Friction as an activated process

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Activated processes

- In the non-regularized Dieterich ruina slid.law., the slip velocity $\dot{u}$ was always nonzero.
- Regularized D.-R. required certain "if" condition ($|\dot{u}| \geq 0$)
- It is possible to reformulate the whole problem as a certain variational principle
- Presence of the yield (activation) condition $\longrightarrow$ non-smoothness of the "potentials"

First order optimality conditions for a convex (non-smooth) function

- Subdifferential of a convex function ∂f

$$\partial f(x) := \{x^\star \in X^\star; \forall y \in X : f(x) + \langle x^\star, y - x\rangle \leq f(y)\}$$

- First-order optimality conditions

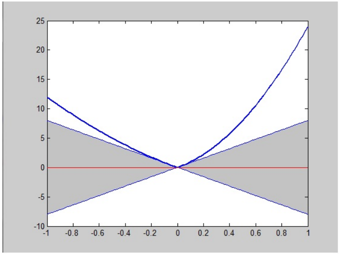
"Smooth" (differentiable case)

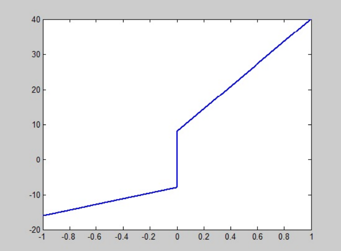
$$df(x) = 0$$

Non-differentiable case

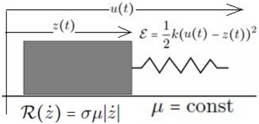
$$0 \in \partial f(x)$$

Example:





Motivation example - a simple model of rate independent friction

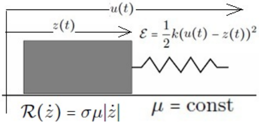


- We define the material by specifying the way it
  - stores energy $\mathcal{E}$
  - dissipates energy $\mathcal{R}$
- Consider a simple model $\mathcal{E} = \hat{\mathcal{E}}(t, u, z) \quad \mathcal{R} = \hat{\mathcal{R}}(\dot{z})$
- Let the dissipation potential be positive 1-homogeneous:

$$\forall \lambda \geq 0, \forall \dot{z} : \mathcal{R}(\lambda \dot{z}) = \lambda \mathcal{R}(\dot{z})$$

- This implies $\langle \partial_{\dot{z}} \mathcal{R}(\dot{z}), \dot{z} \rangle = \mathcal{R}(\dot{z})$
- Analogy with the classical irreversible thermodynamics: rate of entropy production $\sim \sum_i \mathcal{X}_i \mathcal{J}_i \longrightarrow \partial_{\dot{z}} \mathcal{R}$ is the thermodynamic force associated with the flux $\dot{z}$.

Motivation example - a simple model of rate independent friction



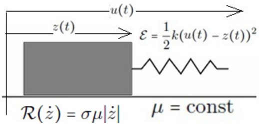
- Define "Lagrangian" $\mathcal{L}(t, u, z, \dot{z}) := \frac{d}{dt}\mathcal{E}(t, u, z) + \mathcal{R}(\dot{z})$
- Evolution corresponds to the following variational problem: minimize $\mathcal{L}$

$$\begin{aligned} \mathcal{L}(t, u, z, \dot{z}) &:= \frac{d}{dt}\mathcal{E}(t, u, z) + \mathcal{R}(\dot{z}) \\ &= \mathcal{E}'_t(t, u, z) + \langle \mathcal{E}'_u(t, u, z), \dot{u} \rangle + \langle \mathcal{E}'_z(t, u, z), \dot{z} \rangle + \mathcal{R}(\dot{z}) \end{aligned}$$

- First-order optimality conditions:

$$\begin{aligned} 0 &= \mathcal{E}'_u(t, u, z) && \text{"minimum energy principle"} \\ 0 &\in \mathcal{E}'_z(t, u, z) + \partial_{\dot{z}}\mathcal{R}(\dot{z}) && \text{"minimum dissipation potential principle"} \end{aligned}$$

Relation with the "maximum-rate-of-entropy-production principle"



