_{\rm i})\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big)\cdot\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big) + {\rm div}_{\rm S}\big(\kappa_{0\rm i}\nabla_{\rm S}\zeta_{\rm i} + \kappa_{1\rm i}|\nabla_{\rm S}\dot\zeta_{\rm i}|^{r_{\rm i}-2}\nabla_{\rm S}\dot\zeta_{\rm i}\big)$$

$$\text{with}\quad \mathfrak{f}_{\rm i}(\dot\zeta_{\rm i}) = \begin{cases} \mathfrak{a}_{\rm i}\dot\zeta_{\rm i} & \text{if } \dot\zeta_{\rm i} > 0,\\ [-\mathfrak{d}_{\rm i},0] & \text{if } \dot\zeta_{\rm i} = 0,\\ \mathfrak{b}_{\rm i}\dot\zeta_{\rm i} - \mathfrak{d}_{\rm i} & \text{if } \dot\zeta_{\rm i} < 0,\end{cases}$$

$$\dot\pi_{\rm i} \in N_{\alpha_{\rm i}(\zeta_{\rm i})P_{\rm i}}\Big(\mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big) - {\rm div}_{\rm S}\nabla_{\rm S}\pi_{\rm i}\Big),$$

$$\dot\varepsilon_{\rm i} = \mathbb{D}_{\rm i}^{-1}(\zeta_{\rm i})\mathbb{C}_{\rm i}(\zeta_{\rm i})\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big),$$

where ${\rm div}_{\rm S} := {\rm trace}(\nabla_{\rm S})$ denotes the $(d-1)$-dimensional "surface divergence".

Remark 1: Non-quadratic and even non-convex stored energies:

the goal: to reflect instabilities:

even detailed data based on observations/experiments are available

mathematically, it would need $\nabla e(u)$-terms in $\mathcal{E}$, i.e. the concept of $2^{\rm nd}$-orded non-simple materials

V.Lyakhovsky, Y.Ben-Zion, A.Agnon, *J. Geophys. Res.*, 1997:

cussed by *Lyakhovsky et al.* [1997], the elastic potential is written as

$$U = \frac{1}{\rho}\left(\frac{\lambda}{2}I_1^2 + \mu I_2 - \gamma I_1\sqrt{I_2}\right), \qquad (11)$$

where λ and μ are Lame constants, $I_1=\varepsilon_{kk}$ and $I_2=\varepsilon_{ij}\varepsilon_{ij}$ are two independent invariants of the strain tensor ε_{ij}, and γ is an additional elastic modulus (summation notation is assumed). The second order term with the new modulus γ accounts for microcrack opening and closure in a damaged material. The term incorporates nonlinear elasticity even for an infinitesimal strain, and it simulates abrupt change in the elastic properties when the loading reverses from compression to tension. Using (8), the stress tensor is derived from (11) as

$$\sigma_{ij} = \left(\lambda - \gamma\frac{\sqrt{I_2}}{I_1}\right)I_1\delta_{ij} + \left(2\mu - \gamma\frac{I_1}{\sqrt{I_2}}\right)\varepsilon_{ij}. \qquad (12)$$

dependencies of the elastic moduli λ, μ, and γ on damage:

$$\lambda = \lambda_0 + \alpha\lambda_r, \quad \mu = \mu_0 + \alpha\mu_r, \quad \gamma = \alpha\gamma_r, \qquad (16)$$

where $\lambda=\lambda_0$, $\mu=\mu_0$, and $\gamma=0$ correspond to initial elastic moduli of the uncracked material. Combining equations (10), (11), and (16) yields an equation of damage evolution

$$\frac{d\alpha}{dt} = -C\rho\left(\frac{\lambda_r}{2}I_1^2 + \mu_r I_2 - \gamma_r I_1\sqrt{I_2}\right). \qquad (17)$$

also e.g. in V.Lyakhovsky, Z.Reches, R.Weinberger, T.E.Scott, *Geophys. J. Int.*, 1997

Remark 2: Relation with the frictional models:

The usual frictional contact: the dissipation rate $\mu\sigma_n|\dot{\pi}_{\rm i}|$ with
$\pi_{\rm i} = \mathbb{T}^{-1}[\![u]\!]_{\rm t}$ and here $[\![u]\!]_{\rm n} = 0$ (no cavities).
serious mathematical difficulties even if μ is constant

$\Rightarrow$ regularization: "penalization" of $[\![u]\!]_{\rm n} = 0$ (=small penetration allowed) or
"penalization" of $\mathbb{T}\pi_{\rm i} = [\![u]\!]_{\rm t}$
which is, in fact, the adhesive concept chosen here.

$\mathbb{C}_{\rm i}$ large and neglecting the Maxwellian slip $\varepsilon_{\rm i} = 0$,
$\Rightarrow [\![u]\!]_{\rm t} \sim \mathbb{T}\pi_{\rm i}$
$P_{\rm i}$ a ball of the radius $r_{\rm i}$
$\Rightarrow$ the dissipation rate $\alpha_{\rm i}(\zeta_{\rm i})\delta^*_{P_{\rm i}}(\dot{\pi}_{\rm i}) \sim \alpha_{\rm i}(\zeta_{\rm i})r_{\rm i}|[\![\dot{u}]\!]_{\rm t}|$,
$\Rightarrow \alpha_{\rm i}(\zeta_{\rm i})r_{\rm i}$ is in the position of the coefficient of friction.
$\zeta_{\rm i}$ in the position of the variable "ageing".

more coefficients state dependent, as e.g. also $\mathfrak{d}_{\rm i} = \mathfrak{d}_{\rm i}(\zeta_{\rm i})$ or $\mathfrak{a}_{\rm i} = \mathfrak{a}_{\rm i}(\zeta_{\rm i})$
$\Rightarrow$ additional fitting possible.

The semi-implicit discretisation: first calculate $(u_\tau^k, \pi_\tau^k, \varepsilon_\tau^k)$:

$$\mathcal{M}'\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}+\mathcal{R}'_{\dot u}\Big(\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_u(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)=0,$$

$$\partial_{\dot\zeta}\mathcal{R}\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\zeta(k\tau,u_\tau^k,\zeta_\tau^k,\pi_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\partial_{\dot\pi}\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\pi(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\mathcal{R}'_{\dot\varepsilon}\Big(\zeta_\tau^{k-1};\frac{\varepsilon_\tau^k-\varepsilon_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\varepsilon(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)=0.$$

It advantageously decouples the problem and keeps the variational structure.

Basic assumptions: uncomplete damage without weakening …

$\mathbb{C}(\cdot), \mathbb{C}_\mathrm{i}(\cdot)$ continuously differentiable, uniformly positive definite,

$\mathbb{C}(\cdot)e{:}e$ and $\mathbb{C}_\mathrm{i}(\cdot)u{:}u$ are convex $\quad(\forall e\in\mathbb{R}^{d\times d}_{\rm sym},\ u\in\mathbb{R}^d)$,

$\mathfrak{c}(\cdot), \mathfrak{c}_\mathrm{i}(\cdot)$ concave

$\Rightarrow \mathcal{E}(t,u,\cdot,\pi,\varepsilon)$ convex.

The semi-implicit discretisation: second calculate ζ_τ^k:

$$\mathcal{M}'\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}+\mathcal{R}'_{\dot u}\Big(\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_u(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)=0,$$

$$\partial_{\dot\zeta}\mathcal{R}\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\zeta(k\tau,u_\tau^k,\zeta_\tau^k,\pi_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\partial_{\dot\pi}\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\pi(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\mathcal{R}'_{\dot\varepsilon}\Big(\zeta_\tau^{k-1};\frac{\varepsilon_\tau^k-\varepsilon_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\varepsilon(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k)=0.$$

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The semi-implicit discretisation:

$$\mathcal{M}'\frac{u_\tau^k-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}+\mathcal{R}'_{\dot u}\Big(\zeta_\tau^{k-1};\frac{u_\tau^k-u_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_u(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)=0,$$

$$\partial_{\dot\zeta}\mathcal{R}\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\zeta(k\tau,u_\tau^k,\zeta_\tau^k,\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\partial_{\dot{\boldsymbol{\pi}}}\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{\boldsymbol{\pi}_\tau^k-\boldsymbol{\pi}_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_{\boldsymbol{\pi}}(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)\ni 0,$$

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The semi-implicit discretisation:

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$$\partial_{\dot\zeta}\mathcal{R}\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\zeta(k\tau,u_\tau^k,\zeta_\tau^k,\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)\ni 0,$$

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$$\mathcal{R}'_{\dot\varepsilon}\Big(\zeta_\tau^{k-1};\frac{\varepsilon_\tau^k-\varepsilon_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\varepsilon(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)=0.$$

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$$\partial_{\dot{\boldsymbol{\pi}}}\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{\boldsymbol{\pi}_\tau^k-\boldsymbol{\pi}_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_{\boldsymbol{\pi}}(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)\ni 0,$$