- Define $\mathfrak{f} := -\mathcal{E}'_z(t,u,z)$
- "Flow rule" for z: $0 \in -\mathfrak{f} + \partial_{\dot z}\mathcal{R}$

$$\forall v, \forall \omega \in \partial R_{\dot z}(v) : \ \langle \omega - \mathfrak{f}, v - \dot z\rangle \geq 0$$

- In particular, for $v = 0$: $\forall \omega \in \partial\mathcal{R}_{\dot z}(0) : \ \langle \omega - \mathfrak{f}, \dot z\rangle \leq 0$

$$\langle \mathfrak{f}, \dot z\rangle = \max_{\omega\in\partial\mathcal{R}_{\dot z}(0)} \langle \omega, \dot z\rangle \qquad \text{"maximum entropy dissipation principle"}$$

Variational reformulation of the rate-and-state evolution

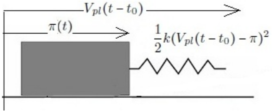
- Motivation:
  - The state- dependence of the friction law and the state evolution lack sufficient thermodynamic reasoning
  - Natural question: Does some of the rate-and-state friction laws correspond to certain dissipation potentials?
- We confine ourselves to the regularized Dieterich-Ruina sliding law:

$$\begin{aligned} \mu(t) &= \mu_0 + A \ln\left(\frac{V_0 + V(t)}{V_\infty + V(t)}\right) + B \ln\left(1 + \theta(t)\frac{V_\infty - V_0}{D_C}\right) \\ \dot{\theta}(t) &= 1 - \theta(t)\frac{V_0 + V(t)}{D_C} \end{aligned}$$

- Can we find $\mathcal{E}$, $\mathcal{R}$, s.t.:

$$\begin{aligned} 0 &\in \partial_\theta \mathcal{E} + \partial_{\dot{\theta}} \mathcal{R} \\ 0 &\in \partial_\pi \mathcal{E} + \partial_{\dot{\pi}} \mathcal{R} \end{aligned}$$

Variational reformulation of the rate-and-state evolution



$$
\begin{aligned}
\mathcal{E}(t,\pi,\theta) &= \frac{1}{2}k(V_{pl}(t-t_0)-\pi)^2 - \kappa\theta \\
\mathcal{R}_1(\theta,\dot\pi,\dot\theta) &= \sigma\left(\mu_0 + A\ln\left(\frac{V_0}{V_\infty}\right) + B\ln\left(1+\theta\,\frac{V_\infty - V_0}{D_C}\right)\right)|\dot\pi| \\
&+ \frac{\kappa\theta}{D_C}\left(\frac{V_0\theta}{D_C} - 1\right)|\dot\pi| + \kappa\,\frac{V_0\theta}{D_C}\,\dot\theta \\
\mathcal{R}_2(\dot\pi,\dot\theta) &= A\sigma\left\{\ln\left(1+\frac{|\dot\pi|}{V_0}\right)(V_0+|\dot\pi|) - \ln\left(1+\frac{|\dot\pi|}{V_\infty}\right)(V_\infty+|\dot\pi|)\right\} \\
&+ \frac{1}{2}\kappa\left(\dot\theta + \frac{\theta}{D_C}|\dot\pi|\right)^2 + \frac{G}{4\beta}\dot\pi^2
\end{aligned}
$$

$$
\begin{aligned}
0 &\in \partial_\theta\mathcal{E} + \partial_{\dot\theta}(\mathcal{R}_1 + \mathcal{R}_2) \\
0 &\in \partial_\pi\mathcal{E} + \partial_{\dot\pi}(\mathcal{R}_1 + \mathcal{R}_2)
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Variational reformulation of the rate-and-state evolution

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$$
\begin{aligned}
\dot\theta &= 1 - \theta\frac{V_0 + |\dot\pi|}{D_C} \\
k(V_{pl}(t-t_0)-\pi) - \frac{G}{2\beta}\dot\pi &\in \sigma\left(\mu_0 + A\ln\left(\frac{V_0+|\dot\pi|}{V_\infty+|\dot\pi|}\right) + B\ln\left(1+\theta\frac{V_\infty - V_0}{D_C}\right)\right)\partial_{\dot\pi}|\dot\pi|
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Variational reformulation of the rate-and-state evolution

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\dot\theta &= 1 - \theta\frac{V_0+|\dot\pi|}{D_C} \\
k(V_{pl}(t-t_0)-\pi) - \frac{G}{2\beta}\dot\pi &= \sigma\left(\mu_0 + A\ln\left(\frac{V_0+|\dot\pi|}{V_\infty+|\dot\pi|}\right) + B\ln\left(1+\theta\frac{V_\infty-V_0}{D_C}\right)\right) && \text{if } \dot\pi > 0 \\
\left|k(V_{pl}(t-t_0)-\pi) - \frac{G}{2\beta}\dot\pi\right| &\le \sigma\left(\mu_0 + A\ln\left(\frac{V_0}{V_\infty}\right) + B\ln\left(1+\theta\frac{V_\infty-V_0}{D_C}\right)\right) && \text{if } \dot\pi = 0 \\
k(V_{pl}(t-t_0)+\pi) - \frac{G}{2\beta}\dot\pi &= -\sigma\left(\mu_0 + A\ln\left(\frac{V_0+|\dot\pi|}{V_\infty+|\dot\pi|}\right) + B\ln\left(1+\theta\frac{V_\infty-V_0}{D_C}\right)\right) && \text{if } \dot\pi < 0
\end{aligned}
$$

Numerics

- Numerically, the variational problem was implemented via a semi-implicit discretization in time:
- First, we discretized the two variational inclusions as follows

$$
\begin{aligned}
0 &\in \partial_\theta \mathcal{E}(t^k, \pi^{k-1}, \theta^k) + \partial_{\dot\theta} \mathcal{R}\left(t^k, \pi^{k-1}, \theta^{k-1}, \frac{\pi^{k-1} - \pi^{k-2}}{\Delta t^{k-1}}, \frac{\theta^k - \theta^{k-1}}{\Delta t^k}\right) \\
0 &\in \partial_\pi \mathcal{E}(t^k, \pi^k, \theta^k) + \partial_{\dot\pi} \mathcal{R}\left(t^k, \pi^{k-1}, \theta^{k-1}, \frac{\pi^k - \pi^{k-1}}{\Delta t^k}, \frac{\theta^k - \theta^{k-1}}{\Delta t^k}\right)
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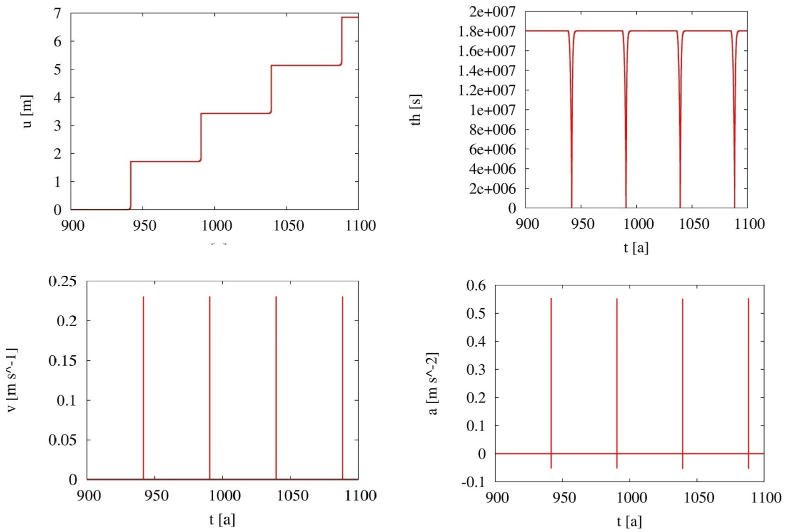
\end{aligned}
$$

- This can be equivalently reformulated as

$$
\begin{aligned}
&\mathrm{minimize}_{\theta^k} \mathcal{E}(t^k, \pi^{k-1}, \theta^k) + \Delta t^k \mathcal{R}\Big(t^k, \pi^{k-1}, \theta^{k-1}, \frac{\pi^{k-1} - \pi^{k-2}}{\Delta t^{k-1}}, \frac{\theta^k - \theta^{k-1}}{\Delta t^k}\Big) \\
&\mathrm{minimize}_{\pi^k} \mathcal{E}(t^k, \pi^k, \theta^k) + \Delta t^k \mathcal{R}\Big(t^k, \pi^{k-1}, \theta^{k-1}, \frac{\pi^k - \pi^{k-1}}{\Delta t^{k-1}}, \frac{\theta^k - \theta^{k-1}}{\Delta t^k}\Big)
\end{aligned}
$$