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$$\partial_{\dot{\boldsymbol{\pi}}}\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{\boldsymbol{\pi}_\tau^k-\boldsymbol{\pi}_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_{\boldsymbol{\pi}}(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)\ni 0,$$

$$\mathcal{R}'_{\dot\varepsilon}\Big(\zeta_\tau^{k-1};\frac{\varepsilon_\tau^k-\varepsilon_\tau^{k-1}}{\tau}\Big)+\mathcal{E}'_\varepsilon(k\tau,u_\tau^k,\zeta_\tau^{k-1},\boldsymbol{\pi}_\tau^k,\varepsilon_\tau^k)=0.$$

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$\mathfrak{c}(\cdot),\mathfrak{c}_{\rm i}(\cdot)$ concave

$\Rightarrow\ \mathcal{E}(t,u,\cdot,\pi,\varepsilon)$ convex

Variational structure: to solve successively
two decoupled convex minimization problems at each time level:

$$\left.\begin{array}{ll}\text{minimize} & \tau^2\mathcal{M}\Big(\dfrac{u-2u_\tau^{k-1}+u_\tau^{k-2}}{\tau^2}\Big) \\ & \quad +\tau\mathcal{R}\Big(\zeta_\tau^{k-1};\dfrac{u-u_\tau^{k-1}}{\tau},0,\dfrac{\boldsymbol{\pi}-\boldsymbol{\pi}_\tau^{k-1}}{\tau},\dfrac{\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_\tau^{k-1}}{\tau}\Big) \\ & \quad +\mathcal{E}\big(k\tau,u,\zeta_\tau^{k-1},\boldsymbol{\pi},\boldsymbol{\varepsilon}\big) \\ \text{subject to} & u\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d), \\ & \boldsymbol{\pi}=(\pi,\pi_{\rm i})\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^{d\times d}_{\rm dev})\times H^1(\Gamma_{\rm C};\mathbb{R}^{d-1}), \\ & \boldsymbol{\varepsilon}=(\varepsilon,\varepsilon_i)\in L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Gamma_{\rm C};\mathbb{R}^{d-1}), \end{array}\right\}$$

and, denoting its (unique) solution by u_τ^k, $\boldsymbol{\pi}_\tau^k$, and $\boldsymbol{\varepsilon}_\tau^k$, further solve:

$$\left.\begin{array}{ll}\text{minimize} & \tau\mathcal{R}\Big(0;0,\dfrac{\zeta-\zeta_\tau^{k-1}}{\tau},0,0\Big)+\mathcal{E}\big(k\tau,u_\tau^k,\zeta,\boldsymbol{\pi}_\tau^k,\boldsymbol{\varepsilon}_\tau^k\big) \\ \text{subject to} & \boldsymbol{\zeta}=(\zeta,\zeta_{\rm i})\in W^{1,r}(\Omega\backslash\Gamma_{\rm C})\times W^{1,r_{\rm i}}(\Gamma_{\rm C}), \end{array}\right\}$$

whose solution will be denoted by ζ_τ^k.

The discrete energy (im)balance:
Test by $u_\tau^k - u_\tau^{k-1}$, $\pi_\tau^k - \pi_\tau^{k-1}$, and $\varepsilon_\tau^k - \varepsilon_\tau^{k-1}$:
Convexity of $\mathcal{E}(t,\cdot,\zeta,\cdot,\cdot)$, 1-homogeneity of $\mathcal{R}(\zeta;\dot u,\dot\zeta,\cdot,\dot\varepsilon)$,
2-homogeneity of $\mathcal{R}(\zeta;\cdot,\dot\zeta,\dot\pi,\cdot) \Rightarrow$

$$\mathcal{M}\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau}\Big) + \tau\mathcal{R}\Big(\zeta_\tau^{k-1};0,0,\frac{\pi_\tau^k - \pi_\tau^{k-1}}{\tau},0\Big) + 2\tau\mathcal{R}\Big(\zeta_\tau^{k-1};\frac{u_\tau^k - u_\tau^{k-1}}{\tau},0,0,\frac{\varepsilon_\tau^k - \varepsilon_\tau^{k-1}}{\tau}\Big)$$
$$+\mathcal{E}\big(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k\big) \le \mathcal{M}\Big(\frac{u_\tau^{k-1} - u_\tau^{k-2}}{\tau}\Big) + \mathcal{E}\big(k\tau,u_\tau^{k-1},\zeta_\tau^{k-1},\pi_\tau^{k-1},\varepsilon_\tau^{k-1}\big).$$

Test the flow-rule by $\zeta_\tau^k - \zeta_\tau^{k-1}$: convexity of $\mathcal{E}(t,u,\cdot,\pi,\varepsilon)$ and $\mathcal{R}(\zeta;\dot u,\cdot,\dot\pi,\dot\varepsilon)$ $\Rightarrow$

$$\Big\langle \partial_{\dot\zeta}\mathcal{R}\Big(0;0,\frac{\zeta - \zeta_\tau^{k-1}}{\tau},0,0\Big),\zeta - \zeta_\tau^{k-1}\Big\rangle + \mathcal{E}\big(k\tau,u_\tau^k,\zeta_\tau^k,\pi_\tau^k,\varepsilon_\tau^k\big) \le \mathcal{E}\big(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k\big).$$

By summing them and by the cancellation of $\pm\mathcal{E}\big(k\tau,u_\tau^k,\zeta_\tau^{k-1},\pi_\tau^k,\varepsilon_\tau^k\big) \Rightarrow$

$$\mathcal{M}(\dot u_\tau(t)) + \mathcal{E}(t,u_\tau(t),\zeta_\tau(t),\pi_\tau(t),\varepsilon_\tau(t)) + \int_0^t \Xi(\underline{\zeta}_\tau(t);\dot u_\tau(t),\dot\zeta_\tau(t),\dot\pi_\tau(t),\dot\varepsilon_\tau(t))\,\mathrm{d}t$$
$$\le \mathcal{M}(v_0) + \mathcal{E}(t,u_0,\zeta_0,\pi_0,\varepsilon_0) + \int_0^t \mathcal{E}_t'(t,\underline{u}_\tau,\underline{\zeta}_\tau,\underline{\pi}_\tau,\underline{\varepsilon}_\tau)\,\mathrm{d}t$$

for all $t = k\tau$, $k = 1,...,T/\tau$.

The power:

$$\mathcal{E}_t'(t,u,\zeta,\pi,\varepsilon) = \int_{\Omega\setminus\Gamma_\mathrm{C}} \mathbb{C}(\zeta)\big(e(u{+}u_\mathrm{D}(t)){-}\pi{-}\varepsilon\big){:}e(\dot u_\mathrm{D}(t)) + \mathbb{D}_0(\zeta)e(u){:}e(\ddot u_\mathrm{D}(t)) - \varrho\,\dddot u_\mathrm{D}(t){\cdot}u\,\mathrm{d}x - \int_{\Gamma_\mathrm{N}} \big(D_0(\zeta)e(\ddot u_\mathrm{D}(t)){+}\mathbb{C}(\zeta)e(\dot u_\mathrm{D}(t))\big)\nu{\cdot}u\,\mathrm{d}S.$$

Assumption: $u_\mathrm{D} \in W^{2,1}(I;H^1(\Omega;\mathbb{R}^d)) \cap W^{3,1}(I;L^2(\Omega;\mathbb{R}^d))$

$\Rightarrow$ by Hölder's + discrete Gronwall's inequalities:

A-priori estimates:

$$\big\|u_\tau\big\|_{H^1(I;H^1(\Omega\setminus\Gamma_\mathrm{C};\mathbb{R}^d))\,\cap\, W^{1,\infty}(I;L^2(\Omega;\mathbb{R}^d))} \le C,$$

$$\big\|\zeta_\tau\big\|_{L^\infty(I;H^1(\Omega\setminus\Gamma_\mathrm{C})\times H^1(\Gamma_\mathrm{C}))\,\cap\,(W^{1,r}(I;W^{1,r}(\Omega))\times W^{1,r_\mathrm{i}}(I;W^{1,r_\mathrm{i}}(\Gamma_\mathrm{C})))} \le C,$$

$$\big\|\pi_\tau\big\|_{L^\infty(I;H^1(\Omega\setminus\Gamma_\mathrm{C};\mathbb{R}^{d\times d}_\mathrm{dev})\times H^1(\Gamma_\mathrm{C};\mathbb{R}^{d-1}))\cap W^{1,1}(I;L^1(\Omega;\mathbb{R}^{d\times d}_\mathrm{dev})\times L^1(\Gamma_\mathrm{C};\mathbb{R}^{d-1}))} \le C,$$

$$\big\|\varepsilon_\tau\big\|_{H^1(I;L^2(\Omega;\mathbb{R}^{d\times d})\times L^2(\Gamma_\mathrm{C};\mathbb{R}^{d-1}))} \le C.$$

Using again the minimization problem for $(u_\tau^k, \pi_\tau^k, \varepsilon_\tau^k)$ and compare its value with a value at $(u_\tau^k, \widetilde{\pi}, \varepsilon_\tau^k)$ with a general $\widetilde{\pi}$ and using the 1-homogeneity of $\mathcal{R}(\zeta; \dot{u}, 0, \cdot, \dot{\varepsilon})$ and thus the corresponding triangle inequality,

we got the so-called discrete semi-stability:

$$\bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau(t), \bar{\zeta}_\tau(t), \bar{\pi}_\tau(t), \bar{\varepsilon}_\tau(t)\big) \leq \bar{\mathcal{E}}_\tau\big(t, \bar{u}_\tau(t), \bar{\zeta}_\tau(t), \widetilde{\pi}, \bar{\varepsilon}_\tau(t)\big) + \mathcal{R}\big(\underline{\zeta}_\tau(t); 0, 0, \widetilde{\pi} - \bar{\pi}_\tau(t), 0\big)$$

for all $t \in [0, T]$ and all $\widetilde{\pi} \in H^1(\Omega \backslash \Gamma_{\rm C}; \mathbb{R}^{d\times d}_{\rm dev}) \times H^1(\Gamma_{\rm C}; \mathbb{R}^{d-1})$.