Numerics - results



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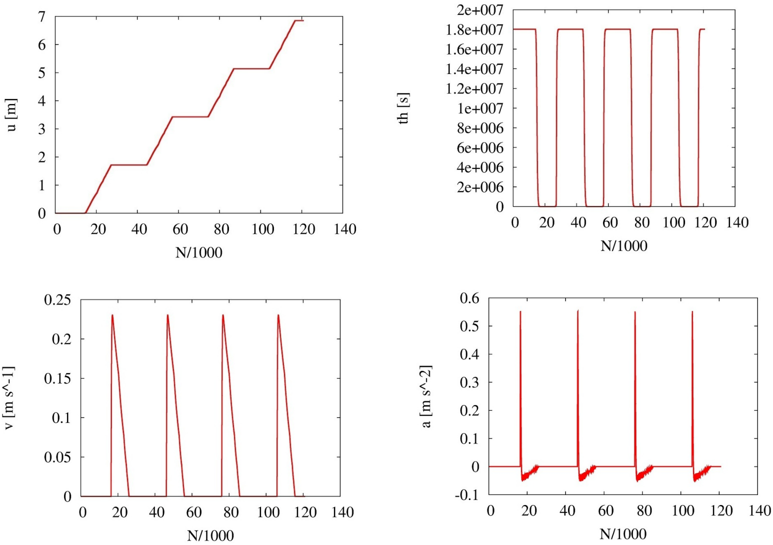

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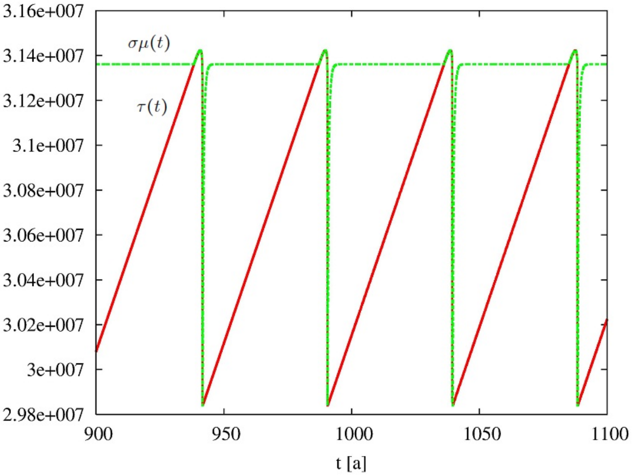
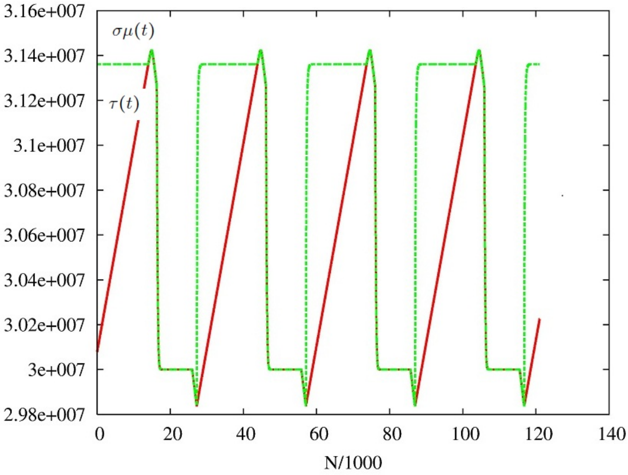
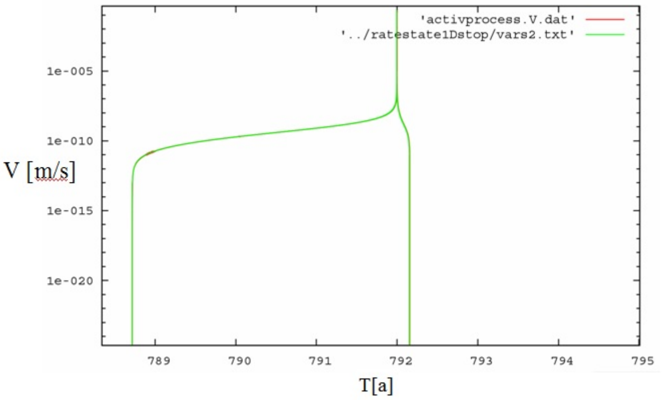


Numerics - results



Numerics - results

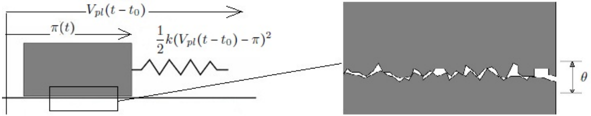
- Comparison with the regularized Dieterich-Ruina model computed using the Runge-Kutta integrator and the stopping criterion



Advantages of the variational formulation

- Variational form gives us some insight into the dissipative mechanisms involved, i.e. a good starting point for further generalizations.
- Rigorous mathematical analysis concerning convergence of the solutions of the discrete problems is achievable for a certain class of models (prof. Roubíček).
- Generalizations for a complex fault structure is straightforward
  - $\mathcal{E} \to \int_\Omega \mathcal{E}\, dx$ (difficulty only here - nonlocal terms - via elastic Green functions)
  - $\mathcal{R} \to \int_\Omega \mathcal{R}\, dx$ (locality of $\mathcal{R}$ - simple "sum" of the dissipative contributions over space)

Interpretation of the variational Dieterich-Ruina sliding law

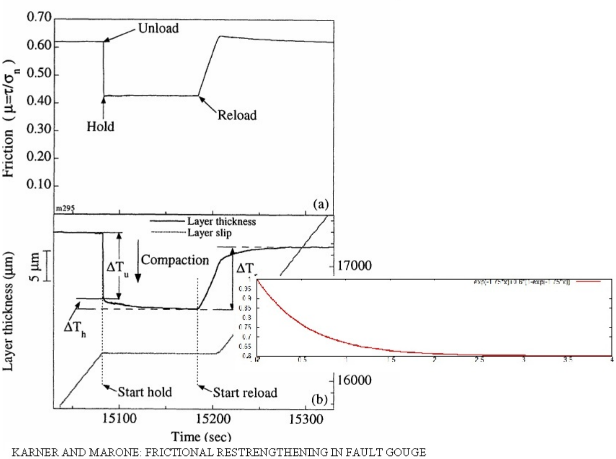


$$\mathcal{E}(t,\pi,\theta) = \frac{1}{2}k(V_{pl}(t-t_0)-\pi)^2 - \kappa\theta$$

- The term $-\kappa\theta$ can be interpreted as the potential energy of the penetration of the surfaces.

Interpretation of the variational Dieterich-Ruina sliding law

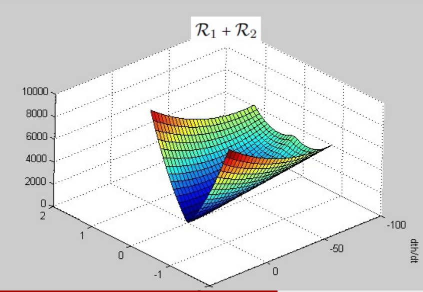
$$\dot{\theta} = 1 - \theta \frac{V_0 + |\dot{\pi}|}{D_c} \qquad |\dot{\pi}| = 0 :$$

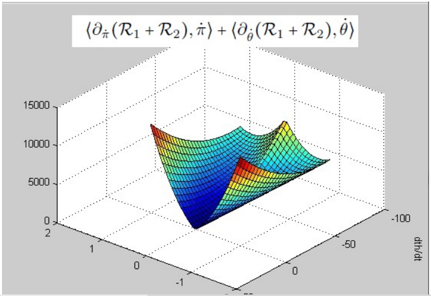


Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the dissipation rate?
- Are θ and π thermodynamically independent?