Proof of convergence towards weak/energetic solutions:

Step 1: selection of subsequences (Banach's + Helly's selection principles):

$$
\begin{array}{lll}
u_\tau \to u & \text{weakly* in } H^1(I; H^1(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)) \cap W^{1,\infty}(\bar{I}; (L^2(\Omega;\mathbb{R}^d)), & \\
\zeta_\tau \to \zeta & \text{weakly in } W^{1,r}(I; W^{1,r}(\Omega)) \times W^{1,r_{\rm i}}(I; W^{1,r_{\rm i}}(\Gamma_{\rm C})), & \\
\boldsymbol{\pi}_\tau \to \boldsymbol{\pi} & \text{weakly* in } L^\infty(I; H^1(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^{d\times d}_{\rm dev}) \times H^1(\Gamma_{\rm C}; \mathbb{R}^{d-1})), & \\
\boldsymbol{\pi}_\tau(t) \to \boldsymbol{\pi}(t) & \text{weakly* in } H^1(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^{d\times d}_{\rm dev}) \times H^1(\Gamma_{\rm C}; \mathbb{R}^{d-1}) & \forall t \in \bar{I}, \\
\varepsilon_\tau \to \varepsilon & \text{weakly in } H^1(I; L^2(\Omega; \mathbb{R}^{d\times d}) \times L^2(\Gamma_{\rm C}; \mathbb{R}^{d-1})), & \\
\bar{\varepsilon}_\tau(t) \to \varepsilon(t) & \text{weakly in } L^2(\Omega; \mathbb{R}^{d\times d}_{\rm dev}) \times L^2(\Gamma_{\rm C}; \mathbb{R}^{d-1}) & \forall t \in \bar{I}.
\end{array}
$$

Step 2: Improved convergence of elastic stresses:

$$\mathbb{C}(\underline{\zeta}_\tau)(e(\bar{u}_\tau+u_{\rm D})-\bar{\pi}_\tau-\bar{\varepsilon}_\tau) \to \mathbb{C}(\zeta)(e(u+u_{\rm D})-\pi-\varepsilon) \quad \text{in } L^p(I;L^2(\Omega;\mathbb{R}^{d\times d})),$$
$$\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})([\![\bar{u}_\tau]\!]-\mathbb{T}(\bar{\pi}_{{\rm i}\tau}+\bar{\varepsilon}_{{\rm i}\tau})) \to \mathbb{C}_{\rm i}(\zeta_{\rm i})([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})) \quad \text{in } L^p(I;L^2(\Gamma_{\rm C};\mathbb{R}^d)),$$

for any $1 \le p < \infty$.

Assume $\mathbb{D}(\cdot)$, $\mathbb{D}_0(\cdot)$, and $\mathbb{D}_{\rm i}(\cdot)$ are constant and $r \ge 3$ and $r_{\rm i} > 2$ (if $d = 3$) or $r > 2$ and $r_{\rm i} > 1$ (if $d = 2$).

Monotonicity of the $\mathbb{C}$-terms between $(\bar{u}_\tau, \bar{\boldsymbol{\pi}}_\tau, \bar{\boldsymbol{\varepsilon}}_\tau)$ and $(u, \boldsymbol{\pi}, \boldsymbol{\varepsilon})$.

Use the discrete equations/inequality tested respectively by $u_\tau - u$, by $\bar{\boldsymbol{\pi}}_\tau - \boldsymbol{\pi}$, and by $\bar{\boldsymbol{\varepsilon}}_\tau - \boldsymbol{\varepsilon}$:

$$\int_{Q\setminus\Sigma_{\rm C}}\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau{-}u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon){:}(e(\bar u_\tau{-}u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon)+\kappa|\nabla\bar\pi_\tau-\nabla\pi|^2\,{\rm d}x{\rm d}t$$
$$+\int_{\Sigma_{\rm C}}\Big(\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau{-}u]\!]-\mathbb{T}(\bar\pi_{{\rm i}\tau}+\pi_{\rm i}-\bar\varepsilon_{{\rm i}\tau}+\varepsilon_{\rm i})\big)\cdot\big([\![\bar u_\tau{-}u]\!]-\mathbb{T}(\bar\pi_{{\rm i}\tau}+\pi_{\rm i}-\bar\varepsilon_{{\rm i}\tau}+\varepsilon_{\rm i})\big)$$
$$+\kappa_{\rm i}|\nabla_{\rm S}\bar\pi_{{\rm i}\tau}-\nabla_{\rm S}\pi_{\rm i}|^2\Big){\rm d}S{\rm d}t$$
$$=\int_{Q\setminus\Sigma_{\rm C}}\Big(\mathbb{D}_0 e(\dot u_\tau){:}e(u{-}\bar u_\tau)+\mathbb{D}\dot\varepsilon_\tau{:}(\varepsilon-\bar\varepsilon_\tau)+\alpha(\underline{\zeta}_\tau)\bar\xi_\tau{:}(\pi-\bar\pi_\tau)-\varrho\big[\dot u_\tau\big]_\tau^{\rm int}\cdot(\dot u_\tau-\dot u)$$
$$-\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau{-}u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon){:}(e(u)-\pi-\varepsilon)-\kappa\nabla(\bar\pi_\tau-\pi){:}\nabla\pi$$
$$-\mathbb{C}(\underline{\zeta}_\tau)(e(\bar u_\tau{-}u)-\bar\pi_\tau+\pi-\bar\varepsilon_\tau+\varepsilon){:}e(u_\tau{-}\bar u_\tau)\Big)\,{\rm d}x{\rm d}t-\int_0^T\big\langle\bar f_{{\rm ext},\tau}(\underline{\zeta}_\tau),u_\tau{-}u\big\rangle{\rm d}t$$
$$+\int_{\Sigma_{\rm C}}\Big(\mathbb{D}_{\rm i}\dot\varepsilon_{{\rm i}\tau}{:}(\varepsilon_{\rm i}-\bar\varepsilon_{{\rm i}\tau})+\alpha_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\bar\xi_{{\rm i},\tau}\cdot(\pi_{\rm i}-\bar\pi_{{\rm i}\tau})-\kappa_{\rm i}\nabla_{\rm S}(\bar\pi_{{\rm i}\tau}-\pi_{\rm i})\cdot\nabla_{\rm S}\pi_{\rm i}$$
$$-\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau{-}u]\!]-\mathbb{T}(\bar\pi_{{\rm i}\tau}+\pi_{\rm i}-\bar\varepsilon_{{\rm i}\tau}+\varepsilon_{\rm i})\big)\cdot\big([\![u]\!]-\mathbb{T}(\pi_{\rm i}+\varepsilon_{\rm i})\big)$$
$$-\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar u_\tau{-}u]\!]-\mathbb{T}(\bar\pi_{{\rm i}\tau}+\pi_{\rm i}-\bar\varepsilon_{{\rm i}\tau}+\varepsilon_{\rm i})\big)\cdot[\![u{-}\bar u_\tau]\!]\Big)\,{\rm d}S{\rm d}t\to 0$$

with some $\bar\xi_\tau\in\partial\delta^*_P(\dot\pi_\tau)$ and $\bar\xi_{{\rm i},\tau}\in\partial\delta^*_{P_{\rm i}}(\dot\pi_{{\rm i}\tau})$.

we use

$$\limsup_{\tau\to 0}\int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}_0 e(\dot u_\tau){:}e(u{-}\bar u_\tau)\,\mathrm{d}x\mathrm{d}t \le \int_{\Omega\backslash\Gamma_{\rm C}}\mathbb{D}_0 e(u_0){:}e(u_0)\,\mathrm{d}x$$

$$-\liminf_{\tau\to 0}\int_{\Omega\backslash\Gamma_{\rm C}}\mathbb{D}_0 e(u_\tau(T)){:}e(u_\tau(T))\,\mathrm{d}x + \lim_{\tau\to 0}\int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}_0 e(\dot u_\tau){:}e(u)\,\mathrm{d}x\mathrm{d}t$$

$$\le \int_{\Omega\backslash\Gamma_{\rm C}}\mathbb{D}_0 e(u_0){:}e(u_0) - \mathbb{D}_0 e(u(T)){:}e(u(T))\,\mathrm{d}x + \int_{Q\backslash\Sigma_{\rm C}}\mathbb{D}_0 e(\dot u){:}e(u)\,\mathrm{d}x\mathrm{d}t = 0$$

where we used $u_\tau(T)\to u(T)$ weakly in $H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ and $\dot u_\tau \to \dot u$ weakly in $L^2(I;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$;

here we also used the assumption $\mathbb{D}_0$ independent of ζ.

Further,

$$\limsup_{\tau\to 0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau:(\varepsilon-\bar{\varepsilon}_\tau)\,\mathrm{d}x\mathrm{d}t \le \int_\Omega \mathbb{D}\varepsilon_0:\varepsilon_0\,\mathrm{d}x$$
$$-\liminf_{\tau\to 0}\int_{\Omega\setminus\Gamma_{\rm C}} \mathbb{D}\varepsilon_\tau(T):\varepsilon_\tau(T)\,\mathrm{d}x + \lim_{\tau\to 0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau:\varepsilon\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \mathbb{D}\varepsilon_0:\varepsilon_0 - \mathbb{D}\varepsilon(T):\varepsilon(T)\,\mathrm{d}x + \int_Q \mathbb{D}\dot{\varepsilon}:\varepsilon\,\mathrm{d}x\mathrm{d}t = 0$$

where we used $\varepsilon_\tau(T)\to\varepsilon(T)$ weakly in $L^2(\Omega;\mathbb{R}^d)$ and
$\dot{\varepsilon}_\tau \to \dot{\varepsilon}$ weakly in $L^2(Q;\mathbb{R}^d)$);
here we used the assumption $\mathbb{D}$ independent of ζ.

By analogous arguments, also $\int_{\Sigma_{\rm C}} \mathbb{D}_{\rm i}\dot{\varepsilon}_{{\rm i}\tau}:(\varepsilon_{\rm i}-\bar{\varepsilon}_{{\rm i}\tau})\,\mathrm{d}S\mathrm{d}t \to 0$.

Moreover, we use the (generalized) Aubin-Lions' theorem which yields $\bar{\pi}_\tau \to \pi$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_{\rm dev})$ so that

$$\int_Q \alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau:(\pi-\bar{\pi}_\tau)\,\mathrm{d}x\mathrm{d}t \to 0$$

because $\alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau$ is bounded in $L^\infty(Q;\mathbb{R}^{d\times d}_{\rm dev})$.

Further,

$$\limsup_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}(\varepsilon-\bar{\varepsilon}_\tau)\,\mathrm{d}x\mathrm{d}t \le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x$$
$$-\liminf_{\tau\to0}\int_{\Omega\setminus\Gamma_\mathrm{C}} \mathbb{D}\varepsilon_\tau(T){:}\varepsilon_\tau(T)\,\mathrm{d}x + \lim_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}\varepsilon\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0 - \mathbb{D}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \mathbb{D}\dot{\varepsilon}{:}\varepsilon\,\mathrm{d}x\mathrm{d}t = 0$$

where we used $\varepsilon_\tau(T) \to \varepsilon(T)$ weakly in $L^2(\Omega;\mathbb{R}^d)$ and $\dot{\varepsilon}_\tau \to \dot{\varepsilon}$ weakly in $L^2(Q;\mathbb{R}^d)$);
here we used the assumption $\mathbb{D}$ independent of ζ.

By analogous arguments, also $\int_{\Sigma_\mathrm{C}} \mathbb{D}_\mathrm{i}\dot{\varepsilon}_{\mathrm{i}\tau}{:}(\varepsilon_\mathrm{i}-\bar{\varepsilon}_{\mathrm{i}\tau})\,\mathrm{d}S\mathrm{d}t \to 0$.

Moreover, we use the (generalized) Aubin-Lions' theorem which yields $\bar{\pi}_\tau \to \pi$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_\mathrm{dev})$ so that

$$\int_Q \alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau{:}(\pi-\bar{\pi}_\tau)\,\mathrm{d}x\mathrm{d}t \to 0$$

because $\alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau$ is bounded in $L^\infty(Q;\mathbb{R}^{d\times d}_\mathrm{dev})$.

Further,

$$\limsup_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}(\varepsilon-\bar{\varepsilon}_\tau)\,\mathrm{d}x\mathrm{d}t \le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x$$

$$-\liminf_{\tau\to0}\int_{\Omega\backslash\Gamma_\mathrm{C}} \mathbb{D}\varepsilon_\tau(T){:}\varepsilon_\tau(T)\,\mathrm{d}x + \lim_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}\varepsilon\,\mathrm{d}x\mathrm{d}t$$

$$\le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0 - \mathbb{D}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \mathbb{D}\dot{\varepsilon}{:}\varepsilon\,\mathrm{d}x\mathrm{d}t = 0$$

where we used $\varepsilon_\tau(T)\to\varepsilon(T)$ weakly in $L^2(\Omega;\mathbb{R}^d)$ and
$\dot{\varepsilon}_\tau\to\dot{\varepsilon}$ weakly in $L^2(Q;\mathbb{R}^d)$);
here we used the assumption $\mathbb{D}$ independent of ζ.

By analogous arguments, also $\int_{\Sigma_\mathrm{C}} \mathbb{D}_\mathrm{i}\dot{\varepsilon}_{\mathrm{i}\tau}{:}(\varepsilon_\mathrm{i}-\bar{\varepsilon}_{\mathrm{i}\tau})\,\mathrm{d}S\mathrm{d}t\to0$.

Moreover, we use the (generalized) Aubin-Lions' theorem which yields $\bar{\pi}_\tau\to\pi$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_\mathrm{dev})$ so that

$$\int_Q \alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau{:}(\pi-\bar{\pi}_\tau)\,\mathrm{d}x\mathrm{d}t\to0$$

because $\alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau$ is bounded in $L^\infty(Q;\mathbb{R}^{d\times d}_\mathrm{dev})$.

Further,

$$\limsup_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}(\varepsilon-\bar{\varepsilon}_\tau)\,\mathrm{d}x\mathrm{d}t \le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0\,\mathrm{d}x$$

$$-\liminf_{\tau\to0}\int_{\Omega\setminus\Gamma_\mathrm{C}} \mathbb{D}\varepsilon_\tau(T){:}\varepsilon_\tau(T)\,\mathrm{d}x + \lim_{\tau\to0}\int_Q \mathbb{D}\dot{\varepsilon}_\tau{:}\varepsilon\,\mathrm{d}x\mathrm{d}t$$

$$\le \int_\Omega \mathbb{D}\varepsilon_0{:}\varepsilon_0 - \mathbb{D}\varepsilon(T){:}\varepsilon(T)\,\mathrm{d}x + \int_Q \mathbb{D}\dot{\varepsilon}{:}\varepsilon\,\mathrm{d}x\mathrm{d}t = 0$$

where we used $\varepsilon_\tau(T)\to\varepsilon(T)$ weakly in $L^2(\Omega;\mathbb{R}^d)$ and
$\dot{\varepsilon}_\tau\to\dot{\varepsilon}$ weakly in $L^2(Q;\mathbb{R}^d)$);
here we used the assumption $\mathbb{D}$ independent of ζ.

By analogous arguments, also $\int_{\Sigma_\mathrm{C}}\mathbb{D}_\mathrm{i}\dot{\varepsilon}_{\mathrm{i}\tau}{:}(\varepsilon_\mathrm{i}-\bar{\varepsilon}_{\mathrm{i}\tau})\,\mathrm{d}S\mathrm{d}t\to0$.

Moreover, we use the (generalized) Aubin-Lions' theorem which yields $\bar{\pi}_\tau\to\pi$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ so that

$$\int_Q \alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau{:}(\pi-\bar{\pi}_\tau)\,\mathrm{d}x\mathrm{d}t\to0$$

because $\alpha(\underline{\zeta}_\tau)\bar{\xi}_\tau$ is bounded in $L^\infty(Q;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. ↖here $\nabla\pi$ needed!

Step 3: Limit passage to the momentum IBVP:

The Aubin-Lions' theorem, $\zeta_\tau \to \zeta$ strongly

also $\underline{\zeta}_\tau \to \zeta$ strongly

and thus also $\mathbb{C}(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau}) \to \mathbb{C}_{\rm i}(\zeta_{\rm i})$ strongly in the corresponding L^p-spaces, $p < \infty$.

Then the convergence in the discrete momentum IBVP easy.