$$
\begin{aligned}
\mathcal{R}_1(\theta,\dot\pi,\dot\theta) &= \sigma\left(\mu_0 + A\ln\left(\frac{V_0}{V_\infty}\right) + B\ln\left(1+\theta\frac{V_\infty - V_0}{D_C}\right)\right)|\dot\pi| \\
&+ \frac{\kappa\theta}{D_C}\left(\frac{V_0\theta}{D_C} - 1\right)|\dot\pi| + \kappa\frac{V_0\theta}{D_C}\dot\theta \\
\mathcal{R}_2(\dot\pi,\dot\theta) &= A\sigma\left\{\ln\left(1+\frac{|\dot\pi|}{V_0}\right)(V_0+|\dot\pi|) - \ln\left(1+\frac{|\dot\pi|}{V_\infty}\right)(V_\infty+|\dot\pi|)\right\} \\
&+ \frac{1}{2}\kappa\left(\dot\theta + \frac{\theta}{D_C}|\dot\pi|\right)^2 + \frac{G}{4\beta}\dot\pi^2
\end{aligned}
$$





Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the rate-of-dissipation?

$$\begin{aligned}
& \langle \partial_{\dot\pi}(\mathcal{R}_1 + \mathcal{R}_2), \dot\pi\rangle + \langle \partial_{\dot\theta}(\mathcal{R}_1 + \mathcal{R}_2), \dot\theta\rangle \\
= \quad & \sigma\left(\mu_0 + A\ln\left(\frac{V_0+|\dot\pi|}{V_\infty+|\dot\pi|}\right) + B\ln\left(1+\theta\frac{V_\infty - V_0}{D_c}\right)\right)|\dot\pi| \\
+ \quad & \kappa\left(\dot\theta + \frac{\theta|\dot\pi|}{D_c}\right)^2 + \kappa\frac{\theta}{D_c}\left(\dot\theta V_0 + |\dot\pi|\left(\frac{\theta V_0}{D_c} - 1\right)\right)
\end{aligned}$$

Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the rate-of-dissipation?

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+ \quad & \kappa \left(\dot{\theta} + \frac{\theta |\dot{\pi}|}{D_C} \right)^2 + \kappa \frac{\theta}{D_C} \left(\dot{\theta} V_0 + |\dot{\pi}| \left(\frac{\theta V_0}{D_C} - 1 \right) \right)
\end{aligned}
$$

Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the rate-of-dissipation?

$$\begin{aligned} & \langle \partial_{\dot\pi}(\mathcal{R}_1+\mathcal{R}_2),\dot\pi\rangle + \langle \partial_{\dot\theta}(\mathcal{R}_1+\mathcal{R}_2),\dot\theta\rangle \\ = \quad & \sigma\left(\mu_0 + A\ln\left(\frac{V_0+|\dot\pi|}{V_\infty+|\dot\pi|}\right) + B\ln\left(1+\theta\frac{V_\infty-V_0}{D_C}\right)\right)|\dot\pi| \\ + \quad & \kappa\left(\dot\theta+\frac{\theta|\dot\pi|}{D_C}\right)^2 + \kappa\frac{\theta}{D_C}\left(\dot\theta V_0 + |\dot\pi|\left(\frac{\theta V_0}{D_C}-1\right)\right) \end{aligned}$$

- Not: $|\dot\pi|=0$:

$$\langle \partial_{\dot\pi}(\mathcal{R}_1+\mathcal{R}_2),\dot\pi\rangle + \langle \partial_{\dot\theta}(\mathcal{R}_1+\mathcal{R}_2),\dot\theta\rangle = \kappa\dot\theta\left(\dot\theta+\frac{\theta V_0}{D_C}\right)$$

$$\langle \partial_{\dot\theta}(\mathcal{R}_1+\mathcal{R}_2),\dot\theta\rangle\left(\dot\pi=0,\dot\theta=-\frac{\theta V_0}{2D_C}\right) = -\kappa\left(\frac{\theta V_0}{2D_C}\right)^2$$

Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the rate-of-dissipation during the evolution?