Step 4: Limit passage to the flow rules for ζ:

Original flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot\zeta_\tau) - \mathfrak{c}'(\zeta_\tau) \ni -\frac{1}{2}\mathbb{C}'(\zeta_\tau)\big(e(u_\tau)-\pi_\tau-\varepsilon_\tau\big):\big(e(u_\tau)-\pi_\tau-\varepsilon_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\zeta_\tau + \kappa_1|\nabla\dot\zeta_\tau|^{r-2}\nabla\dot\zeta_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\setminus\Sigma_{\rm C}} \mathfrak{F}(\widetilde\zeta) + \frac{1}{2}\mathbb{C}'(\bar\zeta_\tau)\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big):\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\big(\widetilde\zeta-\dot\zeta_\tau\big) - \mathfrak{c}'(\bar\zeta_\tau)\big(\widetilde\zeta-\dot\zeta_\tau\big) + \kappa_0\nabla\bar\zeta_\tau\cdot\nabla\big(\widetilde\zeta-\dot\zeta_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde\zeta|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\setminus\Sigma_{\rm C}} \mathfrak{F}(\dot\zeta_\tau) + \frac{\kappa_1}{r}|\nabla\dot\zeta_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde\zeta \in L^\infty(I;W^{1,r}(\Omega\setminus\Gamma_{\rm C}))$.

Like in Step 3, $\mathbb{C}'(\underline\zeta_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline\zeta_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p<\infty$.

Using Step 2, $\mathbb{C}'(\bar\zeta_\tau)\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big):\big(e(\bar u_\tau)-\bar\pi_\tau-\bar\varepsilon_\tau\big)\big(\widetilde\zeta-\dot\zeta_\tau\big) \to$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot\zeta_\tau \to \dot\zeta$ weakly* in $L^r(I;L^\infty(\Omega))$.

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\setminus\Sigma_{\mathrm{C}}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\setminus\Sigma_{\mathrm{C}}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I;W^{1,r}(\Omega\setminus\Gamma_{\mathrm{C}}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p<\infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I;L^\infty(\Omega))$.

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \geq \int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I;W^{1,r}(\Omega\backslash\Gamma_\mathrm{C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p<\infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I;L^\infty(\Omega))$

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\setminus\Sigma_{\mathrm{C}}}\mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}{-}\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}{-}\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau{\cdot}\nabla\big(\widetilde{\zeta}{-}\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\setminus\Sigma_{\mathrm{C}}}\mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I; W^{1,r}(\Omega\setminus\Gamma_{\mathrm{C}}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p < \infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau){-}\bar{\pi}_\tau{-}\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}{-}\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I; L^\infty(\Omega))$

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\backslash\Sigma_{\rm C}}\mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\backslash\Sigma_{\rm C}}\mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta}\in L^\infty(I;W^{1,r}(\Omega\backslash\Gamma_{\rm C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau)\to\mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau)\to\mathfrak{c}'(\zeta)$ strongly in L^p, $p<\infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau\to\dot{\zeta}$ weakly* in $L^r(I;L^\infty(\Omega))$

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I; W^{1,r}(\Omega\backslash\Gamma_\mathrm{C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p < \infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I; L^\infty(\Omega))$

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\setminus\Sigma_\mathrm{C}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \ge \int_{Q\setminus\Sigma_\mathrm{C}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I; W^{1,r}(\Omega\setminus\Gamma_\mathrm{C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p < \infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I; L^\infty(\Omega))$

↖ the term $\mathrm{div}\big(\kappa_1|\nabla\dot{\zeta}|^{r-2}\nabla\dot{\zeta}\big)$ needed!

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \geq \int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I; W^{1,r}(\Omega\backslash\Gamma_\mathrm{C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p < \infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big):\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I; L^\infty(\Omega))$

The resting terms by weak lower semicontinuity (+by part integration).

Step 4: Limit passage to the flow rules for ζ:

Discrete flow rule for bulk damage ζ:

$$\partial\mathfrak{F}(\dot{\zeta}_\tau) - \mathfrak{c}'(\bar{\zeta}_\tau) \ni -\frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big) + \mathrm{div}\big(\kappa_0\nabla\bar{\zeta}_\tau + \kappa_1|\nabla\dot{\zeta}_\tau|^{r-2}\nabla\dot{\zeta}_\tau\big),$$

Discrete flow rule as a variational inequality:

$$\int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\widetilde{\zeta}) + \frac{1}{2}\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) - \mathfrak{c}'(\bar{\zeta}_\tau)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \kappa_0\nabla\bar{\zeta}_\tau\cdot\nabla\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) + \frac{\kappa_1}{r}|\nabla\widetilde{\zeta}|^r\,\mathrm{d}x\mathrm{d}t \geq \int_{Q\backslash\Sigma_\mathrm{C}} \mathfrak{F}(\dot{\zeta}_\tau) + \frac{\kappa_1}{r}|\nabla\dot{\zeta}_\tau|^r\,\mathrm{d}x\mathrm{d}t$$

holds for all $\widetilde{\zeta} \in L^\infty(I; W^{1,r}(\Omega\backslash\Gamma_\mathrm{C}))$.

Like in Step 3, $\mathbb{C}'(\underline{\zeta}_\tau) \to \mathbb{C}(\zeta_\tau)$ and $\mathfrak{c}'(\underline{\zeta}_\tau) \to \mathfrak{c}'(\zeta)$ strongly in L^p, $p < \infty$.

Using Step 2, $\mathbb{C}'(\bar{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big){:}\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)\big(\widetilde{\zeta}-\dot{\zeta}_\tau\big) \rightharpoonup$ in $L^r(L^1(\Omega))$.

We need (and use) $\dot{\zeta}_\tau \to \dot{\zeta}$ weakly* in $L^r(I; L^\infty(\Omega))$

Analogously for the flow rule for the interface damage ζ_i.

Step 5: Limit passage in the energy (im)balance:

From the discrete variant (already displayed):

$$\mathcal{M}(\dot{u}_\tau(t)) + \mathcal{E}(t, u_\tau(t), \zeta_\tau(t), \pi_\tau(t), \varepsilon_\tau(t)) + \int_0^t \Xi(\underline{\zeta}_\tau(t); \dot{u}_\tau(t), \dot{\zeta}_\tau(t), \dot{\pi}_\tau(t), \dot{\varepsilon}_\tau(t))\,\mathrm{d}t \\ \le \mathcal{M}(v_0) + \mathcal{E}(t, u_0, \zeta_0, \pi_0, \varepsilon_0) + \int_0^t \mathcal{E}_t'(t, \underline{u}_\tau, \underline{\zeta}_\tau, \underline{\pi}_\tau, \underline{\varepsilon}_\tau)\,\mathrm{d}t.$$

for $t = k\tau$, $k = 1, ..., T/\tau$,

by the weak lower semicontinuity,

$$\mathcal{M}(\dot{u}(t)) + \mathcal{E}(t, u(t), \zeta(t), \pi(t), \varepsilon(t)) + \int_0^t \Xi(\zeta(t); \dot{u}(t), \dot{\zeta}(t), \dot{\pi}(t), \dot{\varepsilon}(t))\,\mathrm{d}t \\ \le \mathcal{M}(v_0) + \mathcal{E}(t, u_0, \zeta_0, \pi_0, \varepsilon_0) + \int_0^t \mathcal{E}_t'(t, u, \zeta, \pi, \varepsilon)\,\mathrm{d}t$$

holds for all $t \in [0, T]$.

Step 6: Limit passage in the semistability:

From the discrete variant (already displayed): $\forall t,\ \forall\widetilde{\boldsymbol{\pi}}$:

$$\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\boldsymbol{\zeta}}_\tau(t),\bar{\boldsymbol{\pi}}_\tau(t),\bar{\varepsilon}_\tau(t)\big)\le\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\boldsymbol{\zeta}}_\tau(t),\widetilde{\boldsymbol{\pi}},\bar{\varepsilon}_\tau(t)\big)\\+\mathcal{R}\big(\underline{\boldsymbol{\zeta}}_\tau(t);0,0,\widetilde{\boldsymbol{\pi}}-\bar{\boldsymbol{\pi}}_\tau(t),0\big)$$

for $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi},\bar{\pi}_{\mathrm{i},\tau}(t))$:

$$0\le\int_{\Omega\setminus\Gamma_\mathrm{C}}\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big)\\-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big)\\+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x\\=\int_{\Omega\setminus\Gamma_\mathrm{C}}\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\bar{\varepsilon}_\tau(t)\big):\big(\widetilde{\pi}-\bar{\pi}_\tau(t)\big)+\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\widetilde{\pi}:\widetilde{\pi}\\-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\bar{\pi}_\tau(t):\bar{\pi}_\tau(t)+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x.$$

Step 6: Limit passage in the semistability:

From the discrete variant (already displayed): $\forall t,\ \forall\widetilde{\boldsymbol{\pi}}$:

$$\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\zeta}_\tau(t),\bar{\boldsymbol{\pi}}_\tau(t),\bar{\varepsilon}_\tau(t)\big)\le\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\zeta}_\tau(t),\widetilde{\boldsymbol{\pi}},\bar{\varepsilon}_\tau(t)\big)+\mathcal{R}\big(\underline{\zeta}_\tau(t);0,0,\widetilde{\boldsymbol{\pi}}-\bar{\boldsymbol{\pi}}_\tau(t),0\big)$$

for $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi},\bar{\pi}_{\mathrm{i},\tau}(t))$:

$$\begin{aligned}0\le\int_{\Omega\setminus\Gamma_{\rm C}}&\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big)\\&-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big)\\&+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x\\=\int_{\Omega\setminus\Gamma_{\rm C}}&\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\bar{\varepsilon}_\tau(t)\big):\big(\widetilde{\pi}-\bar{\pi}_\tau(t)\big)+\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\widetilde{\pi}:\widetilde{\pi}\\&-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\bar{\pi}_\tau(t):\bar{\pi}_\tau(t)+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x.\end{aligned}$$

Step 6: Limit passage in the semistability:

From the discrete variant (already displayed): $\forall t,\ \forall\widetilde{\boldsymbol{\pi}}$:

$$\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\zeta}_\tau(t),\bar{\boldsymbol{\pi}}_\tau(t),\bar{\varepsilon}_\tau(t)\big)\le\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\zeta}_\tau(t),\widetilde{\boldsymbol{\pi}},\bar{\varepsilon}_\tau(t)\big)+\mathcal{R}\big(\underline{\zeta}_\tau(t);0,0,\widetilde{\boldsymbol{\pi}}-\bar{\boldsymbol{\pi}}_\tau(t),0\big)$$

for $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi},\bar{\pi}_{\mathrm{i},\tau}(t))$:

$$0\le\int_{\Omega\setminus\Gamma_{\rm C}}\frac{1}{2}\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big)$$
$$-\frac{1}{2}\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big)$$
$$+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x$$
$$=\int_{\Omega\setminus\Gamma_{\rm C}}\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t)+\bar{u}_{\mathrm{D},\tau})-\bar{\varepsilon}_\tau(t)\big):\big(\widetilde{\pi}-\bar{\pi}_\tau(t)\big)+\frac{1}{2}\mathbb{C}(\underline{\zeta}_\tau(t))\widetilde{\pi}:\widetilde{\pi}$$
$$-\frac{1}{2}\mathbb{C}(\underline{\zeta}_\tau(t))\bar{\pi}_\tau(t):\bar{\pi}_\tau(t)+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta_P^*(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x.$$

Then weak upper semicontinuity leads to the limit.

Step 6: Limit passage in the semistability:

From the discrete variant (already displayed): $\forall t,\ \forall\widetilde{\boldsymbol{\pi}}$:

$$\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\boldsymbol{\zeta}}_\tau(t),\bar{\boldsymbol{\pi}}_\tau(t),\bar{\varepsilon}_\tau(t)\big)\le\bar{\mathcal{E}}_\tau\big(t,\bar{u}_\tau(t),\bar{\boldsymbol{\zeta}}_\tau(t),\widetilde{\boldsymbol{\pi}},\bar{\varepsilon}_\tau(t)\big)+\mathcal{R}\big(\underline{\boldsymbol{\zeta}}_\tau(t);0,0,\widetilde{\boldsymbol{\pi}}-\bar{\boldsymbol{\pi}}_\tau(t),0\big)$$

for $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi},\bar{\pi}_{\mathrm{i},\tau}(t))$:

$$\begin{aligned}0\le\int_{\Omega\setminus\Gamma_{\rm C}}&\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t){+}\bar{u}_{\mathrm{D},\tau})-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t){+}\bar{u}_{\mathrm{D},\tau}(t))-\widetilde{\pi}-\bar{\varepsilon}_\tau(t)\big)\\
&-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t){+}\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big):\big(e(\bar{u}_\tau(t){+}\bar{u}_{\mathrm{D},\tau}(t))-\bar{\pi}_\tau(t)-\bar{\varepsilon}_\tau(t)\big)\\
&+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta^*_P(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x\\
=\int_{\Omega\setminus\Gamma_{\rm C}}&\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\bar{u}_\tau(t){+}\bar{u}_{\mathrm{D},\tau})-\bar{\varepsilon}_\tau(t)\big):\big(\widetilde{\pi}-\bar{\pi}_\tau(t)\big)+\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\widetilde{\pi}{:}\widetilde{\pi}\\
&-\frac12\mathbb{C}(\underline{\zeta}_\tau(t))\bar{\pi}_\tau(t){:}\bar{\pi}_\tau(t)+\frac{\kappa}{2}|\nabla\widetilde{\pi}|^2-\frac{\kappa}{2}|\nabla\bar{\pi}_\tau(t)|^2+\alpha(\underline{\zeta}_\tau(t))\delta^*_P(\widetilde{\pi}_\tau-\bar{\pi}_\tau(t))\,\mathrm{d}x.\end{aligned}$$

Analogously, $\widetilde{\boldsymbol{\pi}}=(\bar{\pi}_\tau(t),\widetilde{\pi}_{\rm i})$ leads to the interfatial semistability.

Step 7: Limit passage in the Maxwellien dynamics:

The discrete equation

$$\mathcal{R}'_{\dot{\boldsymbol{\varepsilon}}}(\underline{\zeta}_\tau;\dot{\boldsymbol{\varepsilon}}_\tau)+[\bar{\mathcal{E}}_\tau]'_{\boldsymbol{\varepsilon}}(\bar{u}_\tau,\bar{\zeta}_\tau,\bar{\boldsymbol{\pi}}_\tau,\bar{\varepsilon}_\tau)=0$$

involves semilinear equations

$$\mathbb{D}(\underline{\zeta}_\tau)\dot{\varepsilon}=\mathbb{C}(\underline{\zeta}_\tau)\big(e(\bar{u}_\tau)-\bar{\pi}_\tau-\bar{\varepsilon}_\tau\big)$$

and

$$\mathbb{D}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\dot{\varepsilon}_{\rm i}=\mathbb{C}_{\rm i}(\underline{\zeta}_{{\rm i},\tau})\big([\![\bar{u}_\tau]\!]-\mathbb{T}(\bar{\pi}_{{\rm i},\tau}+\bar{\varepsilon}_{{\rm i},\tau})\big),$$

which bears easily the limit passage.

Comments:

- Rate independent flow rule for π and theory of rate-independent processes, in particular the "energetic-solution" concept, used for the definition of the weak solution; i.e. combination of energy inequalitity and (semi)stability.
- As $\pi \mapsto \mathcal{E}(t, u, \zeta, \pi, \varepsilon)$ is convex, no too-early-jump effects.
- Rate dependent flow rule for π would also be possible. Then $\dot{\pi} \in L^2$ and conventional weak formulation works (again, strong convergence of stresses and thus gradient of π needed).

Computational experiments:

we neglect: all inertial/inelastic/viscous effects in the bulk,
the Maxwellian rheology both in the bulk and on the fault,
thus we set $\varepsilon = 0$, $\pi = 0$, $\zeta = 0$, and $\varepsilon_\mathrm{i} = 0$.

The ansatz: $e(u)$ is constant on each particular subdomain, here Ω_1 and Ω_2,
thus, in particular, $u|_{\Omega_1}$ and $u|_{\Omega_2}$ are affine,
π_i and ζ_i are constant along Γ_C.

Symmetric geometry:

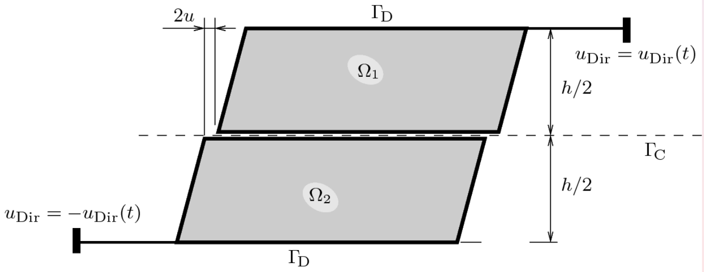


Fig. 3 *A single-degree-of-freedom slider, having 1 d-o-f observable parameter u (other 2 d-o-f are in internal parameters π_i and ζ_i).*

Data (academical, dimensionless):