$$\dot{\theta} = 1 - \theta \frac{|\dot{\pi}| + V_0}{D_C}$$

$$\begin{aligned} & \langle \partial_{\dot{\pi}}(\mathcal{R}_1 + \mathcal{R}_2), \dot{\pi} \rangle + \langle \partial_{\dot{\theta}}(\mathcal{R}_1 + \mathcal{R}_2), \dot{\theta} \rangle \\ = \quad & \kappa\dot{\theta} + \sigma \left(\mu_0 + A \ln \left(\frac{V_0 + |\dot{\pi}|}{V_\infty + |\dot{\pi}|} \right) + B \ln \left(1 + \theta \frac{V_\infty - V_0}{D_C} \right) \right) |\dot{\pi}| \end{aligned}$$

Interpretation of the variational Dieterich-Ruina sliding law

- Non-negativity of the rate-of-dissipation during the evolution?

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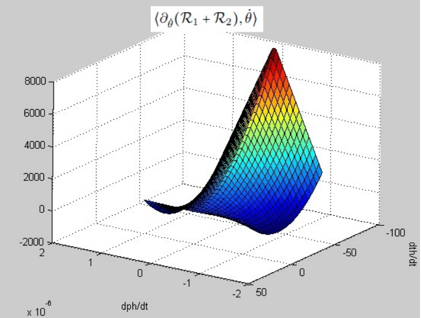
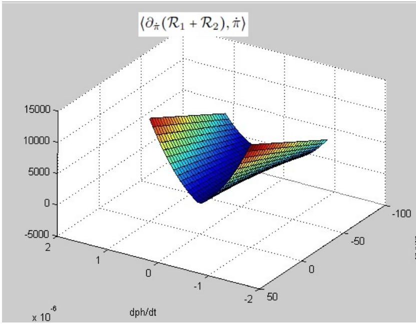
- $\Longrightarrow$ Necessary condition for "stability":

$$0 \leq \kappa \leq V_0 \sigma \left(\mu_0 + A \ln \left(\frac{V_0}{V_\infty} \right) \right)$$

Interpretation of the variational Dieterich-Ruina sliding law

- Independence of the dissipation mechanisms for θ, π?
- Let's inspect the two dissipative contributions





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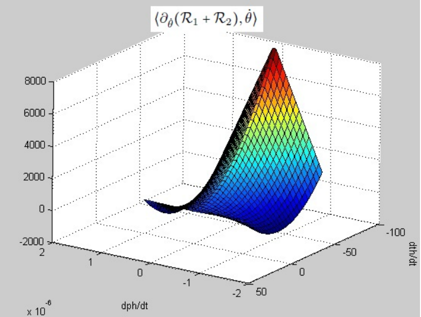

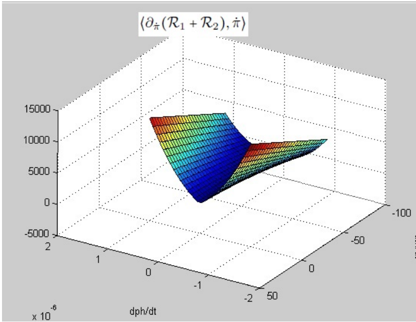

- The lack of separate non-negativity of both contributions suggests that they are **not independent** dissipative mechanisms.

Interpretation of the variational Dieterich-Ruina sliding law - SUMMARY

- We were able to identify the stored energy $\mathcal{E}$ and the dissipation (pseudo)potential $\mathcal{R}$ corresponding to the regularized Dieterich-Ruina model
- The state variable θ was interpreted in terms of the penetration depth of the contact.
- Assuming independence of the dissipative mechanisms for θ, π, the dissipative potentials don't fit the modern thermodynamic viewpoint- we can not ensure non-negativity of the particular dissipations $\langle \partial_{\dot{\pi}}(\mathcal{R}_1+\mathcal{R}_2), \dot{\pi} \rangle$, $\langle \partial_{\dot{\theta}}(\mathcal{R}_1+\mathcal{R}_2), \dot{\theta} \rangle$, nor their sum.
- This suggests that we should try to modify the dissipative potentials in order to still capture the essential dynamics, and simultaneously provide a clear thermodynamic view of the dissipative mechanisms involved.

Thank you for your attention!