$P_\mathrm{i} = [-1, 1]$,
$\alpha_\mathrm{i}(\zeta_\mathrm{i}) := \alpha_{\mathrm{i}0} + \alpha_{\mathrm{i}1}\zeta_\mathrm{i}$ =damage activation threshold
with $\alpha_{\mathrm{i}1} = 1$, $\alpha_{\mathrm{i}0} = 10^{-4}$ ($\in [0, ..., 10^{-3}]$ works equally),
$\mathfrak{c}_\mathrm{i}(\zeta_\mathrm{i}) := \mathfrak{c}_0\zeta_\mathrm{i}$ =stored energy in interface damage ($\sim$fracture toughness)
with $\mathfrak{c}_0$ varying ($= 3.10^{-4}$, 9.10^{-4}, and 27.10^{-4})
and here with constraints $0 \le \zeta_\mathrm{i} \le 1$,
$\mathbb{C}_\mathrm{i}(\zeta_\mathrm{i}) := \mathbb{C}_{\mathrm{i}0} + \mathbb{C}_{\mathrm{i}1}\zeta_\mathrm{i}$ =interfatial elastic modulus ($\mathbb{C}_{\mathrm{i}0} = 0.1$, $\mathbb{C}_{\mathrm{i}1} = 1$),
$\mathfrak{b}_\mathrm{i} = 0.1$ (prescibing rate of damage),
$\mathfrak{a}_\mathrm{i} = 20$ (prescibing healing rate $\mathfrak{c}_0/\mathfrak{a}_\mathrm{i}$ for stress-free state),
$\mathfrak{d}_\mathrm{i} = 0$ (no "dead" zone, only either healing or damaging),
linearly increasing prescribed horizontal shift $u_\mathrm{Dir}(t) = 10^{-4}t$
with $t \in [0, T]$, $T = 8.10^7$,
$h\mathbb{C} = 10^{-4}$ (if not considered varying),

bulk stored energy: $\frac{1}{2}h\mathbb{C}|u - u_\mathrm{Dir}(t)|^2$
interfacial stored energy: $\frac{1}{2}\mathbb{C}_\mathrm{i}(\zeta_\mathrm{i})|u - \pi_\mathrm{i}|^2 - \mathfrak{c}_\mathrm{i}(\zeta_\mathrm{i})$

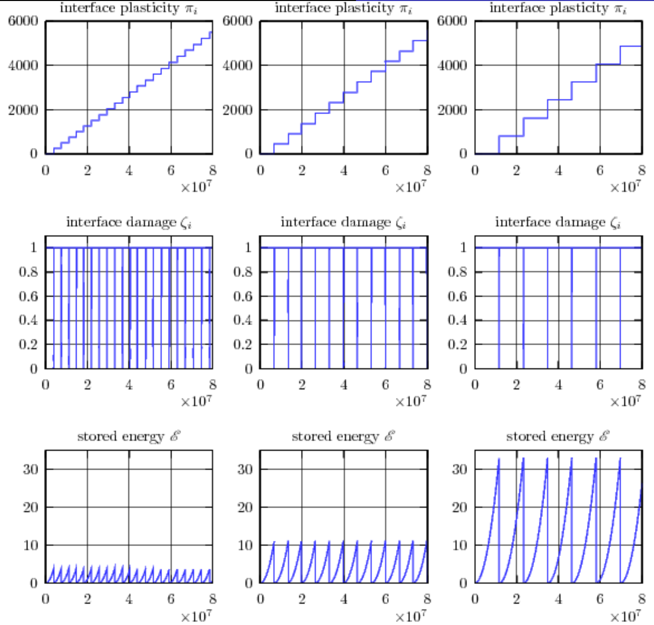
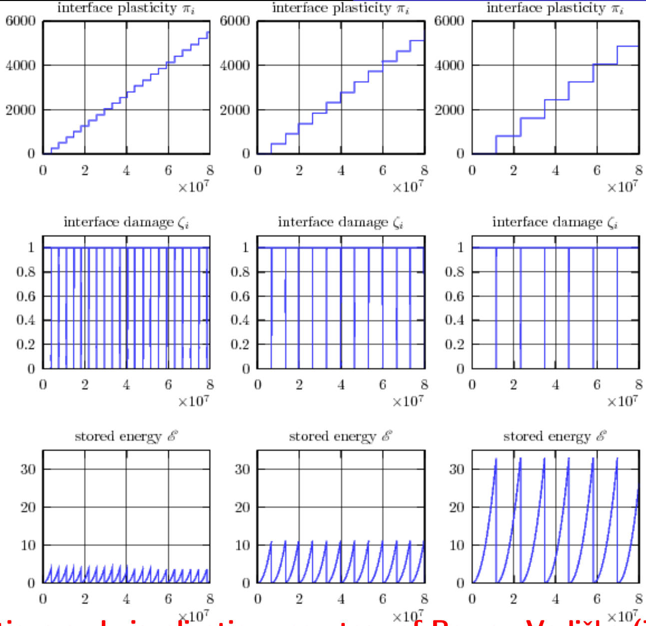


FIGURE 4. Oscillatory response of ζ_i, π_i, and $\mathscr{E}$ in time on the linearly increasing load u_Dir displayed for three different values of $\mathfrak{c}_\mathrm{i}$, namely (from left to to right) $\mathfrak{c}_\mathrm{i} = 3.10^{-4}$, 9.10^{-4}, and 27.10^{-4}.



Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

FIGURE 4. Oscillatory response of $\zeta_{\rm i}$, $\pi_{\rm i}$, and $\mathscr{E}$ in time on the linearly increasing load $u_{\rm Dir}$ displayed for three different values of $\mathfrak{c}_{\rm i}$, namely (from left to to right) $\mathfrak{c}_{\rm i} = 3.10^{-4}$, 9.10^{-4}, and 27.10^{-4}.

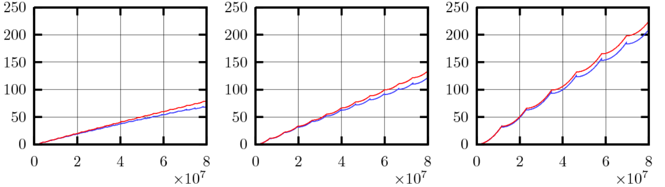
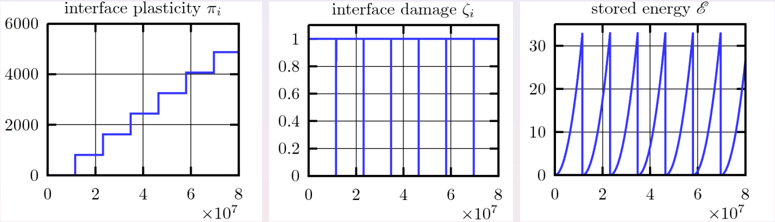


FIGURE 5. The energies on the left- and the right-hand sides in the energy balance as functions of time for the three values of $\mathfrak{c}_i$ used also in Fig. 4; the difference has been used for the refinement/corsening of time step τ during earthquakes/healing periods, respectively.

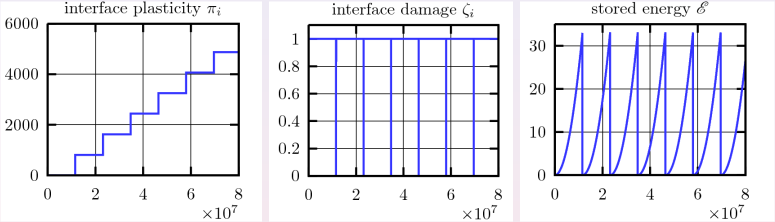
Multiscale problem in time: time-step variation needed.
We varied τ to keep the error in energy balance uniformly small:

Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

Right colomn of Fig.4 one again:



Right colomn of Fig.4 one again:



and its detail for one the particular (=4^{th}) jump:

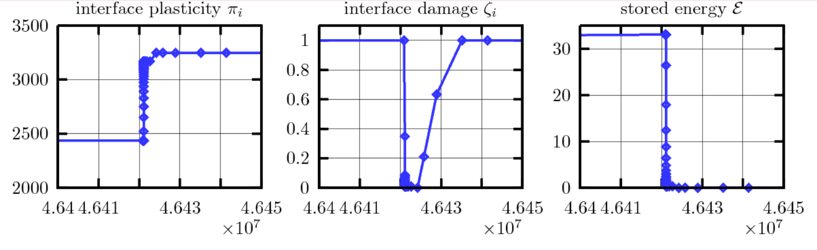
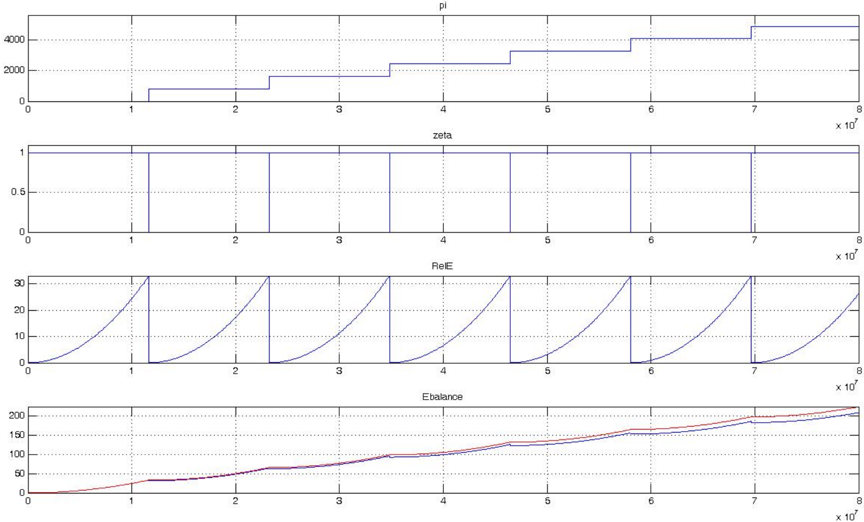


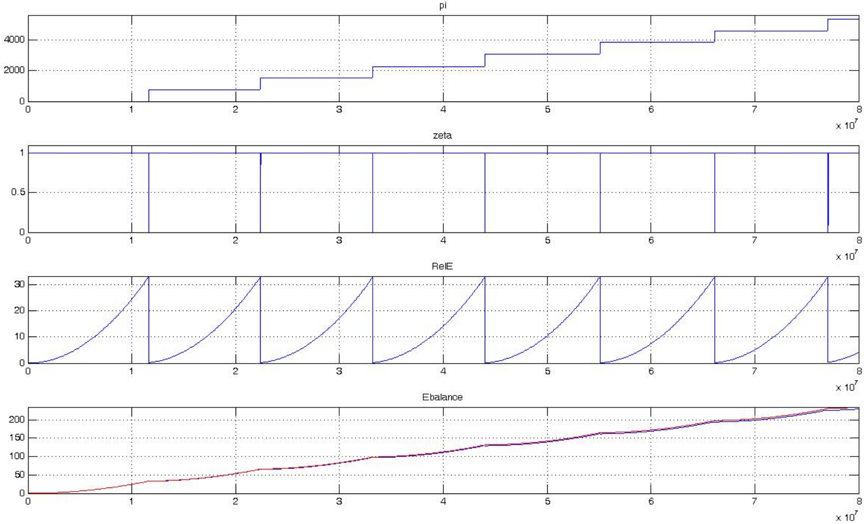
FIGURE 6. Time-zoom of $\zeta_{\rm i}$, $\pi_{\rm i}$, and $\mathscr{E}$ during one particular "earthquake" from Fig. 4(right).

Illustration of convergence: tolerance per time step = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}:



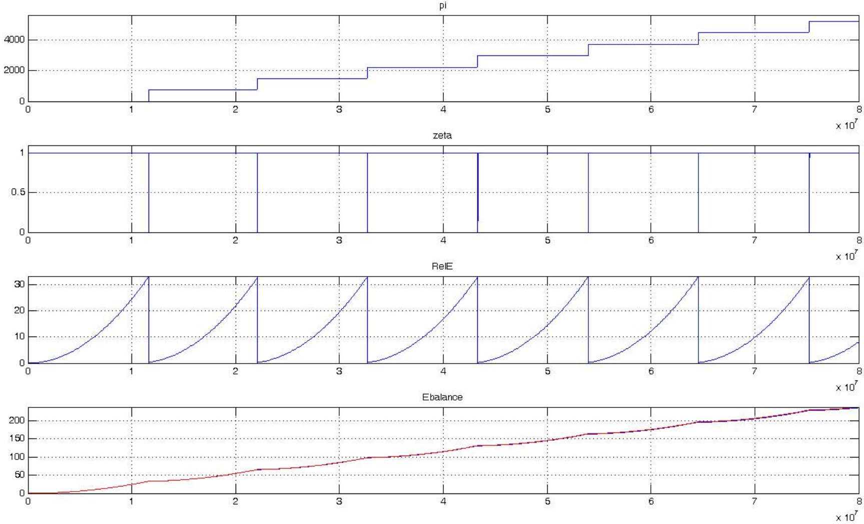
Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

Illustration of convergence: tolerance per time step $= 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}:



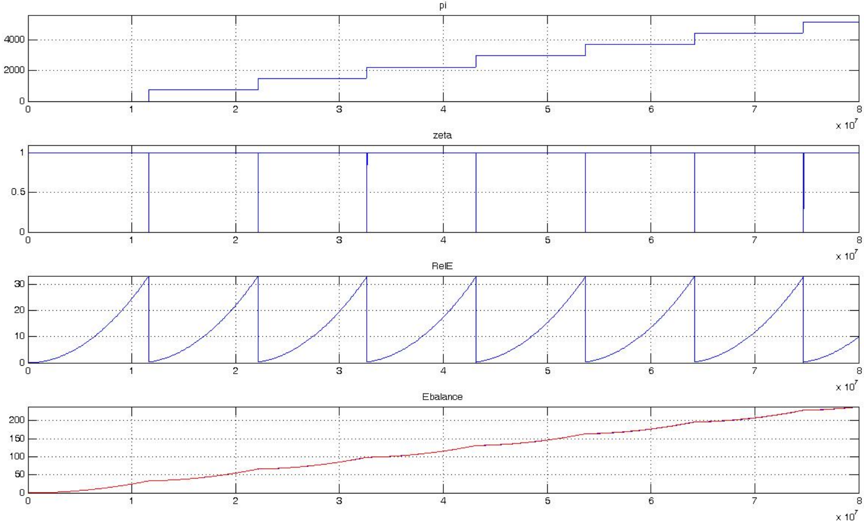
Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

Illustration of convergence: tolerance per time step $= 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}:



Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

Illustration of convergence: tolerance per time step $= 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}:



Calculations and visualization: courtesy of Roman Vodička (T.U. Košice).

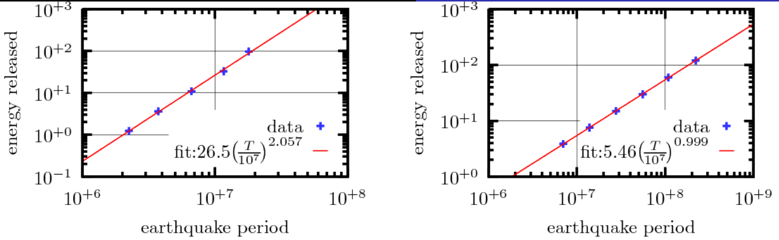
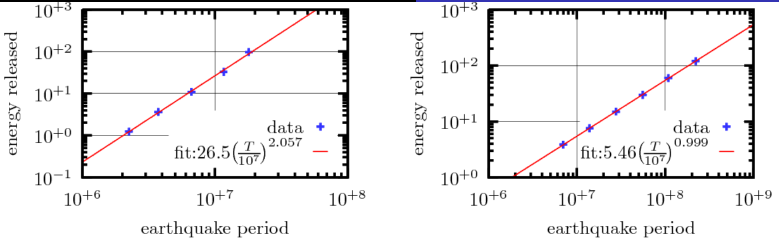


FIGURE 7. Variation of stored energy versus periods between particular earthquakes depicted in logarithmic scale:
Left: the activation energy $\mathfrak{c}_\mathrm{i}$ (=fault fracture toughness) varies as $3^i.10^{-4}$ for $i = 0, ..., 4$; the slope is close to 2.
Right: the plate height h varies: $h\mathbb{C} = 2^i.10^{-6}$ for $i = 0, ..., 5$; the slope is ~ 1.



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EARTHQUAKE MAGNITUDE, INTENSITY, ENERGY, AND ACCELERATION

(Second Paper)

By B. GUTENBERG AND C. F. RICHTER

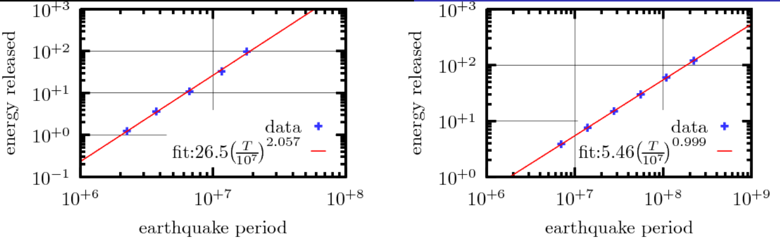
ABSTRACT

This supersedes Paper 1 (Gutenberg and Richter, 1942). Additional data are presented. Revisions involving intensity and acceleration are minor. The equation $\log a = I/3 - \frac{1}{2}$ is retained. The magnitude-energy relation is revised as follows:

$$\log E = 9.4 + 2.14\,M - 0.054\,M^2 \qquad (20)$$

In: *Bull. Seismol Soc. Amer.* **46** (1956) 105-145.

Paper 1: *Bull. Seismol. Soc. Amer.* **32** (1942) 163-191.



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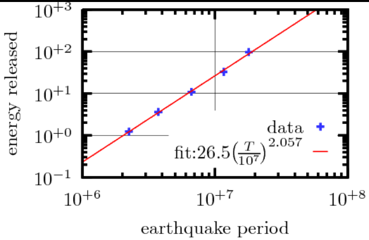
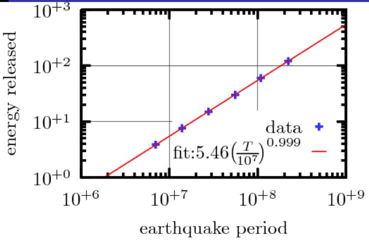
"More seismic" interpretation:

paralel arrangement of many 1-DOF sliders with uniformly distributed $\mathfrak{c}_\mathrm{i}$

$\Rightarrow$ magnitude-M earthquake occurence frequency $\sim 1/M$,
amplitude $\sim M$,
energy released $\sim M^2$.





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Very roughly speaking: 1 Richter $\sim$ 2 Bell

Thanks a lot for your attention.